Chapter one
CHAPTER 1

ON THE MAIN INVARIANT OF ELEMENTS ALGEBRAIC OVER A HENSELIAN VALUED FIELD

§1.1. MOTIVATION OF THE PROBLEM AND STATEMENT OF THE MAIN RESULT

Let $v$ be a Krull valuation of a field $K$ with value group $G$ and $\bar{v}$ be a fixed prolongation of $v$ to an algebraic closure $\bar{K}$ of $K$ with value group $\bar{G}$. In 1936 MacLane [MAC] gave an iterative method of describing those real valuations $w$ of a simple transcendental extension $K(x)$ of $K$ which extend $v$, when $v$ is a discrete real valuation. In the general case, using some ideas of MacLane, Alexandru et al gave a description of extensions of $v$ to $K(x)$ by means of minimal pairs (see [A-P-Z2,3]). A pair $(\alpha, \delta) \in \bar{K} \times \bar{G}$ is said to be minimal (with respect to $K$ and $\bar{v}$) if whenever $\beta \in \bar{K}$ satisfies $\bar{v}(\alpha - \beta) \geq \delta$, then $[K(\alpha) : K] \leq [K(\beta) : K]$. It is clear that when $\alpha \in K$, then $(\alpha, \delta)$ is a minimal pair for each $\delta \in \bar{G}$ and that a pair $(\alpha, \delta)$ in $(\bar{K} \setminus K) \times \bar{G}$ is minimal if and only if $\delta$ is strictly greater than each element of the set $M(\alpha, K)$ defined by

$$M(\alpha, K) = \{\bar{v}(\alpha - \beta) \mid \beta \in \bar{K}, [K(\beta) : K] < [K(\alpha) : K]\}. \quad (1)$$

¹The results of this chapter are to appear in the paper with the same name which has been accepted for publication in Proc. Edinburgh Math. Soc.
This led to the invariant $\delta_K(\alpha)$ defined for those $\alpha \in \overline{K}/K$ for which $M(\alpha, K)$ has an upper bound in $\overline{G}$, by

$$\delta_K(\alpha) = \sup \{ \bar{v}(\alpha - \beta) \mid \beta \in \overline{K}, [K(\beta) : K] < [K(\alpha) : K] \} \quad (2)$$

where for the sake of supremum, $\overline{G}$ may be viewed as a subset of its Dedekind order completion as defined in [BOU, Chap. III, Sec. 1, Ex. 15]. In 1995, Popescu and Zaharescu proved that if $(K, v)$ is a complete discrete rank 1 valued field, then $\delta_K(\alpha)$ belongs to $M(\alpha, K)$ for each $\alpha \in \overline{K}/K$ (see [A-P-Z2, Theorem 3.9], [P-Z, p. 105]). Examples are known when $\delta_K(\alpha) \in \overline{G}$ but fails to belong to $M(\alpha, K)$ (see Example 1.2.1). This has led us to consider the following problem.

How can we characterize those henselian valued fields $(K, v)$ for which to each $\alpha \in \overline{K}/K$, there corresponds $\beta \in \overline{K}$ satisfying $[K(\beta) : K] < [K(\alpha) : K]$ and $\delta_K(\alpha) = \bar{v}(\alpha - \beta)$?

In the present chapter, we solve this problem by proving

**THEOREM 1.1.1** Let $v$ be a henselian valuation of any rank of a field $K$ and $(\overline{K}, \bar{v})$ be as above. The following two statements are equivalent.

(i) To each $\alpha \in \overline{K}/K$, there corresponds $\beta \in \overline{K}$ with $[K(\beta) : K] < [K(\alpha) : K]$ such that $\delta_K(\alpha) = \bar{v}(\alpha - \beta)$.

(ii) For each $\theta \in \overline{K}$, $K(\theta)/K$ is a defectless extension with respect to the valuation obtained by restricting $\bar{v}$.  

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Recall that a finite extension \((K', v')\) of a henselian valued field \((K, v)\) is said to be \textbf{defectless} if \([K' : K] = ef\), where \(e\), \(f\) are respectively the index of ramification and the residual degree of \(v'/v\).

The above theorem, in turn has given rise to the following problem.

\textit{For a henselian field} \((K, v)\), \textit{if} \(K(\theta)/K\) \textit{is defectless for each} \(\theta \in \overline{K}\), \textit{then is it true that every finite extension of} \((K, v)\) \textit{is defectless?}

An example has been given in the last section of this chapter to show that the answer to the above question is “no” in general.

\section*{§1.2. DEFINITIONS, NOTATIONS AND SOME PRELIMINARY RESULTS}

Throughout \(v\) is a henselian valuation defined on a field \(K\) with value group \(G\) of any rank and \(\overline{v}\) is (unique) extension of \(v\) to a fixed algebraic closure \(\overline{K}\) of \(K\) with value group \(\overline{G}\). For an overfield \(L\) of \(K\) contained in \(\overline{K}\), \(R(L)\) and \(G(L)\) will respectively stand for the residue field and the value group of the valuation of \(L\) obtained by restricting \(\overline{v}\). For a finite extension \(L/K\), \(\text{def}(L/K)\) will stand for the \textbf{defect} of the valued field extension \(L/K\), with respect to the valuation \(v_L\).
obtained by restricting \( \tilde{v} \) to \( L \), i.e.,

\[
def(L/K) = [L : K]/ef,
\]

where \( e, f \) are the index of ramification and the residual degree of \( v_L/v \). By the
degree (over \( K \)) of an element \( \alpha \in \overline{K} \), we shall mean the degree of the extension
\( K(\alpha)/K \). For any \( \xi \) in the valuation ring of \( \tilde{v} \), \( \xi^* \) will denote its \( \tilde{v} \)- residue, i.e.,
the image of \( \xi \) under the canonical homomorphism from the valuation ring of \( \tilde{v} \)
onto its residue field.

If \( f(x) \) is a fixed nonzero polynomial in \( K[x] \), then using the Euclidean
algorithm, each \( F(x) \in K[x] \) can be uniquely represented as a finite sum
\[
\sum_{i \geq 0} F_i(x)f(x)^i,
\]
where for any \( i \), the polynomial \( F_i(x) \) is either 0 or has degree less
than that of \( f(x) \). The above representation will be referred to as the
\( f \)- expansion of \( F(x) \).

We need the following theorem, which is already known (see [KH-SA2]);
its proof is omitted.

**THEOREM 1.2.A** Let \((K, \nu)\) be a henselian valued field of any rank.
Let \( \alpha, \beta \in \overline{K} \) be such that \( \tilde{v}(\alpha - \beta) > \tilde{v}(\alpha - \gamma) \) for any \( \gamma \in \overline{K} \) satisfying
\([K(\gamma) : K] < [K(\alpha) : K]\). Then

(i) \( G(K(\alpha)) \subseteq G(K(\beta)) \);

(ii) \( R(K(\alpha)) \subseteq R(K(\beta)) \);

(iii) \( \text{def}(K(\alpha)/K) \) divides \( \text{def}(K(\beta)/K) \).
Let \((K, v), (\overline{K}, \overline{v})\) be as above and \((\alpha, \delta)\) belonging to \(K \times \overline{G}\) be a minimal pair. The valuation \(\overline{w}_{\alpha, \delta}\) of \(\overline{K}(x)\), defined on \(\overline{K}[x]\) by

\[
\overline{w}_{\alpha, \delta}(\sum c_i (x - \alpha)^i) = \min\{v(c_i) + i\delta\}, \quad c_i \in \overline{K}
\]

will be referred to as the valuation defined by the pair \((\alpha, \delta)\). The description of \(\overline{w}_{\alpha, \delta}\) on \(K[x]\) is given by the already known theorem stated below (see [A-P-Z1, Theorem 2.1], [KH2, Theorem 1.4]).

**Theorem 1.2.B** Let \(\overline{w}_{\alpha, \delta}\) be the valuation of \(\overline{K}(x)\) defined by a minimal pair \((\alpha, \delta)\) and \(w_{\alpha, \delta}\) the valuation of \(K(x)\) obtained by restricting \(\overline{w}_{\alpha, \delta}\). Let \(f(x)\) be the minimal polynomial of \(\alpha\) over \(K\) of degree \(n\) and \(\lambda\) be an element of \(\overline{G}\) such that \(w_{\alpha, \delta}(f(x)) = \lambda\). One has

(i) For any \(F(x)\) belonging to \(K[x]\) with \(f\)-expansion \(\sum F_i(x)f(x)^i\),

\[
w_{\alpha, \delta}(F(x)) = \min_i \{v(F_i(\alpha)) + i\lambda\};
\]

(ii) if \(g(x) \in K[x]\) is a polynomial of degree less than \(n\), then the \(\overline{w}_{\alpha, \delta}\)-residue of \(g(x)/g(\alpha)\) equals one;

(iii) let \(e\) be the smallest positive integer such that \(e\lambda \in G(K(\alpha))\) and \(h(x) \in K[x]\) be a polynomial of degree less than \(n\) with \(w_{\alpha, \delta}(h(x)) = e\lambda\), then the \(w_{\alpha, \delta}\)-residue \((f^e/h)^*\) of \(f(x)^e/h(x)\) is transcendental over the residue field of \(v\) and the residue field of \(w_{\alpha, \delta}\) is \(R(K(\alpha))((f^e/h)^*)\).
The following theorem is proved in [KH-PO-RO, Theorem 3.1]. It is an immediate consequence of the well known fact that completion of a henselian valued field is henselian and of [A-P-Z2, Corollary 3.10]. Its proof is omitted.

**THEOREM 1.2.C** Let \((\tilde{K}, \tilde{v})\) be completion of a henselian valued field \((K, v)\) and \(\alpha\) be an element of \(\bar{K}\setminus K\). Then \(M(\alpha, K)\) has an upper bound in \(\tilde{G}\), if and only if \([K(\alpha) : K] = [\bar{K}(\alpha) : \bar{K}]\).

It may be pointed out that the supremum of \(M(\alpha, K)\) being in \(G\) does not necessarily imply that it belongs to \(M(\alpha, K)\). Here is an example to support this assertion.

**EXAMPLE 1.2.1** Let \(k_0\) be the algebraic closure of the finite field \(F_2\) of 2 elements and \(K_0 = k_0((T))\) be the field of Laurent series in \(T\) with valuation \(v_0\) given by \(v_0(T) = 1\). Let \(\bar{v}_0\) be the extension of \(v_0\) to an algebraic closure \(\bar{K}_0\) of \(K_0\). Let \(K\) be the inseparable closure of \(K_0\) in \(\bar{K}_0\), with valuation \(v\) which is the restriction of \(\bar{v}_0\). Then \((K, v)\) being an algebraic extension of a complete rank 1 valued field is henselian. Let \(\alpha\) be a root of the polynomial \(x^2 - x - T^{-1} = 0\). As shown in [KH1], there does not exist any \(c \in K\) such that \(\bar{v}(\alpha - c) \geq 0\), whereas \(\delta_K(\alpha) = 0\) by virtue of a result of Ax [AX, Lemma 6] stated below and the fact that \(\Delta_K(\alpha) = 0\).
LEMMA 1.2.D (Ax) Let \( v \) be a henselian valuation defined on a perfect field \( K \) of characteristic \( p > 0 \) and \( \bar{v} \) be as above. If \( \alpha \) is algebraic of degree \( p \) over \( K \), then for each positive integer \( j \), there exists \( \beta_j \in K \) such that

\[
\bar{v}(\alpha - \beta_j) \geq \left( \frac{p-1}{p} \right)^j \bar{v}(\alpha) + \frac{1}{p} \left( 1 + \frac{p-1}{p} + \ldots + \left( \frac{p-1}{p} \right)^j \right) \Delta_K(\alpha).
\]

We prove two lemmas; the first one is well known [KH2, Lemma 2.1(ii)]. For the sake of completion, we prove it here.

LEMMA 1.2.1 Let \((\alpha, \delta)\) be a minimal pair (with respect to \( K \) and \( \bar{v} \)) and \( \theta \) be an element of \( \overline{K} \) with \( \bar{v}(\theta - \alpha) \geq \delta \). Let \( h(x) \in K[x] \) be a polynomial such that for each root \( \beta \) of \( h(x) \), \( \bar{v}(\alpha - \beta) < \delta \). Then \( \bar{v}(h(\theta) - h(\alpha)) > \bar{v}(h(\alpha)) \).

Proof. Write \( h(x) = c \prod_j (x - \beta_j) \). Then

\[
\frac{h(\theta)}{h(\alpha)} = \prod_j \left( \frac{\theta - \beta_j}{\alpha - \beta_j} \right) = \prod_j \left( 1 + \frac{\theta - \alpha}{\alpha - \beta_j} \right).
\]

By hypothesis

\[
\bar{v} \left( \frac{\theta - \alpha}{\alpha - \beta_j} \right) \geq \delta - \bar{v}(\alpha - \beta_j) > 0.
\]

Therefore \( \bar{v} \left( \frac{h(\theta)}{h(\alpha)} - 1 \right) > 0 \) as desired.

LEMMA 1.2.2 Let \((K, v)\) be henselian and \( \theta \) be an element of \( \overline{K} \setminus K \) such that \( \delta_K(\theta) \) defined by (2) belongs to \( M(\theta, K) \). If \( \alpha \in \overline{K} \) is an element of smallest degree over \( K \) such that \( \bar{v}(\theta - \alpha) = \delta_K(\theta) \), then
(a) \((\alpha, \delta_K(\theta))\) is a minimal pair,

(b) \(\bar{w}_{\alpha, \delta}(G(x)) = \bar{v}(G(\theta))\) for any polynomial \(G(x) \in K[x]\) of degree less than the degree of \(\theta\) over \(K\), where the valuation \(\bar{w}_{\alpha, \delta}\) is as defined by (3) with \(\delta = \delta_K(\theta)\).

**Proof.**  
(a) We show that for every \(\gamma \in \bar{K}\) with \(deg\gamma < deg\alpha\), the inequality \(\bar{v}(\alpha - \gamma) < \delta_K(\theta)\) holds. For such an element \(\gamma\), the choice of \(\alpha\) gives \(\bar{v}(\theta - \gamma) < \bar{v}(\theta - \alpha)\), which by virtue of the strong triangle law implies that

\[
\bar{v}(\alpha - \gamma) = \min\{\bar{v}(\alpha - \theta), \bar{v}(\theta - \gamma)\} = \bar{v}(\theta - \gamma).
\]

Consequently \(\bar{v}(\alpha - \gamma) < \delta_K(\theta)\) as desired.

(b) Write \(G(x) = c \prod (x - \beta_i)\). By virtue of (3)

\[
\bar{w}_{\alpha, \delta}(G(x)) = \bar{v}(c) + \sum_i \min\{\bar{v}(\alpha - \beta_i), \delta\}. \tag{4}
\]

Keeping in view (4), the desired assertion is proved once it is shown that for each root \(\beta_i\) of \(G(x)\),

\[
\min\{\bar{v}(\alpha - \beta_i), \delta\} = \bar{v}(\theta - \beta_i). \tag{5}
\]

Since \(degG(x)\) is less than \([K(\theta): K]\), it follows that

\[
\bar{v}(\theta - \beta_i) \leq \delta_K(\theta) = \delta.
\]

Using the above inequality and the fact that \(\bar{v}(\theta - \alpha) = \delta\), one can quickly verify (5).
§1.3. PROOF OF (i) IMPLIES (ii) IN THEOREM 1.1.1

Assuming (i) we prove assertion (ii) of Theorem 1.1.1 by induction on the degree of the extension \( K(\theta)/K \). Clearly it is enough to prove that to each \( \theta \in \overline{K}\backslash K \), there corresponds \( \alpha \in \overline{K} \) such that \( [K(\alpha) : K] < [K(\theta) : K] \) and \( \text{def}(K(\alpha)/K) = \text{def}(K(\theta)/K) \). Indeed we prove

THEOREM 1.3.1 Let \((K, v)\) be a henselian valued field and \( \theta \) be an element of \( \overline{K}\backslash K \). Let \( \alpha \in \overline{K} \) be such that \( \delta_K(\theta) = v(\theta - \alpha) \) and \( [K(\alpha) : K] < [K(\theta) : K] \). Assume that \( \alpha \) is an element of smallest degree over \( K \) with this property. If \( f(x) \) is the minimal polynomial of \( \alpha \) over \( K \), then

(i) \( \text{def}(K(\theta)/K) = \text{def}(K(\alpha)/K) \);

(ii) \( G(K(\theta)) = G(K(\alpha)) + \mathbb{Z}v(f(\theta)) \);

(iii) \( R(K(\theta)) = R(K(\alpha)) \left( \left( \frac{f(\theta)^e}{h(\alpha)} \right)^* \right) \), where \( e \) is the smallest positive integer such that \( e\overline{v}(f(\theta)) = \overline{v}(h(\alpha)) \) is in \( G(K(\alpha)) \);

Proof. Let \( m, n \) denote respectively the degrees of the extensions \( K(\theta)/K \) and \( K(\alpha)/K \). By the choice of \( \alpha \) if an element \( \gamma \) in \( \overline{K} \) has degree strictly less than the degree of \( \alpha \), then \( \overline{v}(\theta - \gamma) < \overline{v}(\theta - \alpha) \); consequently

\[ \overline{v}(\alpha - \gamma) = \min\{\overline{v}(\alpha - \theta), \overline{v}(\theta - \gamma)\} = \overline{v}(\theta - \gamma) < \overline{v}(\alpha - \theta). \]

Therefore Theorem 1.2.A implies that

\[ G(K(\alpha)) \subseteq G(K(\theta)), \quad R(K(\alpha)) \subseteq R(K(\theta)), \quad \text{def}(K(\alpha)/K) \leq \text{def}(K(\theta)/K). \quad (6) \]
By hypothesis $e$ is the smallest positive integer such that $ev(f(\theta)) \in G(K(\alpha))$; in fact $ev(f(\theta)) = \psi(h(\alpha))$, where $h(x) \in K[x]$ is a polynomial of degree less than $n$. By Lagrange’s theorem of groups, $e$ divides $[G(K(\theta)) : G(K(\alpha))]$. It now follows from (6) that $en$ divide $m$. We denote $m/en$ by $l$ and $f(\theta)^e/h(\alpha)$ by $\xi$. Then

$$l = \left[ R(K(\theta)) : R(K(\alpha)) \right] \left( \frac{[G(K(\theta)) : G(K(\alpha))]}{e} \right) \left( \frac{def(K(\theta)/K)}{def(K(\alpha)/K)} \right). \tag{7}$$

The theorem is proved as soon as we show that the element $\xi^*$ of $R(K(\theta))$ is algebraic over $R(K(\alpha))$ of degree $l$, for then by virtue of (7) we shall have

$$[G(K(\theta)) : G(K(\alpha))] = e,$$
$$R(K(\theta)) = R(K(\alpha))\langle \xi^* \rangle,$$
$$def(K(\theta)/K) = def(K(\alpha)/K).$$

Suppose to the contrary that $\xi^*$ is algebraic over $R(K(\alpha))$ of degree $q < l$. Then there exist polynomials $A_i(x) \in K[x]$, $A_0(\alpha)^* \neq 0$ such that

$$\left( \frac{f(\theta)^e}{h(\alpha)^e} \right)^q + A_{q-1}(\alpha)^* \left( \frac{f(\theta)^e}{h(\alpha)^e} \right)^{q-1} + \ldots + A_0(\alpha)^* = 0. \tag{8}$$

For $0 \leq i < q - 1$, we write $A_i(\alpha)/h(\alpha)^i$ as $B_i(\alpha)$ and $h(\alpha)^{-q}$ as $B_q(\alpha)$, where each $B_i(x) \in K[x]$ is of degree less than $n$. So (8) can be rewritten as

$$(B_q(\alpha)f(\theta)^{\psi(\alpha)})^* + (B_{q-1}(\alpha)f(\theta)^{\psi(\alpha-1)})^* + \ldots + B_0(\alpha)^* = 0 \tag{9}$$

with $B_0(\alpha)^* = A_0(\alpha)^* \neq 0$. Recall that by virtue of Lemma 1.2.2(a), $(\alpha, \delta)$ is a minimal pair with $\delta = \psi(\theta - \alpha)$. So by Lemma 1.2.1, $(B_i(\alpha)/B_i(\theta))^* = 1$. Therefore
(9) shows that
\[ v(B_q(\theta)f(\theta)^{e_q} + B_{q-1}(\theta)f(\theta)^{e(q-1)} + \ldots + B_0(\theta)) > 0. \] (10)

Set
\[ G(x) = B_q(x)f(x)^{e_q} + B_{q-1}(x)f(x)^{e(q-1)} + \ldots + B_0(x). \] (11)

Observe that
\[ \text{deg } G(x) < eqn + n \leq e(l - 1)n + n = m - en + n \leq m. \]

As the expansion of \( G(x) \) given by (11) is its \( f \)-expansion, it follows from Theorem 1.2.B(i) that
\[ \min_{0 \leq i \leq q} \{ v(B_i(\alpha)) + i\epsilon v_{\alpha,\delta}(f(x)) \} \leq v(B_0(\alpha)) = 0. \] (12)

Keeping in view that \( \text{deg } G(x) < m \), we conclude by Lemma 1.2.2(b) and (10) that \( \bar{v}_{\alpha,\delta}(G(x)) = v(G(\theta)) > 0 \), which contradicts (12).

This contradiction proves that \( (f(\theta)^{\epsilon}/h(\alpha))^\ast \) is algebraic of degree \( l \) over \( R(K(\alpha)) \) as desired.

The following corollaries, which have actually been obtained during the course of the proof of the above theorem, are also immediate consequences of the theorem.

**COROLLARY 1.3.2** Let \( \theta, \alpha, e \) be as the foregoing theorem. Then \( e(\deg \alpha) \) divides \( \deg \theta \).
COROLLARY 1.3.3 If \( \theta, \alpha \) are as in the above theorem, then with notations as in the theorem, \( \left( \frac{f(\theta)^*}{h(\alpha)} \right)^* \) is algebraic of degree \( m/\text{en} \) over \( R(K(\alpha)) \).

§1.4. PROOF OF (ii) IMPLIES (i) IN THEOREM 1.1.1

For this, we prove

THEOREM 1.4.1 Let \((K, v)\) be a henselian valued field. If \( \alpha \in \overline{K}\setminus K \) is such that \( K(\alpha)/K \) is defectless, then \( \delta_K(\alpha) \in M(\alpha, K) \).

Proof. Since \( K(\alpha)/K \) is defectless, it follows that \( [K(\alpha) : K] = [\overline{K}(\alpha) : \overline{K}] \), where \((\overline{K}, \overline{v})\) is completion of \((K, v)\). Therefore by virtue of Theorem 1.2.C, \( M(\alpha, K) \) has an upper bound in \( \overline{G} \); consequently \( \delta_K(\alpha) \) is defined in the Dedekind order completion of \( \overline{G} \). Assume that \( \delta_K(\alpha) \notin M(\alpha, K) \).

We shall obtain the desired contradiction by showing that \( K(\alpha)/K \) is not defectless. Since \( M(\alpha, K) \) is totally ordered without last element, it contains a well ordered cofinal subset (cf. [HAL, P.68]). So we can choose a net \( \{\delta_i\}_{i \in I} \) in \( M(\alpha, K) \) satisfying

1) \( \{\delta_i\}_{i \in I} \) is cofinal in \( M(\alpha, K) \) and \( \delta_i < \delta_j \) for \( i < j, i, j \in I \);

2) \( \delta_i = \overline{v}(\alpha - \beta_i), \beta_i \in \overline{K} \) is such that \( \text{deg} \beta_i < n = \text{deg} \alpha \) and whenever \( \gamma \in \overline{K} \) has degree less than \( \text{deg} \beta_i \), then \( \overline{v}(\alpha - \gamma) < \delta_i \).

If necessary on replacing \( \{\delta_i\}_{i \in I} \) by a subnet, we may assume that all \( \beta_i \)
are of the same degree (say \( s \)) over \( K \). Keeping in view that \( \delta_i < \delta_j \) for \( i < j \), we have

\[
\overline{v}(\beta_i - \beta_j) \geq \min\{\overline{v}(\beta_i - \alpha), \overline{v}(\alpha - \beta_j)\} = \delta_i,
\]

and for any \( \gamma \in \overline{K} \) with \( \deg \gamma < s \)

\[
\overline{v}(\beta_i - \gamma) = \overline{v}(\beta_i - \alpha + \alpha - \gamma) = \overline{v}(\alpha - \gamma) < \delta_i.
\]

Consequently it follows from Theorem 1.2.A, that

\[
G(K) \subseteq G(K(\beta_i)) \subseteq G(K(\beta_j)), \ i < j, \tag{13}
\]

\[
R(K) \subseteq R(K(\beta_i)) \subseteq R(K(\beta_j)), \ i < j, \ i, j \in I. \tag{14}
\]

As all the extensions \( K(\beta_i)/K \) are of the same degree \( s < n \), it is clear from (13) and (14) that there exists \( j_0 \in I \) such that

\[
G(K(\beta_j)) = G(K(\beta_{j_0})), \ R(K(\beta_j)) = R(K(\beta_{j_0})) \quad \text{for} \ j \geq j_0
\]

Therefore

\[
\bigcup_{i \in I} G(K(\beta_i)) = G(K(\beta_{j_0})), \ \bigcup_{i \in I} R(K(\beta_i)) = R(K(\beta_{j_0})). \tag{15}
\]

We are going to prove that

\[
G(K(\alpha)) = \bigcup_{i \in I} G(K(\beta_i)), \ R(K(\alpha)) = \bigcup_{i \in I} R(K(\beta_i)). \tag{16}
\]

As the extension \( K(\alpha)/K \) is of degree \( n > s \), (15) and (16) immediately imply that \( K(\alpha)/K \) is not defectless leading to the desired contradiction.
To prove (16), let $F(x) \in K[x]$ be any polynomial of degree less than $n$. It is enough to prove that there exists $k \in I$ such that $\bar{v}(F(\alpha) - F(\beta_k)) > \bar{v}(F(\beta_k))$.

Let $\gamma$ be a root of $F(x)$. Since $\bar{v}(\alpha - \gamma) \in M(\alpha, K)$, it follows from property 1 of the net $\{\delta_i\}_{i \in I}$ that there exists $k \in I$ such that $\bar{v}(\alpha - \gamma) < \delta_k$. Choosing $k$ sufficiently large, we may assume that

$$\bar{v}(\alpha - \gamma_i) < \delta_k$$

for each root $\gamma_i$ of $F(x)$. Write $F(x) = c \prod (x - \gamma_i)$. Then

$$\frac{F(\alpha)}{F(\beta_k)} = \prod_i \left( \frac{\alpha - \gamma_i}{\beta_k - \gamma_i} \right) = \prod_i \left( 1 + \frac{\alpha - \beta_k}{\beta_k - \gamma_i} \right).$$

(18)

Since $\bar{v}(\alpha - \gamma_i) < \delta_k$ by (17) and $\bar{v}(\alpha - \beta_k) = \delta_k$ by choice of $\delta_k$, we see that

$$\bar{v}(\beta_k - \gamma_i) = \min\{\bar{v}(\beta_k - \alpha), \bar{v}(\alpha - \gamma_i)\} = \bar{v}(\alpha - \gamma_i).$$

Consequently (18) gives $\bar{v} \left( \frac{F(\alpha)}{F(\beta_k)} - 1 \right) > 0$, which proves (16) and completes the proof of the theorem.

The following example shows that the converse of Theorem 1.4.1 is not true in general.

**EXAMPLE 1.4.2** Let $Q_2$ be the field of $2$-adic numbers with the usual valuation $v_2$ characterized by $v_2(2) = 1$. Let $K = \bigcup_{n=0}^{\infty} K_n$ where $K_0 = Q_2$, $K_1 = K_0(\xi_1)$ with $\xi_1^2 = 2$ and by induction $K_n = K_{n-1}(\xi_n)$ with $\xi_n^2 = \xi_{n-1}$. Since $K/Q_2$ is an algebraic extension, $v_2$ has a unique prolongation to $K$ which
will be denoted by $v$. As usual, $\bar{v}$ will stand for the prolongation of the henselian valuation $v$ to the algebraic closure $\bar{K}$ of $K$.

Let $i$ denote a primitive fourth root of unity. It has been shown in [RIB, §6.3] that $K(i)/K$ is an immediate extension of degree 2 and that the residue field of $v$ is the field $F_2$ of 2 elements.

Consider $\theta \in \bar{K}$ defined by

$$\theta = -1 + i(1 + \sqrt{3}) = i + 2\omega,$$

where $\omega$ is a primitive 3rd root of unity. We shall prove that $K(\theta)/K$ is an extension of degree 4, $\delta_K(\theta) \in M(\theta, K)$ and that $K(i) \subseteq K(\theta)$, which shows that $K(\theta)/K$ is not defectless.

For proving that $[K(\theta) : K] = 4$, it is enough to show that $\omega \notin K(i)$. Suppose to the contrary $\omega \in K(i)$. The fact, that the residue field of $\bar{v}$ restricted to $K(i)$ is $F_2$, together with the supposition implies that the $\bar{v}$-residue $\omega^*$ of $\omega$ equals 1*, i.e.

$$\bar{v}(\omega - 1) = \bar{v}(\frac{-3 + i\sqrt{3}}{2}) > 0.$$

On taking norm, the last inequality gives $\bar{v}(6/4) > 0$, which is a contradiction. Hence $\omega \notin K(i)$.

In order to prove that $\delta_K(\theta) \in M(\theta, K)$, we are going to show that if $\alpha = \frac{1}{\sqrt{3}} + 4$, then

$$\bar{v}(\theta - \alpha) = \delta_K(\theta).$$

Since $\theta - \alpha = \frac{2\omega(1 + \sqrt{3})}{\sqrt{3}} - 4$ and $\bar{v}(1 + \sqrt{3}) = \frac{1}{2}$, it follows by virtue of the
strong triangle law that

\[ \nu(\theta - \alpha) = \min \left\{ \nu \left( \frac{2\omega(1 + \sqrt{3})}{\sqrt{3}} \right), \nu(4) \right\} = 3/2. \quad (19) \]

Let \( \theta_1 = \theta, \theta_2 = -i + 2\omega, \theta_3 = i + 2\omega^2, \theta_4 = -i + 2\omega^2 \) be all the \( K \)-conjugates of \( \theta \) and \( \omega_K(\theta) \) denotes the Krasner’s constant defined by

\[ \omega_K(\theta) = \max\{\nu(\theta - \theta_2), \nu(\theta - \theta_3), \nu(\theta - \theta_4)\}. \]

A simple calculation shows that

\[ \omega_K(\theta) = \max\{\nu(2i), \nu(2\sqrt{3}i), \nu(2i + 2\sqrt{3}i)\} = \frac{3}{2}. \]

As \( \delta_K(\theta) \leq \omega_K(\theta) \) by Krasner’s Lemma, it follows from (19) and the above equation that

\[ \nu(\theta - \alpha) = \delta_K(\theta) = \frac{3}{2}. \quad (20) \]

It only remains to be shown that \( i \in K(\theta) \). Since \( \theta + 1 = i(1 + \sqrt{3}) \), on taking squares, we see that \( \sqrt{3} \in K(\theta) \) and hence \( i = \frac{\theta + 1}{1 + \sqrt{3}} \) belongs to \( K(\theta) \) as desired.

§1.5. AN EXAMPLE

We give an example to show that the assumption "\( K(\alpha)/K \) defectless for each \( \alpha \in \overline{K} \)" does not imply in general that every finite extension of a henselian valued field \( (K, \nu) \) is defectless. The construction of the field \( (K, \nu) \) given below appears in a different context in [KUF2, Chapter 8.4].
Let $F_p((t))$ be the field of Laurent series in an indeterminate $t$ with coefficients from the finite field $F_p$ of $p$ elements, $p$ prime, with the valuation $v_t$ given by $v_t(t) = 1$. Fix $x, y \in F_p((t))$ both of $v_t$-valuation 1, which are algebraically independent over $F_p(t)$. Set $s = x^p + ty^p$ and $L = F_p(s, t)$. Then $s, t$ are algebraically independent over $F_p$, because $F_p(s^{1/p}, t^{1/p}, x) = F_p(t^{1/p}, x, y)$ is of transcendence degree 3 over $F_p$.

Let $K$ be the algebraic closure of $L$ in $F_p((t))$ with valuation $v$, which is the restriction of $v_t$. Claim is that $K/L$ is a separable extension, which shows that $K$ being the separable closure of $L$ in a complete discrete rank 1 valued field is henselian (see [END, 17.18]). Since $K/L$ is a normal extension, the claim is proved once we show that whenever an element $\alpha$ of $F_p((t))$ belongs to $L^{1/p} = F_p(s^{1/p}, t^{1/p})$, then $\alpha \in L$. Write $\alpha = P/Q$, with $P, Q$ in $F_p[s^{1/p}, t^{1/p}]$. Since $Q^p \in F_p[s, t] \subseteq L$, on replacing $\alpha$ by $\alpha Q^p$ we may assume that $\alpha \in F_p[s^{1/p}, t^{1/p}]$. Write

$$\alpha = \sum_{i,j} a_{ij} s^{i/p} t^{j/p}, \quad a_{ij} \in F_p, \quad a_{ij} \neq 0. \quad (21)$$

It is to be shown that $p$ divides each $i$ and $j$. Suppose this is false. On replacing $\alpha$ by $\alpha - c$ for some $c \in F_p[s, t]$, we may assume that for each pair $(i, j)$ appearing in (21), either $p$ does not divide $i$ or $p$ does not divide $j$. Let $a_{mn} s^{m/p} t^{n/p}$ be the smallest degree monomial in the variables $s^{1/p}, t^{1/p}$ occurring in (21) in which the exponent of $s^{1/p}$ is also the smallest, i.e., if $a_{ij} s^{i/p} t^{j/p}$ occurs in (21), then $i \geq m$ and $i + j \geq m + n$. On dividing $\alpha$ by a suitable integral power of $s$, we may further assume that $0 \leq m \leq p - 1$. 24
We now prove that
\[ v_t(\alpha) = \bar{v}_t(a_{mn} \ s^{m/p} \ t^{n/p}) = m + (n/p). \] (22)

The second equality in (22) holds by virtue of the fact that \( v_t(s) = p \). The first equality is verified once we show that each monomial \( a_{ij} \ s^{i/p} \ t^{j/p} \neq a_{mn} \ s^{m/p} \ t^{n/p} \) occurring in (21) has \( \bar{v}_t \) - valuation strictly greater than \( m + (n/p) \). Keeping in view that for a such monomial, \( i + j \geq m + n \) and either \( i > m \) or \( j > n \), one concludes that \( i + (j/p) > m + (n/p) \), for otherwise \( i - m \leq (n - j)/p \) and consequently \( n - j \leq i - m \leq (n - j)/p \), which is possible only when \( i = m, j = n \). This proves the desired assertion.

As \( \alpha \in F_p((t)) \), \( v_t(\alpha) \) is an integer. Hence by (22), \( p \) divides \( n \). So our supposition gives \( p \) does not divide \( m \), i.e., \( 0 < m < p \). On recalling that \( s^{1/p} = x + t^{1/p}y \), we can rewrite \( \alpha \) as
\[ \alpha = A_0(x, y, t) + A_1(x, y, t) \ t^{1/p} + \ldots + A_{p-1}(x, y, t) \ t^{(p-1)/p} \] (23)
where each \( A_j(x, y, t) \in F_p[x, y, t] \). It is clear that all the non-zero summands on the right hand side of (23) have different \( \bar{v}_t \)-valuation. It may be pointed out that \( A_m(x, y, t) \neq 0 \); in fact we show \( a_{mn} y^m \ t^{n/p} \) is the monomial of smallest degree (in \( y, t \)) among those monomials of \( A_m(x, y, t) \) which are free from \( x \). By virtue of (21), any monomial in the variables \( y, t \) which actually occurs in \( A_m(x, y, t) \) is of the type \( cy^i t^{(i+j-m)/p} \) for some \( c \in F_p \) with \( i \geq m, i+j \geq m+n \) and therefore has (in \( y, t \)) degree \( i+(i+j-m)/p > m+(n/p) \) except when \( i = m, j = n \). Keeping in view that \( m \geq 1 \), we conclude that
$v_t(\alpha - A_0(x, y, t))$ is not a rational integer, which is not so. This contradiction proves that $K/L$ is a separable extension.

We next show that for each $\alpha$ algebraic over $K$, $K(\alpha)/K$ is a defectless extension. Since a complete discrete valued field is defectless [END, 18.8], the above assertion is proved as soon as it is shown that

$$[K(\alpha) : K] = [\tilde{K}(\alpha) : \tilde{K}],$$

where $\tilde{K} = F_p((t))$ being the completion of $K$. To verify the above equation, observe that if $g(x)$ is the minimal polynomial of $\alpha$ over $\tilde{K}$, then the coefficients of $g(x)$ are algebraic over $K$ and these coefficients being in $\tilde{K}$ must belong to $K$, for $K$ is algebraically closed in $K = F_p((t))$.

Finally it may be pointed out that $K(s^{1/p}, t^{1/p})/K$ has defect $p$. Since $K/L$ is a separable extension, we have

$$[K(s^{1/p}, t^{1/p}) : K] = [L(s^{1/p}, t^{1/p}) : L] = p^2.$$

Keeping in view that $K(s^{1/p}, t^{1/p}) \subseteq F_p((t^{1/p}))$, it now follows that $K(s^{1/p}, t^{1/p})/K$ has index of ramification $p$, residual degree 1 and defect $p$. So $(K, v)$ is not a defectless field.