CHAPTER-2

BASIC EQUATIONS AND BOUNDARY CONDITIONS

2.1 Introduction

In this chapter, we discuss the relevant basic equations, boundary conditions, non-dimensional parameters, nomenclature arising in the mathematical modeling under investigation and numerical method used, in a general manner.

2.2 Basic Equations

The basic equations of fluid flow for different physical situations are discussed in detail in the following section. Further, this section also incorporates the different non-Newtonian fluids, namely, power law fluid model and Eyring-Powell fluid.

Equation of state

The equation of state can be derived by expanding density $\rho(T)$ by Taylor's series at $T = T_0$ and doing we get

$$\rho(T) = \rho(T_0) + \left(\frac{\partial \rho}{\partial T}\right)_{T=T_0} \left(T - T_0\right) + \left(\frac{\partial^2 \rho}{\partial T^2}\right)_{T=T_0} \frac{\left(T - T_0\right)^2}{2!} + ----$$

neglecting the second and higher order terms, we get

$$\rho(T) = \rho(T_0) + \left(\frac{\partial \rho}{\partial T}\right)_{T=T_0} \left(T - T_0\right).$$

By the definition of coefficient of thermal expansion we have

$$\alpha = -\frac{1}{\rho_0}\left(\frac{\partial \rho}{\partial T}\right)_{T=T_0}, \Rightarrow \left(\frac{\partial \rho}{\partial T}\right)_{T=T_0} = -\rho_0 \alpha$$

therefore, $\rho = \rho_0\left[1 - \alpha\left(T - T_0\right)\right]$, where $\rho(T) = \rho(T_0)$.

Equation of continuity – Conservation of mass

This equation expresses that the rate of generation of mass within a given volume is entirely due to the net inflow of mass through the surface enclosing the given volume (assuming that there are no internal sources). It amounts to the basic physical law that the matter is conserved; it is neither being created nor destroyed.

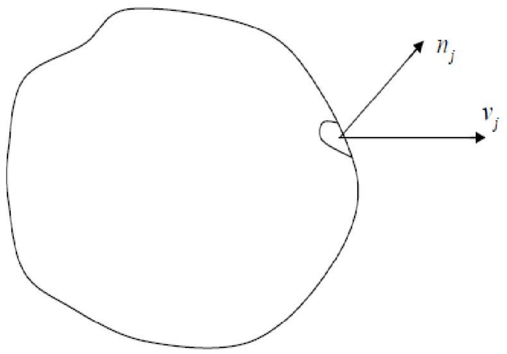

Fig 2.1. A closed surface S on closing an arbitrary fixed volume V in the region of a moving fluid.

Let us consider a closed surface S, enclosing a fixed (arbitrarily chosen) volume V in the region occupied by the moving fluid. If n_j is the unit vector in the direction of the outward drawn normal to the element dS of the surface S and v_j be the velocity of the fluid at that point, then the inward normal velocity is $\left(-\mathrm{v}_j n_j\right)$. Thus the mass of the fluid entering, per unit time, through the element dS is equal to $\rho\left(-\mathrm{v}_j n_j\right)dS$. From the above it follows that the mass of the fluid entering, per unit time, in the controlled surface S is $-\int_S \rho\,\mathrm{v}_j n_j\ dS.$ (2.1)

Also the mass of the fluid within S is $\int_V \rho\ dV.$ (2.2)

Therefore, the rate at which the enclosed mass increases is simply

$$\frac{\partial}{\partial t}\int_V \rho\ dV, \text{ or } \quad \int_V \frac{\partial \rho}{\partial t}\ dV, \tag{2.3}$$

the differentiation and integration being interchangeable because a fixed volume is considered. Now, by the conservation of mass; the expression in (2.1) should be the same as in (2.3). Hence $\int_V \frac{\partial \rho}{\partial t}\ dV = -\int_S \rho\,\mathrm{v}_j n_j\ dS$ (2.4)

Applying Gauss theorem, we get

$$\int_V \frac{\partial \rho}{\partial t}\ dV = -\int_V \frac{\partial}{\partial x_j}\left(\rho\,\mathrm{v}_j\right) dV, \text{ or } \quad \int_V \left\{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho\,\mathrm{v}_j\right)\right\} dV = 0. \tag{2.5}$$

Since, V is an arbitrary chosen volume, we deduce that

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_j}\left(\rho \mathrm{v}_j\right)=0, \tag{2.6}$$

which is the required equation of continuity in the Cartesian tensor notations. In vector notation it can be written as $\frac{\partial \rho}{\partial t}+div(\rho\vec{V})=0.$ (2.7)

In case of steady compressible flow the equation of continuity reduces to

$$\frac{\partial}{\partial x_j}\left(\rho \mathrm{v}_j\right)=0, \text{ or } \quad div(\rho\vec{V})=0. \tag{2.8}$$

If the fluid is considered as incompressible it further reduces to

$$\frac{\partial \mathrm{v}_j}{\partial x_j}=0, \text{ or } \quad div(\vec{V})=0. \tag{2.9}$$

Equations of motion (Navier-Stokes' equations) – Conservation of momentum

The equations of motion are derived from Newton`s second law of motion which states that Rate of change of linear momentum = Total surface. Let us consider a closed surfaces S, as earlier (see Fig 2.1) enclosing a volume V in the region occupied by the moving fluid. The rate at which momentum entering the element dS is $\mathrm{v}_i\left(-\rho dS \mathrm{v}_j n_j\right)$. Therefore, the rate at which the momentum enters the controlled surface S is

$$-\int_S \mathrm{v}_i\left(-\rho \mathrm{v}_j n_j dS\right). \tag{2.10}$$

Also the rate at which the momentum increases in the enclosed volume V is

$$\frac{\partial}{\partial t}\int_V \rho \mathrm{v}_i \ dV. \tag{2.11}$$

Hence, from (2.10) and (2.11), the rate of change of linear momentum is given by

$$\frac{\partial}{\partial t}\int_V \rho \mathrm{v}_i \ dV+\int_S \mathrm{v}_i\left(\rho \mathrm{v}_j n_j\right)dS. \tag{2.12}$$

In fluid motion it is necessary to consider the following two classes of forces,

- forces acting throughout the mass of the body of fluid, such as gravitational forces, known as body forces, and

- forces acting on the boundary, the fluid *stresses*, and are known as *surface stresses*. If f_i denotes the body forces per unit mass and P_i the forces on the boundary per unit area, the equation of motion can be written as

$$\frac{\partial}{\partial t}\int_V \rho \mathrm{v}_i \; dV + \int_S \mathrm{v}_i\left(\rho \mathrm{v}_j n_j\right) dS = \int_V \rho f_i \; dV \quad + \int_S P_i dS. \tag{2.13}$$

where the stress vector P_i is given by

$$P_i = \sigma_{ij}\, n_j \text{, and} \quad \sigma_{ij} = -p\,\delta_{ij} + \tau_{ij}\,. \tag{2.14}$$

Substituting (2.14) in equation (2.13) and changing the surface integrals into volume integrals by Gauss' theorem and noting that V is an arbitrary chosen volume, we get the equation of motion as

$$\frac{\partial}{\partial t}\left(\rho\,\mathrm{v}_i\right) + \frac{\partial}{\partial x_j}\left(\rho \mathrm{v}_i\, \mathrm{v}_j\right) = \rho f_i - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \tag{2.15}$$

or , using the equation of continuity (2.10),

$$\rho\left[\frac{\partial \mathrm{v}_i}{\partial t} + v_j \frac{\partial \mathrm{v}_i}{\partial x_j}\right] = \rho f_i - \frac{\partial p}{\partial x_i} + \frac{\partial v_{ij}}{\partial x_j}. \tag{2.16a}$$

It should be kept in mind that equation (2.16a) is valid for any continuous fluid medium. In order to use these equations to determine velocity distribution, however, we must insert expressions for the viscous stresses in terms of velocity gradients and fluid properties. For isotropic Newtonian fluid these expressions are given by the constitutive equation $\tau_{ij} = 2\mu e_{ij} - \frac{2}{3}\mu e_{kk}\,\delta_{ij}$ where

$$e_{ij} = \frac{1}{2}\left(\frac{\partial \mathrm{v}_i}{\partial x_j} + \frac{\partial \mathrm{v}_j}{\partial x_i}\right) \tag{2.16b}$$

Substituting (2.16b), in equation (2.16a), we finally get

$$\rho\left[\frac{\partial \mathrm{v}_i}{\partial t} + \mathrm{v}_j \frac{\partial \mathrm{v}_i}{\partial x_j}\right] = \rho f_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial \mathrm{v}_i}{\partial x_j} + \frac{\partial \mathrm{v}_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial \mathrm{v}_k}{\partial x_k}\right)\right] \tag{2.17}$$

These are known as *Navier-Stokes equation*s for the motion of a viscous compressible fluid and are three in number. In case of incompressible fluid flow the equation of continuity is $\frac{\partial \mathrm{v}_k}{\partial x_k} = 0,$ and if μ is also regarded as constant, the equation (2.17) can be further simplified to

$$\rho\left[\frac{\partial \mathrm{v}_i}{\partial t}+\mathrm{v}_j\frac{\partial \mathrm{v}_i}{\partial x_j}\right]=\rho f_i-\frac{\partial p}{\partial x_i}+\mu\frac{\partial^2 \mathrm{v}_i}{\partial x_j \partial x_j}, \tag{2.18}$$

keeping in view that

$$\frac{\partial}{\partial x_j}\left(\frac{\partial \mathrm{v}_j}{\partial x_i}\right)=\frac{\partial}{\partial x_i}\left(\frac{\partial \mathrm{v}_j}{\partial x_j}\right)=0 \tag{2.19}$$

Equation (2.18) in vector notation can be written as

$$\rho\frac{D\vec{V}}{Dt}=\rho\vec{F}-\nabla p+\mu\nabla^2\vec{V}. \tag{2.20}$$

where $\frac{D}{Dt}=\frac{\partial}{\partial t}+(\vec{V}.\nabla)$ (2.21)

and is known as the 'material derivative'.

Equation of energy-Conservation of energy

To obtain the energy equation, the law of conservation of energy is used, which states that, the difference in the rate of supply of energy to a controlled surface S enclosing a volume V in the region occupied by a moving fluid and the rate at which the energy goes out through S must be equal to the net rate of increase of energy in the enclosed volume V. Thus the following equation in cartesian tensors is obtained:

$$\int_v \frac{\partial Q}{\partial t}dV+\int_S \mathrm{v}_i\left(\sigma_{ij}n_j\right)dS-\int_S E_t\rho \mathrm{v}_j n_j\, dS-\int_S q_j n_j\, dS=\frac{\partial}{\partial t}\int_V \rho E_t dV \tag{2.22}$$

where $E_t=$ Total energy of the system per unit mass

$$=\frac{1}{2}\mathrm{v}_i\mathrm{v}_j+K+I,=(K.E)(P.E)(I.E)$$

and the heat flux vector q_j is given by the generalized heat conduction law

$$q_j=-k\frac{\partial T}{\partial x_i} \tag{2.23}$$

Substituting (2.23) in (2.22) and changing the surface integrals by Gauss theorem and noting that V is an arbitrary chosen volume, we get the equation of energy as

$$\frac{\partial Q}{\partial t}+\frac{\partial}{\partial x_i}\left(\mathrm{v}_i\sigma_{ij}\right)-\frac{\partial}{\partial x_j}\left(E_t\rho \mathrm{v}_j\right)+\frac{\partial}{\partial x_j}\left(k\frac{\partial T}{\partial x_i}\right)-\frac{\partial}{\partial t}\left(E_t\rho\right)=0 \tag{2.24}$$

The following relations will help us to simplify the equation (2.24). Combining the third and fifth terms of equation (2.24), we get

$$\frac{\partial}{\partial x_j}\left(E_t\rho \mathrm{v}_j\right)+\frac{\partial}{\partial t}\left(E_t\rho\right)=E_t\left\{\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x_j}\left(\rho \mathrm{v}_j\right)\right\}+\rho\left(\frac{\partial E_s}{\partial t}+\mathrm{v}_j\frac{\partial E_t}{\partial x_j}\right)=\rho\frac{DE_t}{Dt}, \quad (2.25)$$

where the equation of continuity (2.6) has been taken into consideration and

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+\mathrm{v}_j\frac{\partial}{\partial x_j}, \quad (2.26)$$

is the material derivative. The right hand side of (2.25), with the help of (2.23), can be written as $\rho\frac{DE_t}{Dt}=\rho\left(\mathrm{v}_i\frac{D\mathrm{v}_i}{Dt}+\frac{DI}{Dt}+\mathrm{v}_j\frac{\partial K}{\partial x_j}\right),$ (2.27)

Since $\frac{\partial K}{\partial t}=0$ (Potential energy is independent of time; it depends only on space coordinates) we have $\sigma_{ij}=-p\delta_{ij}+\tau_{ij}$ (2.28)

The equation of motion (2.16), in view of (2.28), is $\rho\frac{Dv_i}{Dt}=\rho f_i+\frac{\partial}{\partial x_j}\left(\sigma_{ij}\right)$

but $\quad f_i=-\frac{\partial K}{\partial x_i}$ (since K is the P.E.) (2.29)

$$\frac{\partial}{\partial x_j}\left(\sigma_{ij}\right)=\rho\left(\frac{D\mathrm{v}_i}{Dt}+\frac{\partial K}{\partial x_i}\right) \quad (2.30)$$

Now the second term of equation (2.24), in view of (2.30), can be written as

$$\frac{\partial}{\partial x_j}\left(\mathrm{v}_i\sigma_{ij}\right)=\rho \mathrm{v}_i\left(\frac{D\mathrm{v}_i}{Dt}+\frac{\partial K}{\partial x_i}\right)+\left(-p\delta_{ij}+\tau_{ij}\right)\frac{\partial \mathrm{v}_i}{\partial x_j}=\rho \mathrm{v}_i\left(\frac{D\mathrm{v}_i}{Dt}+\frac{\partial K}{\partial x_i}\right)-p\frac{\partial \mathrm{v}_i}{\partial x_i}+\varphi, \quad (2.31)$$

where $\quad \varphi=\tau_{ij}\frac{\partial \mathrm{v}_i}{\partial x_j}=2\mu\left\{\left(e_{ij}-\frac{1}{3}e_{kk}\delta_{ij}\right)\right\}\frac{\partial \mathrm{v}_i}{\partial x_j}$

$$=\mu\left\{\left(\frac{\partial \mathrm{v}_i}{\partial x_j}+\frac{\partial \mathrm{v}_j}{\partial x_i}\right)-\frac{2}{3}\frac{\partial \mathrm{v}_k}{\partial x_k}\delta_{ij}\right\}\frac{\partial \mathrm{v}_i}{\partial x_j}, \quad (2.32)$$

is the heat generated due to frictional forces and is usually known as 'dissipation function'. Thus the equation of energy (2.24), with the help of equations (2.25), (2.27) and (2.31), can be simplified to

$$\frac{\partial Q}{\partial t}+\rho \mathrm{v}_i\frac{D\mathrm{v}_i}{Dt}+\rho \mathrm{v}_i\frac{\partial K}{\partial x_i}-p\frac{\partial \mathrm{v}_i}{\partial x_i}+\phi-\rho \mathrm{v}_i\frac{D\mathrm{v}_i}{Dt}-\rho\frac{DI}{Dt}-\rho \mathrm{v}_i\frac{\partial K}{\partial x_i}+\frac{\partial}{\partial x_j}\left(k\frac{\partial T}{\partial x_i}\right)=0 \quad (2.33)$$

or, $\frac{\partial Q}{\partial t}-p\frac{\partial \mathrm{v}_i}{\partial x_i}-\rho\frac{DI}{Dt}+\frac{\partial}{\partial x_j}\left(k\frac{\partial T}{\partial x_i}\right)+\phi=0$. (2.34)

Further, making use of the equation of continuity (2.6),

$$p\frac{\partial \mathrm{v}_i}{\partial x_i} = -\frac{p}{\rho}\frac{Dp}{Dt} = p\,\rho\frac{D}{Dt}\left(\frac{1}{\rho}\right) \tag{2.35}$$

Therefore, the energy equation (2.34) is written as

$$\rho\left[\frac{DI}{Dt} + p\frac{D}{Dt}\left(\frac{1}{\rho}\right)\right] = \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x_j}\left(k\frac{\partial T}{\partial x_i}\right) + \phi \tag{2.36}$$

For most fluid problems, it is convenient to have the energy equation in terms of fluid temperature and heat capacity rather than the internal energy. For a perfect gas, we have the following relations:

$$p = \rho RT, \qquad c_p - c_0 = R, \tag{2.37}$$

$$I = c_0 T = \left(c_p - R\right)T = c_p T - \frac{p}{\rho} \tag{2.38}$$

where c_0 and c_p are specific heats at constant pressure and at constant volume respectively. Hence for a perfect gas the terms on the left hand side of equation (2.32) may be written as follows:

$$\rho\left[\frac{DI}{Dt} + p\frac{D}{Dt}\left(\frac{1}{\rho}\right)\right] = \rho\left[\frac{D}{Dt}\left(c_p T\right) - \frac{D}{Dt}\left(\frac{p}{\rho}\right) + p\frac{D}{Dt}\left(\frac{1}{\rho}\right)\right] = \rho\frac{D}{Dt}\left(c_p T\right) - \frac{Dp}{Dt}, \tag{2.39}$$

and the energy equation (2.32) may, then, be written as

$$\rho\frac{D}{Dt}\left(c_p T\right) = \frac{Dp}{Dt} + \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right) + \phi. \tag{2.40}$$

In case of an in compressible fluid, $\frac{\partial \mathrm{v}_i}{\partial x_i} = 0$; $I = c_0 T$, the equation of energy (2.34), with constant viscosity and heat conductivity, becomes

$$\rho c_v\frac{DT}{Dt} = \frac{\partial Q}{\partial t} + k\frac{\partial}{\partial x_i}\left(\frac{\partial T}{\partial x_i}\right) + \phi, \tag{2.41}$$

where $$\phi = \mu\left(\frac{\partial \mathrm{v}_i}{\partial x_j} + \frac{\partial \mathrm{v}_j}{\partial x_i}\right)\frac{\partial \mathrm{v}_i}{\partial x_j} \tag{2.42}$$

2.3 Boundary layer equations

The theory of laminar boundary layer was initiated by German Scientist *Ludwig Prandtl* in 1904. *Prandtl*, in the paper on liquid motion with very small friction made a hypothesis that for liquids with very small viscosity the flow about a solid body can be divided in to two regions: (i) A very thin layer in the immediate neighborhood of the body known as the boundary layer in which the viscous effects may be considered to be predominant. (ii) The region outside this layer where the viscous effects may be considered as negligible and the liquid is regarded as inviscid. With the aid of this hypothesis the Navier-Stokes equations are simplified to a mathematically tractable from, which are then called boundary layer equations (see *Schlichting* 1968). We shall begin by establishing the Prandtl boundary layer equations for a two-dimensional flow along a plane wall.

Two-dimensional boundary layer equations for flow over a plane wall

The Navier-Stokes equations, for a viscous incompressible fluid, in a two-dimensional flow are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\mathrm{v}\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{2.43}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+\mathrm{v}\frac{\partial \mathrm{v}}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\left(\frac{\partial^2 \mathrm{v}}{\partial x^2}+\frac{\partial^2 \mathrm{v}}{\partial y^2}\right) \tag{2.44}$$

and the equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial \mathrm{v}}{\partial y}=0 \tag{2.45}$$

where the (x, y)-plane is the plane of motion, the x-axis is along the wall and y-axis perpendicular to it. Thus due to no-slip condition and since the wall is solid (no porous) both u and v will vanish at y=0.

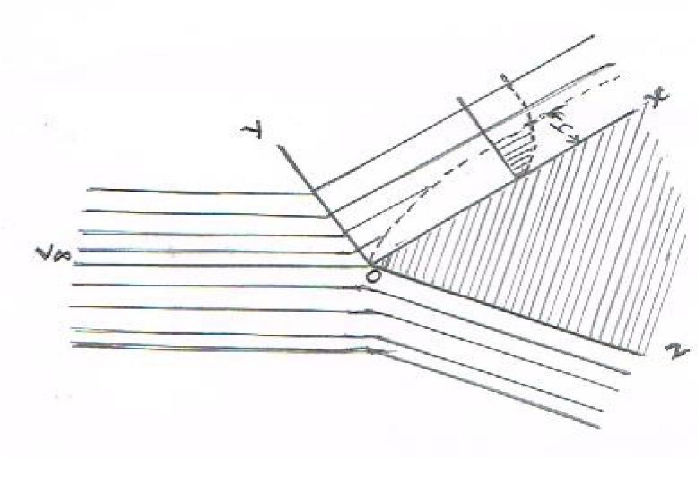

Fig 2.2. Coordinate system for two dimensional boundary layer flow over a plane wall.

The velocity component u parallel to the wall in the boundary layer rises rapidly from a value zero at the wall to a value U in the main stream within a short distance, δ (say) the thickness of the boundary layer, from the wall. Taking *t, x, u* as quantities of 0(1) and *y* of 0(δ), where $\delta << 1$, we find that $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$ are each of 0(1), $\frac{\partial u}{\partial y}$ is of 0(δ^{-1}), and $\frac{\partial^2 u}{\partial x^2}$ is of 0(δ^{-2}) in the boundary layer. It then follows from the equation of continuity that $\frac{\partial \mathrm{v}}{\partial y}$ is of 0(1), and since *y* is of 0(δ) the velocity component v should also be of 0(δ). Hence $\frac{\partial \mathrm{v}}{\partial t}$, $\frac{\partial \mathrm{v}}{\partial x}$, $\frac{\partial^2 \mathrm{v}}{\partial x^2}$ are each of 0(δ), and $\frac{\partial^2 \mathrm{v}}{\partial y^2}$ is of 0(δ^{-1}).Thus in equation (2.43) $\frac{\partial^2 u}{\partial x^2}$ may be neglected in comparison with $\frac{\partial^2 \mathrm{v}}{\partial y^2}$ and then the equation becomes

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \mathrm{v}\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \upsilon\frac{\partial^2 u}{\partial y^2}. \qquad (2.46)$$

It is now supposed that the viscous term is of the same order as the inertia terms, i.e., of order unity, so that υ is of 0(δ^2) or in other words δ is of 0($\sqrt{\upsilon}$). This fact is already confirmed by some exact solutions. The equation (2.44) then gives

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} \text{ is of } 0(\delta). \qquad (2.47)$$

Therefore, the pressure increase across the boundary layer is 0(δ^2) and may be neglected. Hence the pressure is taken, practically, constant in a direction normal to the boundary layer and may be assumed equal to that at the outer edge of the boundary layer where it is determined by the inviscid flow (Potential flow). We may, therefore, write

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x}, \qquad (2.48)$$

where *U* is the potential flow velocity. Hence the Prandtl boundary layer equations for a two dimensional unsteady incompressible flow are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \mathrm{v}\frac{\partial u}{\partial y} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \upsilon\frac{\partial^2 u}{\partial y^2} \qquad (2.49)$$

and $\frac{\partial u}{\partial x} + \frac{\partial \mathrm{v}}{\partial y} = 0$. (2.50)

The boundary conditions, under which these equations are usually integrated, are

$y = 0\ ;\ u = \mathrm{v} = 0\ ;\ y \to \infty\ ;\ u = U(x,t)\,,$ (2.51)

In which the first is the no-slip condition and the condition of non-porous wall and the second is obtained from the consideration that the velocity u, in the boundary layer, must join smoothly on to the main stream velocity.

Boussinesq approximation

To illustrate this let us consider the Navier-Stokes equation in its simplest form and as required for the explanation of Boussinesq approximation $\rho \frac{d\vec{q}}{dt} = -\nabla p + \rho \vec{g} + \mu \nabla^2 \vec{q}$. If p and ρ are now expanded about the values of p_0 and ρ_0 in a reference state of hydrostatic equilibrium for which $\nabla p_0 = \rho_0 \vec{g}$ (i.e., one sets $p = p_0 + p'$ and $\rho = \rho_0 + \rho'$), the above equation becomes $(\rho_0 + \rho') = -\nabla p' + \rho' \vec{g} + \mu \nabla^2 \vec{q}'$. This equation implies that only differences of density ρ' from some standard value are relevant in determining the effect of gravity. In the approximation to be considered now the density variation ρ' is assumed to be small compared to ρ_0. Rewriting the above equation we get $\left(1 + \frac{\rho'}{\rho_0}\right) \frac{d\vec{q}}{dt} = -\frac{1}{\rho_0} \nabla p' + \frac{\rho'}{\rho_0} \vec{g} + \upsilon \nabla^2 \vec{q}'$.

We see that the density ratio ρ'/ρ_0 appears twice, in the first (inertia) term and in the buoyancy term. When ρ'/ρ_0 is small, it produces only a small correction to the inertia compared to a liquid density ρ_0, but it is of primary importance in the buoyancy term. The approximation introduced by *Boussinesq* [1903] consists essentially of neglecting variation of density in so far as they affect inertia, but retaining them in the buoyancy term. When viscosity and diffusion are included, variations of liquid properties are also neglected in this approximation (see *Turner* [1973]). There are other restrictions necessary in compressible liquids, which are often included by the name 'Boussinesq approximation'; these are discussed fully by *Spiegel and Veronis* [1960] and only the results will be quoted here. First one must replace density by potential density. The limitation of small density deviations from a standard ρ_0 implies two things; the vertical scale of the mean motion must be much smaller than the scale height and fluctuating density changes due to local pressure variations must also be negligible. The latter is the most important of the extra conditions; it implies that the liquid can be treated as incompressible and it therefore excludes sound and shock waves. Finally, the ratio of the length to the time scales of any variation in an unsteady flow should be much smaller than the velocity of sound, to ensure that information about pressure changes is transmitted effectively and instantaneously, as it is in an incompressible liquid.

- The gravity acts vertically downwards (see *Peterssen* [1971]).
- Linear equation of state: The temperature of the liquid is everywhere below the boiling point leading to a liberalized equation of state where the density of the liquid is a linear function of temperature.
- The liquid properties namely, the kinematic viscosity, the magnetic viscosity, the thermal diffusivity, the coupling viscosity, shear kinematic viscosity are considered in the problem.
- The usual MHD approximations are taken in to account.

Magnetohydrodynamic (MHD) equations

The basic equations of MHD, when the displacement currents and free charges are neglected, are

Maxwell's Equations

$$Div\,\vec{E} = 0$$

$$Div\,\vec{H} = 0$$

$$Curl\,\vec{E} = -\mu_m \frac{\partial \vec{H}}{\partial t}$$

$$Curl\,\vec{H} = \vec{J}$$

$$Div\,\vec{J} = 0$$

$$\vec{J} = \sigma\left(\vec{E} + \mu_m\left(\vec{q}\times\vec{H}\right)\right)$$

The generalized Navier-Stokes equation of motion for MHD is

$$\frac{d\vec{q}}{dt} = -\frac{1}{\rho}\nabla p + \upsilon\nabla^2\vec{q} + \mu_m\frac{\vec{J}\times\vec{H}}{\rho}$$

When the condition current $\sigma \vec{E}$ is negligible compared to $\mu_m \sigma \left(\vec{q} \times \vec{H} \right)$, Ohm's law gives $\vec{J} = \mu_m \sigma \left(\vec{q} \times \vec{H} \right)$

When $Rm << 1$ (Hartmann formulation), the lorentz force takes form $\mu_m \vec{J} \times \vec{H} = -\mu_m^2 \sigma H_o^2 \vec{q}$.

Power law model

The simplest and most common type of model is the power law fluid (Ostwald-de Waele model) for which the rheological equation of the state between the stress components τ_{ij} and strain components e_{ij} is defined by, $\tau_{ij} = -p\delta_{ij} + K \left| \sum_{m=1}^{3} \sum_{l=1}^{3} e_{lm} e_{ml} \right|^{\frac{n-1}{2}} e_{ij}$, where p is the pressure, δ_{ij} is the Kroneckar delta, K and n are respectively, the consistency coefficient and the power-law index of the fluid.

Eyring-Powell model

The theory of rate processes is used to derive the Eyring-Powell model for describing the shear of non-Newtonian flow. The shear tensor in an Eyring Powell model is given by

$$\boldsymbol{T} = \mu \nabla \mathbf{V} + \frac{1}{\beta_2} \sinh^{-1} \left(\frac{1}{c} \nabla \mathbf{V} \right)$$

where μ is the dynamic viscosity, β_2 and c are the fluid parameters of the Eyring-Powell model. We take the second order approximation of function as

$$\sinh^{-1} \left(\frac{1}{c} \nabla \mathbf{V} \right) = \frac{1}{c} \nabla \mathbf{V} - \frac{1}{6} \left(\frac{1}{c} \nabla \mathbf{V} \right)^3, \qquad \left| \frac{1}{c} \nabla \mathbf{V} \right| << 1.$$

Equation of motion for two dimensional flows in Cartesian form

Continuity equation: $\frac{\partial u}{\partial x} + \frac{\partial \mathrm{v}}{\partial y} = 0$

Momentum equation:

- $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \mathrm{v} \frac{\partial u}{\partial y} = \upsilon \frac{\partial^2 u}{\partial y^2}$ (Newtonian fluid)

- $\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\mathrm{v}\frac{\partial u}{\partial y}=-\gamma\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)^{n}$ (Power law fluid)
- $u\frac{\partial u}{\partial x}+\mathrm{v}\frac{\partial u}{\partial y}=\left(\gamma+\frac{1}{\rho\beta_1 c}\right)\frac{\partial^2 u}{\partial y^2}-\frac{1}{2\beta_1 c^3}\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}$ (Eyring-Powell fluid)

Heat transfer equation: $\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+\mathrm{v}\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}$

Mass transfer equation: $\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+\mathrm{v}\frac{\partial C}{\partial y}=D(C)\frac{\partial^2 C}{\partial y^2}$,

2.4 Boundary conditions

The boundary conditions on velocity and temperature are depend on the nature of the liquid flow and geometry of the boundary wall. The mathematical forms of velocity and temperature boundary conditions and boundary geometry are as follows: Here, the flow is generated due to the stretching of an elastic porous/non-porous sheet caused by the simultaneous application of two equal and opposite forces along the *x*-axis, keeping the origin fixed and considering the flow to be confined to the region $y>0$. The continuous stretching sheet is assumed to have a velocity $U_w(x,t)$, prescribed surface temperature $T_w(x,t)$, prescribed species diffusion $C_w(x,t)$ and the suction/ blowing velocity across the stretching sheet $\mathrm{v_w}$, which vary with the coordinate *x* and time *t*, the relavent boundary conditions are

Prescribed surface temperature (PST):

$u=U_w=bx$, $\mathrm{v}=\pm\mathrm{v_w}$, $T=T_w=T_\infty+E_1(x/l)$, $C=C_w=C_\infty+E_2(x/l)$ at $y=0$

$u\to 0$, $T\to T_\infty, C\to C_\infty$ as $y\to\infty$.

Prescribed heat flux (PHF) :

$u=U_w=bx$, $\mathrm{v}=\pm\mathrm{v_w}$, $-k\frac{\partial T}{\partial y}=q_w=D\left(\frac{x}{l}\right)^2$ at $y=0$

$u\to 0$, $T\to T_\infty$ as $y\to\infty$.

Boundary condition for thin liquid film

The continuous sheet is assumed to have a surface velocity $U_s(x,t)$ and prescribed surface temperature $T_s(x,t)$, prescribed species diffusion $C_s(x,t)$ and the suction/ blowing velocity across the stretching sheet v_s, which vary with the horizontal coordinate *x* and time *t*.

$$u = U_s = bx/(1-\alpha t), \quad \mathrm{v} = \mathrm{v}_s \ , \quad T = T_s = T_0 + T_{ref}\frac{b^{2-n}\,x^2}{2\gamma}(1-\alpha t)^{n-\frac{5}{2}},$$

$$C = C_s = C_0 + C_{ref}\frac{b^{2-n}\,x^2}{2\gamma}(1-\alpha t)^{n-\frac{5}{2}}, \quad \text{at} \quad y = 0$$

$$\frac{\partial u}{\partial y} = \frac{\partial T}{\partial y} = \frac{\partial C}{\partial y} = 0, \quad \mathrm{v} = \frac{dh}{dt} \ \text{ as } \ y \to h(t).$$

Here $h(t)$ is the free surface elevation of the liquid film i.e., the film thickness.

2.5 Dimensionless parameters

Dimensional analysis provides information on qualitative behavior of the physical problem. The dimensionless parameter helps us to understand the physical significance of a particular phenomenon associated with the problem. There are usually two general methods for obtaining dimensionless parameters namely,

- the inspectional analysis
- the dimensionless analysis.

In the inspection analysis, reduce the fundamental equations to a non-dimensional form and obtain the non-dimensional parameters from the resulting equations. In dimensional analysis, form non-dimensional parameters from the physical quantities occurring in the problem, even when the knowledge of the governing equations is missing. The dimensionless parameters used in the thesis are below;

Magnetic parameter

It is defined as the ratio of magnetic force to the force due to density of the fluid $Mn = \sigma B_0 / \rho b$, where σ is the electrical conductivity and B_0 is the electric field, ρ is the density of the fluid, *b* is a constant.

Reynolds number

The dimensionless quantity Reynolds number is defined as $\mathrm{Re} = UL\rho/\mu$, where *U*, *L*, ρ and μ are some characteristic values of the velocity, length, density and viscosity of the fluid respectively. . The concept was introduced by *George Gabriel Stokes* in 1851, but the Reynolds number is named after *Osborne Reynolds*, who popularized its use in 1883. Reynolds number is also defined as the ratio of inertial to viscous force. It is in fact a parameter for viscosity, for if Re is small the viscous forces will be predominant and the effect of viscosity will be felt in the whole flow field. On the other hand, if Re is large the inertial forces will be predominant and in such a case the effect of viscosity can be considered to be confined in a thin layer, known as boundary layer, adjacent to a solid boundary. However , if Re is very large the flow ceases to be laminar and becomes turbulent.

Froude number

The ratio of inertial forces to the gravity forces is given by the non-dimensional parameter which is usually known as Froude number. Thus we have $Fr = U^2/gL$. It is important only when there is free surface, example, in an open channel flow problem. In such a cases too the force due to gravity may be neglected in comparison to the inertial force if $Fr/\mathrm{Re} >> 1$.

Grashoff number

The dimensionless quantity G_r, which characterizes the free convection is known as Grashoff number and is defined as $G_r = gL^3\left(T_w - T_\infty\right)/\gamma^2 T_\infty$, where g is the acceleration due to gravity and T_w and T_∞ are two representative temperatures.

Modified Grashoff number

The dimensionless quantity G_c which characteristics the mass diffusion species is known as modified Grashoff number and it is defined as $G_c = g\beta^*\left(C_w - C_\infty\right)xb^{-n}/\gamma\,\mathrm{Re}$ **,** where β^* is the co-efficient of species expansion and C_w and C_∞ are the two representative of species concentration, γ is the kinematic viscosity.

Suction or injection parameter

Suction or injection parameter (f_w) is defined as $f_w = -\left(\mathrm{v}_s/U\right)\left(2n/n+1\right)\mathrm{Re}^{\frac{1}{n+1}}$, where v_s is the suction/ blowing velocity across the stretching sheet. (here, $f_w > 0$ corresponds to suction whereas $f_w < 0$ corresponds to injection).

Fluid viscosity parameter

It describes the internal friction of a moving fluid. A fluid with large viscosity resists motion because its molecular makeup gives it a lot of internal friction. It is defined as $\theta_r = 1/\left[\gamma\left(T_w - T_\infty\right)\right]$ where T_w is the sheet temperature and T_∞ is the temperature far away from the sheet, $(T_w - T_\infty)$ is the temperature difference.

Prandtl number

The ratio of the kinematic viscosity to the thermal diffusivity of the fluid, that is, $P_r = \mu C_p / k$ is called as the Prandtl number named after the German scientist *Ludwig Prandtl*. It is a measure of the relative importance of heat conduction and viscosity of the fluid. The Prandtl number, like the viscosity and thermal conductivity is a material property and it thus varies from fluid to fluid.

Eckert number

The dimensionless quantity E_c is defined as $E_c = U / C_p T$ is known as Eckert number named after German scientist *E R G Eckert* . In compressible fluids it determines the relative rise in the temperature of the fluid due to adiabatic compression. It can also be retained in incompressible flow, if the frictional heat is to be considered, but the interpretation with reference to adiabatic compression will no longer be true.

Heat source/sink parameter

It is defined as the ratio of volumetric heat to the density of the fluid at specific gravity at constant pressure. It is designed to supply heat consistently and safely over a wide range of extreme conditions. A heat sink is an object that transfers thermal energy from a higher temperature to a lower temperature fluid medium.(Heat sink is a negative parameter). $\beta = Q_S / b \rho c_p$, where Q_S represents the temperature-dependent volumetric rate of heat source when $Q_S > 0$ and heat sink when $Q_S < 0$. These deal with the situation of exothermic and endothermic chemical reactions respectively.

Thermal radiation parameter

The thermal radiation is electromagnetic radiation emitted from a material which is due to the heat of the material, the characteristics of which depend on its temperature. An example of thermal radiation is the infrared radiation emitted by a common household radiator of electric heater. (It rapidly increases in power, and also increases in frequency, with increasing temperature). It is defined as $Nr = 16\sigma^* T_\infty^3 / 3\rho c_p k_2 k$, where k is the thermal conductivity at the slit, σ^* and k_2

are the Stephan-Boltzman constant and the Roseland mean absorption coefficient respectively.

Schmidt number

Schmidt number is a dimensionless number defined as the ratio of momentum diffusivity (viscosity) and mass diffusivity, and is used to characterize fluid flows in which there are simultaneous momentum and mass diffusion convection processes. It was named after the German engineer *Ernst Heinrich Schmidt* (1892-1975). i.e., $Sc = \gamma / D$, where γ is the kinematic viscosity and D is the species diffusion respectively.

Reaction rate parameter

The non-dimensional variable δ is known as reaction rate parameter is defined as $\delta = kL/b$, where k is the reaction rate constant, b is a constant and L is the characteristic length.

Skin friction

When the boundary layer equations are integrated the velocity distribution can be declared and the position of the point of separation can be determined. This in turn permits us to calculate the viscous drag and is known as skin friction. The shearing stress at the wall is given by $C_f = \tau_w / \left(\rho U^2 / 2 \right)$, where τ_w is the surface shear stress.

Nusselt number

The most important dimensionless parameter associated with heat transfer problems is the Nusselt number and it is defined as the ratio of convective to conductive heat transfer across (normal to) the boundary. $Nu_x = L\, q_c / \varepsilon$ where q_c is the convective heat transfer coefficient.

Sherwood number

The Sherwood number (or the mass transfer Nusselt number) is a dimensionless number used in mass transfer analysis. It represents the ratio of convective to diffusive mass transport, and is named in honour of *Thomas Kilgore Sherwood*. It is defined as $Sh_x = L\, q_m / D$, where q_m is the mass transfer coefficient, D is the mass diffusion.

Mach number

The ratio of the flow velocity to the velocity of the sound, that is , $Ma = U\big/\sqrt{\gamma p/\rho}$, is designated as the Mach number named after German scientist *Ernst Mach*. It is a measure of the compressibility of the fluid.

2.6 Numerical method

The mathematical modeling of the physical problem is nothing but the system of coupled linear/non-linear form of partial differential equations. Under certain physical constraints these governing equations are transformed into a system of non-linear coupled ordinary differential equations with/without variable coefficients by using similarity transformation and in turn solved numerically/analytically by different methods namely, shooting method, implicit finite difference method, finite element method, regular perturbation method, and Cranck-Nicolson method, etc. and in few cases, we can solve analytically by series solution. For numerical computing we are using different programming languages namely, C-programming, MAT Lab, FORTRAN, Mathematica, and Maple. In this present thesis, the following numerical methods are used to solve the coupled boundary value problem. The numerical method for the coupled boundary value problem is described briefly as follows;

Shooting method

The shooting method is described in this section applies equally well to linear and nonlinear problems. Since, there is no guarantee of convergence, but the method is easy to apply, and it is usually more efficient than the any other methods. A simple example of a second order boundary-value problem (*Simmons* [2003]) is

$$y''(x) = y(x) \qquad y(0) = 0 \qquad y(1) = 1. \tag{2.52}$$

apply the initial value methods, but to do so both $y(0)$ and $y'(0)$ must know. Since $y'(0)$ is not prescribed, we consider it as an unknown parameter, say α , which must be determined so that the resulting solution yields the prescribed value $y(1)$ to some desired accuracy. Therefore guess at the initial slope and set up an iterative procedure for converging to the correct slope. Let α_0, α_1 be two guesses at the initial slope $y'(0)$, and let $y(\alpha_0;1), y(\alpha_1;1)$ be the values of y at x = 1 obtained from integrating the differential equation. $y(\alpha;1)$ is plotted as a function of α. A normally better approximation to α can now be obtained by linear interpolation. The intersection of the line joining p_0 to p_1 with the line $y(1)$=1 has its α coordinate given by

$$\alpha_2 = \alpha_0 + (\alpha_1 - \alpha_0)\frac{y(1) - y(\alpha_0;1)}{y(\alpha_1;1) - y(\alpha_0;1)}. \tag{2.53}$$

Now integrate the differential equation using the initial values $y(0) = 0$, $y'(0) = \alpha_2$, to obtain $y(\alpha_2;1)$. Again, using linear interpolation based on α_1, α_2 then find the next approximation α_3. The process is repeated until convergence has been obtained, i.e., until $y(\alpha_1;1)$ agrees with $y(1) = 1$ to the desired number of places. There is no guarantee that this iterative procedure will converge. The rapidity of convergence will clearly depend upon how good the initial guesses are. Estimates are sometimes available from physical considerations, and sometimes from simple graphical representations of the solution.

For a general second order boundary value problem

$$y'' = f(x, y, y'),\ y(0) = y_0,\ \ y(b) = y_b, \tag{2.54}$$

the procedure is summarized in the algorithm.

Algorithm: The shooting method for second-order boundary value problems:

1. Let α_k be an approximation to the unknown initial slope $y'(0) = \alpha_2$.

   (choose the first two α_0, α_1 using physical intuition)

2. Solve the initial-value problem $y'' = f(x, y, y')$ $y(0) = y_0$ $y'(0) = \alpha_k$ from x = 0 to x = b, using any of the methods. Call the solution $y(\alpha_k, b)$ at $x = b$.

3. Obtain the next approximation from the linear interpolation

   $$\alpha_{k+1} = \alpha_k + (\alpha_k - \alpha_{k-1})\frac{y(1) - y(\alpha_0;1)}{y(\alpha_1;1) - y(\alpha_0;1)} \quad k = 1, 2, ...$$

4. Repeat step 2 and 3 until $\left|y(\alpha_k, b) - y_0\right| < \varepsilon$ for a prescribed ε.

The iteration used in the above is an application of the secant method. For systems of equations of higher order, this procedure becomes considerably more complicated, and convergence more difficult to obtain. The general situation for a nonlinear system may be represented as follows: Here the system of four equations in four unknowns are considered;

$$\begin{aligned} x' &= f(x,y,z,w,t), \quad y' = g(x,y,z,w,t) \\ z' &= h(x,y,z,w,t), \quad w' = l(x,y,z,w,t) \end{aligned} \qquad (2.55)$$

where now *t* represents the independent variable. Here are given two conditions at *t* = 0, say $x(0) = x_0$ $y(0) = y_0$ and two conditions at *t*=*T*, say $z(T) = z_T$ $w(T) = w_T$. Let $z(0) = \alpha$, $w(0) = \beta$ be the correct initial values of $z(0)$, $w(0)$, and let α_0, β_0 be guesses for these initial values. Now integrate the system (2.55), and denote the values of *z* and *w* obtained at *t*=*T* by $z(\alpha_0, \beta_0; T)$ and $w(\alpha_0, \beta_0; T)$. Since *z* and *w* at *t*=*T* are clearly functions of α and β, we may expand $z(\alpha_0, \beta_0; T)$ and $w(\alpha_0, \beta_0; T)$ into a Taylor series for two variables through linear terms:

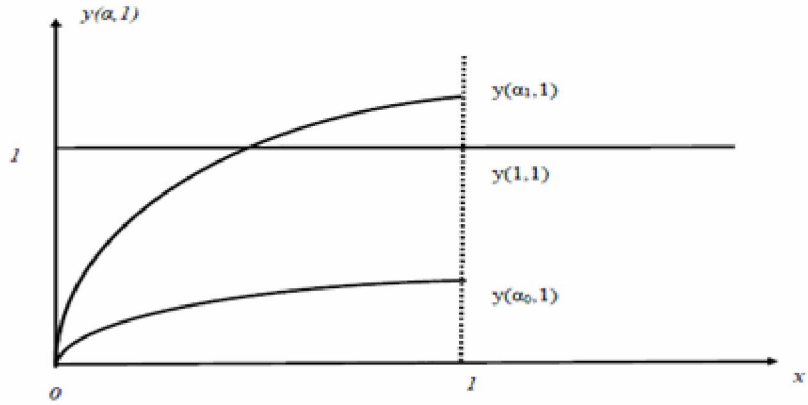

Fig 2.3. Approximate solution curves at $x = 1$

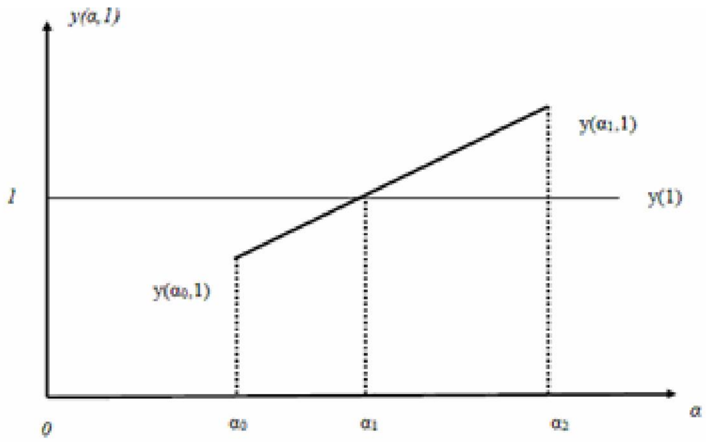

Fig 2.4. Approximations to α

$$z(\alpha,\beta;T)=z(\alpha_0,\beta_0;T)+(\alpha-\alpha_0)\frac{\partial z}{\partial\alpha}(\alpha_0,\beta_0;T)+(\beta-\beta_0)\frac{\partial z}{\partial\beta}(\alpha_0,\beta_0;T) \tag{2.56}$$

$$w(\alpha,\beta;T)=w(\alpha_0,\beta_0;T)+(\alpha-\alpha_0)\frac{\partial w}{\partial\alpha}(\alpha_0,\beta_0;T)+(\beta-\beta_0)\frac{\partial w}{\partial\beta}(\alpha_0,\beta_0;T).$$

Set $z(\alpha_0,\beta_0;T)$ and $w(\alpha_0,\beta_0;T)$ to their desired values z_r and w_r , but before solve (2.56) for the corrections $\alpha-\alpha_0$ and $\beta-\beta_0$ and obtain the partial derivatives in (2.56). Here the solutions z and w are not known and therefore cannot find these derivatives analytically. However, find approximate numerical values for them. To do so, we solve (2.55) once with the initial conditions $x_0, y_0, \alpha_0, \beta_0$, once with the conditions $x_0, y_0, \alpha_0+\Delta\alpha_0, \beta_0$, and then with conditions $x_0, y_0, \alpha_0, \beta_0+\Delta\beta_0$,where $\Delta\alpha_0$ and $\Delta\beta_0$ are small increments. Omitting the variables x_0, y_0 which remain fixed, then form the difference quotients:

$$\frac{z\left(\alpha_0,\beta_0+\Delta\beta_0;T\right)-z(\alpha_0,\beta_0;T)}{\Delta\beta_0}\approx\frac{\partial z}{\partial\beta}(\alpha_0,\beta_0;T)$$

$$\frac{w\left(\alpha_0,\beta_0+\Delta\beta_0;T\right)-w(\alpha_0,\beta_0;T)}{\Delta\beta_0}\approx\frac{\partial w}{\partial\beta}(\alpha_0,\beta_0;T)$$

$$\frac{z\left(\alpha_0+\Delta\alpha_0,\beta_0;T\right)-z(\alpha_0,\beta_0;T)}{\Delta\alpha_0}\approx\frac{\partial z}{\partial\alpha}(\alpha_0,\beta_0;T)$$

$$\frac{w\left(\alpha_0+\Delta\alpha_0,\beta_0;T\right)-w(\alpha_0,\beta_0;T)}{\Delta\alpha_0}\approx\frac{\partial w}{\partial\alpha}(\alpha_0,\beta_0;T).$$

After replacing $z(\alpha,\beta;T)$ by z_r and $w(\alpha,\beta;T)$ by w_r , then solve (2.56) for the corrections $\delta\alpha_0=\alpha-\alpha_0$ and $\delta\beta_0=\beta-\beta_0$, to obtain new estimates $\alpha_1=\alpha_0+\delta\alpha_0$ and $\beta_1=\beta_0+\delta\beta_0$ for the parameters α and β . The entire process is now repeated, starting with $x_0, y_0, \alpha_1, \beta_1$ as the initial conditions. Each iteration thus consists in solving the system (2.55) three times. In general, if there are n unknown initial parameters, each iteration will require n+1 solutions of the original system. The method used here is equivalent to a modified Newton's method for finding the roots of equations in several variables. Boundary value problems constitute one of the most difficult classes of problems to solve on a computer. Convergence is by no means assured, good initial guesses must be available and considerable trial and errors, as well as large amounts of machine time, are usually required.

Keller-box method

An alternative implicit method due to Keller is now described and is referred to as the Box method. This method has several very desirable features that make it appropriate for the solution of all parabolic partial differential equations. The main features of this method are

- Slightly more arithmetic to solve than the Crank-Nicolson method.
- Second-order accuracy with arbitrary (non-uniform) x and y spacing.
- Allows very rapid variations.
- Allows easy programming of the solution of large numbers of coupled equations.

The solution of an equation by this method can be obtained by the following four steps.

- Reduce the equation or equations to a first-order system.
- Write difference equations using central differences.
- Linearize the resulting algebraic equations (if they are nonlinear), and write them in matrix-vector form.
- Solve the linear system by the block-tridiagonal-elimination method.

For example, consider the equation $\frac{\nu}{\Pr}\frac{\partial^2 T}{\partial y^2} = u\frac{\partial T}{\partial x}$.

To solve the above equation by numerically, first express it in terms of a system of two first order equations by letting $T' = p$ and $p' = \frac{\Pr}{\nu}u\frac{\partial T}{\partial x}$. (2.57)

Here the primes denote differentiation with respect to y. The finite difference form of the ordinary differential equation (2.57) is written for the midpoint $\left(x_n, y_{j-1/2}\right)$ of the segment $p_1 p_2$ shown in fig 2.5, and the finite difference form of the partial differential equation (2.57) is written for the midpoint $\left(x_{n-1/2}, y_{j-1/2}\right)$ of the rectangle $p_1 p_2 p_3 p_4$. This gives

$$\frac{T_j^n - T_{j-1}^n}{h_j} = \frac{p_j^n + p_{j-1}^n}{2} = p_{j-1/2}^n, \frac{1}{2}\left(\frac{p_j^n - p_{j-1}^n}{h_j} + \frac{p_j^{n-1} - p_{j-1}^{n-1}}{h_j}\right) = \frac{\Pr}{\nu}u_{j-1/2}^{n-1/2}\frac{T_{j-1/2}^n - T_{j-1/2}^{n-1}}{k_n}. \quad (2.58)$$

Rearranging both expressions in the form

$$T_j^n - T_{j-1}^n - \frac{h_j}{2}\left(p_j^n - p_{j-1}^n\right) = 0,\left(S_1\right)_j p_j^n + \left(S_2\right)_j p_{j-1}^n + \left(S_3\right)_j\left(T_j^n - T_{j-1}^n\right) = R_{j-1/2}^{n-1} \qquad (2.59)$$

Here $\left(S_1\right)_j = 1,\ \left(S_2\right)_j = -1,\ \left(S_3\right)_j = -\lambda_j/2,\ R_{j-1/2}^{n-1} = -\lambda_j + T_{j-1/2}^{n-1} + p_{j-1}^{n-1} - p_j^{n-1}$ (2.60)

$$\lambda_j = \frac{2\,\mathrm{Pr}}{\nu} u_{j-1/2}^{n-1/2} \frac{h_j}{k_n}.$$

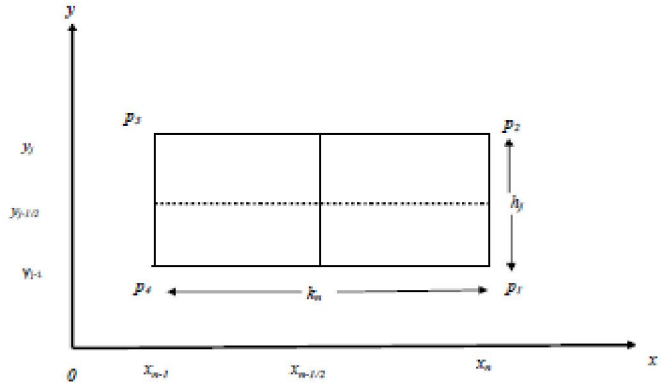

Fig 2.5. Finite difference grid for the Box method. Note that both h and k can be non-uniform. Here $x_{n-1/2} = 1/2\left(x_n + x_{n-1/2}\right)$ and $y_{j-1/2} = 1/2\left(y_j + y_{j-1}\right)$.

As before, the superscript on $u_{j-1/2}$ is not necessary but is included for generality. Equations (2.59) are imposed for $j = 1, 2, ..., J-1$. $A_j = 0$ and for J, we have

$$T_0 = T_w\ ,T_J = T_e \qquad (2.61)$$

respectively. Since equation (2.59) are linear as are the corresponding boundary conditions given by equation (2.61), the system may be written at once in matrix vector form as shown below without the linearization needed in the case of the finite difference equations for the velocity field.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & & \\ -1 & \frac{-h_1}{2} & 1 & \frac{-h_1}{2} & & \\ \left(S_3\right)_j & \left(S_2\right)_j & \left(S_3\right)_j & \left(S_1\right)_j & 0 & 0 \\ 0 & 0 & -1 & \frac{-h_{j+1}}{2} & 1 & \frac{-h_{j+1}}{2} \\ & & \left(S_3\right)_J & \left(S_2\right)_J & \left(S_3\right)_J & \left(S_1\right)_J \\ & & 0 & 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} \begin{pmatrix} T_0 \\ p_0 \end{pmatrix} \\ \begin{pmatrix} T_j \\ p_j \end{pmatrix} \\ \begin{pmatrix} T_J \\ p_J \end{pmatrix} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} (r_1)_0 \\ (r_2)_0 \end{pmatrix} \\ \begin{pmatrix} (r_1)_j \\ (r_2)_j \end{pmatrix} \\ \begin{pmatrix} (r_1)_J \\ (r_2)_J \end{pmatrix} \end{bmatrix}. \qquad (2.62)$$

$$\left(r_1\right)_0 = T_w \ , \left(r_1\right)_j = R_{j-1/2}^{n-1}, \quad 1 \le j \le J \quad \left(r_2\right)_j = 0, \ 1 \le j \le J-1, \quad \left(r_2\right)_J = T_e. \tag{2.63}$$

The system of equations given by equation (2.62) can be written as

$$A\delta = r, \tag{2.64}$$

Where

$$\begin{bmatrix} A_0 & C_0 & & & & \\ B_1 & A_1 & C_1 & & & \\ & & B_J & A_j & C_j & \\ & & & B_{J-1} & A_{J-1} & C_{J-1} \\ & & & & B_J & A_J \end{bmatrix}, \ \delta = \begin{bmatrix} \delta_0 \\ \delta_1 \\ . \\ \delta_j \\ . \\ \delta_J \end{bmatrix}, \ r = \begin{bmatrix} r_0 \\ r_1 \\ . \\ r_j \\ . \\ r_J \end{bmatrix} \tag{2.65}$$

$$\delta_j = \begin{bmatrix} T_j \\ p_j \end{bmatrix}, \ r_j = \begin{bmatrix} \left(r_1\right)_j \\ \left(r_2\right)_j \end{bmatrix} \tag{2.66a}$$

and A_j, B_j, C_j are 2×2 matrices defined as follows

$$A_0 \equiv \begin{bmatrix} 1 & 0 \\ -1 & \frac{-h_1}{2} \end{bmatrix}, \ A_j \equiv \begin{bmatrix} \left(S_3\right)_j & \left(S_1\right)_j \\ -1 & \frac{-h_{j+1}}{2} \end{bmatrix}, 1 \le j \le J-1$$

$$A_J \equiv \begin{bmatrix} \left(S_3\right)_J & \left(S_1\right)_J \\ -1 & 0 \end{bmatrix}, B_J \equiv \begin{bmatrix} \left(S_3\right)_J & \left(S_2\right)_J \\ 0 & 0 \end{bmatrix}, 1 \le j \le J, \ C_j \equiv \begin{bmatrix} 0 & 0 \\ 1 & \frac{-h_{j+1}}{2} \end{bmatrix}, 1 \le j \le J-1 \tag{2.66b}$$

Note that, as in the Crank-Nicolson method, the implicit nature of the method has again generated a tridiagonal matrix, but the entries are 2×2 blocks rather than scalars.

The solution of equation (2.64) by the block-elimination method consists of two sweeps. In the forward sweep we compute Γ_j, Δ_j, and w_j from the recursion formulas given by

$$\Delta_0 = A_0, \ \Gamma_j \Delta_{j-1} = B_j, \ \Delta_j = A_j - \Gamma_j C_{j-1} \quad 1 \le j \le J, \tag{2.67}$$

$$w_0 = r_0 \ , w_j = r_j - \Gamma_j w_{j-1} \quad 1 \le j \le J. \tag{2.68}$$

Here Γ_j has the same structure as B_j, that is,

$\Gamma_j \equiv \begin{bmatrix} \left(\gamma_{11}\right)_j & \left(\gamma_{12}\right)_j \\ 0 & 0 \end{bmatrix}$, and although the second row of Δ_j has the same structure as the second row of A_j, $\Delta_j \equiv \begin{bmatrix} (\alpha_{11})_j & (\alpha_{12})_j \\ -1 & \frac{-h_{j+1}}{2} \end{bmatrix}$.

For generality we write it as $\Delta_j \equiv \begin{bmatrix} (\alpha_{11})_j & (\alpha_{12})_j \\ (\alpha_{21})_j & (\alpha_{22})_j \end{bmatrix}$. In the backward sweep, δ_j is computed from the following recursion formulas:

$$\Delta_J \delta_J = w_J, \Delta_j \delta_j = w_j - C_j \delta_{j+1}, \quad j = J-1, J-2, ..., 0.$$

2.7 Nomenclature

A	Constant
b	Stretching rate or positive constant
B_0	Magnetic field strength
c	Fluid parameter of Eyring-Powell model
C_f	Specific heat of fluid
C_s	Species diffusion
C_0	Concentration at the wall
C_{ref}	Reference species diffusion
C_{fx}	Local skin friction
C_w	Species concentration at the plate
C_∞	Species concentration far away from the plate
C_p	Specific heat at constant pressure
D	Coefficient of thermal diffusing species
$D_B(C)$	Concentration dependent mass diffusivity
D_{B0}	Brownian diffusion co-efficient
D_T	Thermophoretic diffusion co-efficient
E	Constant
E_c	Eckert number
e_{ij}	Strain components
f	Dimensionless stream function
f_w	Suction/blowing parameter
g	Acceleration due to gravity
G_r	Grashoff number
G_c	Modified Grashoff number
$h(x)$	Heat transfer co-efficient
K	Consistency co-efficient of viscosity (absolute viscosity)
k	Thermal conductivity
k_0	Viscoelastic parameter
k_1	Reaction rate parameter

k_2	Roseland mean absorption coefficient
k_∞	Thermal conductivity far away from sheet
l	Characteristic length
Le	Lewis number
m	Velocity exponent parameter
M_n	Magnetic parameter
n	Power law index of the fluid
Nu_x	Local Nusselt number
Np_r	Modified Prandtl number for power law fluid
N_b	Generalized Brownian motion parameter
N_t	Generalized thermophoresis parameter
N_r	Thermal radiation parameter
NS_c	Modified Schmidt number for power law fluid
P	Pressure
p_r	Prandtl number
Q	Volumetric rate of heat generation/absorption
q_m	Surface mass flux
q_w	Local heat flux
q_r	Radiative heat flux
Re_x	Local Reynolds number
r	Temperature exponent parameter
r_1	Species concentration of wall parameter
S	Unsteady parameter
S_c	Schmidt number
Sh_x	Sherwood number
T	Temperature of the fluid
T_{ref}	Reference temperature
T_s	Temperature of the stretching sheet
T_0	Temperature at the slot
t	Dimensionless time
T_w	Temperature at the wall

T_∞	Temperature far away from the sheet
u	Velocity in horizontal direction
U	Velocity of sheet
v	Velocity in vertical direction
$\mathrm{v}_s / \mathrm{v}_w$	Suction/ blowing velocity across the stretching sheet
x	Horizontal distance
y	Vertical distance

Greek symbols

α	Thermal diffusivity
α_1	Real positive root
α_∞	Thermal diffusivity of the fluid far away from sheet
$\alpha(T)$	Temperature dependent thermal conductivity
β	Heat source/sink parameter
β^*	Co-efficient of volumetric expansion
β_1	Thermal expansion co-efficient
β_2	Fluid parameter of Eyring-Powell model
β_c	Concentration expansion coefficient
γ	Kinematic viscosity
ε	Thermal conductivity parameter
ε_1	Variable species diffusivity parameter
η	Similarity variable
θ	Dimensionless temperature function
θ_r	Fluid viscosity parameter
μ	Consistency co-efficient of viscosity (absolute viscosity)
μ_1	Shear viscosity
μ^*	Consistency index of the power law fluid
μ_0	Magnetic permeability
ρ	Fluid density

ρ_p	Density of nano particle
ρ_f	Density of fluid
$(\rho C)_p$	Effective heat capacity of nano particle
$(\rho C)_f$	Effective heat capacity of fluid
$\phi\ \sigma$	Fluid electrical conductivity
σ^*	Stefan Boltzmann constant
τ_{ij}	Stress tensor
τ_w	Local wall shear stress
ϕ	Dimensionless concentration function
ψ	Stream function
ξ	Similarity variable
$f(\xi)$	Dimensionless similarity function
δ	Reaction rate parameter
δ_{ij}	Kroneckar delta
ΔT	Sheet temperature

Subscripts/Superscripts

∞	Boundary condition at infinity
w	Denotes the wall
*	Dimensionless parameter
'	Denotes differentiation with respect to ξ