Chapter 4

The Operator sum-difference representation of a quantum noise channel

4.1 Introduction

Any practical use of a quantum operation involves taking into account the effect of the ambient environment, and the systematic study of such an effect constitutes the theory of open quantum systems. This now pervades a vast arena of studies, see for e.g., the Refs. [6, 110] for two distinct flavors of the subject. The effect of the environment, interchangeably called the reservoir or the bath, affects the system dynamics, in general, in two ways depending upon the commutability of the system and interaction Hamiltonian. If the two commute, then the process is a quantum nondemolition one, that is, there is dephasing without any energy exchange [20]; while if they do not commute, then there is dephasing along with dissipation [13, 67]. These effects have been brought within the ambit of practical implementation by a number of very impressive experiments, for e.g., [73] involving ion traps and [111] using high-Q cavity quantum electrodynamics.

As a result, the use of ideas from open quantum systems has become widespread in quantum information processing [11]. A very useful tool in this regard is the Kraus representation of the noise channel [8], which encodes the effect of the environment on the system of interest. The Kraus operators are canonically derived by spectrally decomposing the Choi matrix $\mathcal{B}$, which is proportional to the density matrix obtained by acting the channel on one half of a maximally entangled bipartite system. The map on density operators corresponding to a noise channel is completely positive (CP) if and only if the associated Choi matrix is positive. A map $\mathcal{E}$ is defined CP if it is positive, and furthermore, its extension $\mathbb{I} \otimes \mathcal{E}$ to a larger Hilber space, is also positive [11]. Negative eigenvalues in the Choi matrix would imply that $\mathcal{E}$ is non-completely-positive (NCP).

Partial trace is a familiar NCP map.

A quantum noise model posits a system S interacting with the environment E via a Hamiltonian $H_S + H_E + H_{SE}$, where H_S (H_E) is the free Hamiltonian for the system (environment) and H_{SE} is the interaction Hamiltonian [20, 13]. The Schrödinger equation for the evolution of the joint state ρ_{SE} of the system and environment is solved. Tracing out the environment, yields the map on ρ_S, the reduced density operator of the system. For sufficiently complicated noisy situations, the model is only numerically solvable. When the noise model is exactly solvable, we obtain an analytical expression for $\mathcal{E}$. Even so, when we attempt to derive the Kraus operators, we may encounter a problem which has its origin in the Abel-Galois irreducibility theorem [112].

If any Choi matrix $\mathcal{B}$ has an analytic spectral decomposition, that would give a recipe to obtain the eigenvalues analytically. However, the eigenvalues are solutions to the eigenvalue problem $\mathcal{B}\vec{v} = \lambda\vec{v}$. This is equivalent to solving the characteristic polynomial associated with matrix $\mathcal{B}$. However, the above theorem proves that for quintic and higher degree polynomial equations, there are in general no analytic solutions. Consequently, analytic spectral decomposition will also not be possible in such a case. In practice, one way to circumvent the problem is to derive Kraus operators numerically, but in this case it is doubtful that the Kraus representation offers any advantage over other representations of the noise channel. Here[1], we present a method for an analytic derivation of a Kraus-like representation even when the rank of channel is 5 or above, provided the model is exactly solvable. This is accomplished at the cost of introducing more Kraus-like operators than would have sufficed if an analytic diagonalization of $\mathcal{B}$ were possible, and furthermore our method will also include negative Kraus-like terms. Accordingly, we call this the 'operator sum-difference representation' (OSDR). We will call as *tractable* the map or noise channel if the associated Choi matrix is analytically diagonalizable. Otherwise the map is called *intractable*. The key insight of our work is that by exploiting linearity of the map, one can always express a (possibly) intractable map as a linear combination of tractable 'submaps'.

We illustrate our method by applying it to a two-qubit model much discussed in the literature [27], whose associated Choi matrix is of rank 9 and intractable in the Abel-Galois sense. The model considered is that of two qubits interacting with a bath, e.g., an electromagnetic field in a squeezed thermal state, via the dipole interaction, which has been considered in detail for both pure dephasing [21] as well as dissipative [19] system-reservoir interactions. The system-reservoir coupling constant is dependent upon the position of the qubit, leading to interesting dynamical consequences. In particular, we consider here the special case of a vacuum bath, because this is exactly solvable [19]. One can continuously shift the noise from the independent to the collective regime, elaborated below in Section 4.4. The former noise is tractable and admits a conventional

[1]The results included in this chapter are on arXiv.org [10]

Kraus representation, while the latter requires an OSD representation.

The plan of the chapter is as follows. In the next section, we show how any Hermitian map (CP or non-CP) can be always expressed as a sum of Hermitian submaps, each of which is tractable, even though the parent map may be intractable. This yields the OSD representation of the complete noise channel, which may correspond to a CP or NCP map. This representation is not unique and will in general be different from the canonical Kraus representation. In Section 4.3, we point out that the representation of NCP maps as the difference of two CP maps may be considered as implementing a special case of OSDR. Further, we point out that the OSDR even of a CP map may include negative Kraus-like terms, because the submaps are only Hermitian and not necessarily positive. In Section 4.4, we apply this method to derive the OSD representation of the two-qubit amplitude damping (2AD) channel. This is followed by a discussion on how our method can be extended to general linear maps in Section 4.5. Finally, we make our conclusions in Section 4.6.

4.2 Hermitian decomposition

A (state-independent) linear transformation of a quantum state is a Hermitian map if and only if it maps Hermitian operators ρ in its domain to Hermitian operators. It is described according to the prescription [113]:

$$\rho \longrightarrow \mathcal{E}^{\rm H}(\rho) = \sum_j c_j A_j \rho A_j^\dagger, \quad c_j \in \mathbb{R}, \tag{4.1}$$

where ρ is the density operator and $A_j^\dagger A_j$ are positive operators that satisfy the completeness condition $\sum_j c_j A_j^\dagger A_j = \mathbb{I}$. In Eq. (4.1), if $c_j \geq 0$, then we obtain a completely positive (CP) map $\mathcal{E}^{\rm CP}$ [7]. This arises when S interacts via interaction $U(t)$ with its environment E and the initial state $\rho_{SE}(0)$ of the joint system S-E has the product form $\rho_{SE} \equiv \rho_S \otimes \rho_E$. The CP map is obtained by tracing out E after its interaction: $\mathcal{E}^{\rm CP} = {\rm Tr}_E[U(t)\rho_{SE}(0)U^\dagger(t)]$. If this positivity requirement on c_j is not met, the result is a non-completely positive (NCP) map, which implies that $\rho_{SE}(0)$ lacks the product form. An NCP map can be represented as the difference of two CP maps [114].

Here we show that any CP or NCP map can be decomposed into simpler Hermitian sub-maps. This decomposition is useful when a given channel is exactly solvable but intractable by the Abel-Galois theorem, in which case one can arrange for the Hermitian 'sub-maps' in the decomposition to be of sufficiently low-degree as to be tractable. Given a Hermitian map $\mathcal{E}^H$, we can form a decomposition of the kind

$$\mathcal{E}^{\rm H}(\rho) = \sum_{a=1}^{p} \mathcal{E}_a^{\rm H}(\rho) = \sum_a \left(\sum_{j_a=1}^{\mu} c_a^{(j_a)} A_a^{(j_a)} \rho \left(A_a^{(j_a)}\right)^\dagger \right), \quad c_a^{(j_a)} \in \mathbb{R}, \tag{4.2}$$

where $\mathcal{E}_a^{\mathrm{H}}$ are tractable, and the operators $A_a^{(j)}$ satisfy the completeness condition

$$\sum_a \sum_{j_a} c_a^{(j_a)} \left(A_a^{(j_a)}\right)^\dagger A_a^{(j_a)} = \mathbb{I}, \tag{4.3}$$

where $\mathbb{I}$ is the identity operator. If the noise $\mathcal{E}^H$ is exactly solvable, then the operators A_{j_a} have a closed, analytic form, even when $\mathcal{E}^H$ is intractable.

Theorem 2 *Any Hermitian map $\mathcal{E}^{\mathrm{H}}$ can always be decomposed as the sum of tractable Hermitian maps $\mathcal{E}_a^{\mathrm{H}}$ in the form Eq. (4.2).*

Proof. The decomposition (4.2) is not unique. We only require that each sub-map in the decomposition be analytically diagonalizable. To see this, we essentially extend the method of Choi to an arbitrary Hermitian decomposition of $\mathcal{B}$ [7]. Consider the bipartite unnormalized state $\tilde{\Phi} = \sum_{j,k}^n |j\rangle\langle k| \otimes |j\rangle\langle k| = n|\phi\rangle\langle\phi|$, where $|\phi\rangle = \frac{1}{\sqrt{n}} \sum_j |j,j\rangle$ is the maximally entangled state in the Hilbert space $\mathcal{H}_n \otimes \mathcal{H}_n$, represented in the basis $\{|j\rangle\}$ for $\mathcal{H}_n$. Now $\tilde{\Phi}$ can be considered as an $n \times n$ array of $n \times n$ blocks, where the (j,k)th block is $|j\rangle\langle k|$ (cf. Ref. [65]). We will now construct two equivalent representations of the Choi matrix $\mathcal{B} \equiv (\mathcal{I} \otimes \mathcal{E})[\tilde{\Phi}]$.

(i) In the first construction, $(\mathcal{I} \otimes \mathcal{E})[\tilde{\Phi}] = \sum_{j,k} |j\rangle\langle k| \otimes \mathcal{E}\left(|j\rangle\langle k|\right)$ is an array of $n \times n$ array of $n \times n$ blocks, such that the (j,k)th block is $\mathcal{E}(|j\rangle\langle k|)$:

$$(\mathcal{I} \otimes \mathcal{E})[\tilde{\Phi}]) = \left(\begin{array}{c|c|c}
\mathcal{E}\begin{pmatrix} 1 & 0 & \cdot & 0 \\ 0 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 \end{pmatrix} & \mathcal{E}\begin{pmatrix} 0 & 1 & \cdot & 0 \\ 0 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 \end{pmatrix} & \begin{matrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{matrix} \\ \hline
\mathcal{E}\begin{pmatrix} 0 & 0 & \cdot & 0 \\ 1 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 \end{pmatrix} & \mathcal{E}\begin{pmatrix} 0 & 0 & \cdot & 0 \\ 0 & 1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 \end{pmatrix} & \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \\ \hline
\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} & \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} & \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}
\end{array}\right) \tag{4.4}$$

(ii) In the second construction, we spectrally decompose each matrix into the Hermitian matrix corresponding to each submap $\mathcal{E}_a^{\mathrm{H}}$. We then have

$$\mathcal{B} \equiv \sum_a \mathcal{B}_a = \sum_{a=1} U_a D_a U_a^\dagger = \sum_a \left(\sum_{j_a} c_a^{(j_a)} |j_a\rangle\langle j_a| \right), \tag{4.5}$$

where D_a is a diagonal matrix of real numbers, which are the eigenvalues c_{j_a} of $\mathcal{B}_a$.

Now divide the column (row) vector $|j_a\rangle$ ($\langle j_a|$) into n segments of length n. In the columnar (row) case, define a $n \times n$ matrix $A_a^{(j_a)}$ ($A_a^{(j_a)\dagger}$) whose lth column (row) is the lth segment. Then, in $|j_a\rangle\langle j_a|$, the (j,k)th block will be $A_a^{(j_a)}|j\rangle\langle k|A_a^{(j_a)\dagger}$, where $|j\rangle$ and

$|k\rangle$ are vectors in the n-dimensional space. To see this, we note that

$$
\begin{aligned}
&(\mathcal{I}\otimes\mathcal{E})[\tilde{\Phi}] = \\
&\sum_a \sum_{j_a} c_a^{(j_a)} \left[\begin{array}{c} A_a^{(j_a)}\begin{pmatrix}1\\0\\0\\\vdots\end{pmatrix} \\ \hline A_a^{(j_a)}\begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix} \\ \hline A_a^{(j_a)}\begin{pmatrix}0\\0\\1\\\vdots\end{pmatrix} \\ \hline \vdots \end{array}\right] \times \left[\ \boxed{\left(\begin{array}{c|c|c|c} 1 & 0 & 0 & \cdots \end{array}\right) A_a^{(j_a)\dagger}} \quad \boxed{\left(\begin{array}{c|c|c|c} 0 & 1 & 0 & \cdots \end{array}\right) A_a^{(j_a)\dagger}} \quad \cdots \right] \\
&= \sum_a \sum_{j_a} c_a^{(j_a)} \left[\begin{array}{c} A_a^{(j_a)}|1\rangle \\ \hline A_a^{(j_a)}|2\rangle \\ \hline \vdots \\ \hline A_a^{(j_a)}|n\rangle \end{array}\right] \times \left[\ \boxed{\langle 1|A_a^{(j_a)\dagger}} \quad \boxed{\langle 2|A_a^{(j_a)\dagger}} \quad \cdots \quad \boxed{\langle n|A_a^{(j_a)\dagger}}\ \right] \\
&= \sum_a \sum_{j_a} c_a^{(j_a)} \begin{array}{|c|c|c|c|} \hline A_a^{(j_a)}|1\rangle\langle 1|A_a^{(j_a)} & A_a^{(j_a)}|1\rangle\langle 2|A_a^{(j_a)} & \cdots & A_a^{(j_a)}|1\rangle\langle n|A_a^{(j_a)} \\ \hline A_a^{(j_a)}|2\rangle\langle 1|A_a^{(j_a)} & A_a^{(j_a)}|2\rangle\langle 2|A_a^{(j_a)} & \cdots & A_a^{(j_a)}|2\rangle\langle n|A_a^{(j_a)} \\ \hline \cdots & \cdots & \cdots & \cdots \\ \hline A_a^{(j_a)}|n\rangle\langle 1|A_a^{(j_a)} & A_a^{(j_a)}|n\rangle\langle 2|A_a^{(j_a)} & \cdots & A_a^{(j_a)}|n\rangle\langle n|A_a^{(j_a)} \\ \hline \end{array}
\end{aligned}
\tag{4.6}
$$

Comparing the two constructions for their description of the action on $|j\rangle\langle k|$, we find that

$$\mathcal{E}(|j\rangle\langle k|) = \sum_a \sum_{j_a} c_a^{(j_a)} A_a^{(j_a)} |j\rangle\langle k| A_a^{(j_a)\dagger}, \tag{4.7}$$

from which Eq. (4.2) follows. To complete the proof, we note that a Hermitian matrix

$\mathcal{B} \equiv \{B_{jk}\}$ for any dimension N can always be decomposed into a diagonal matrix plus $\frac{N(N-1)}{2}$ Hermitian matrices of at most rank 2, which can certainly be diagonalized. To see this note that for any pair of basis vectors $|j\rangle, |k\rangle$ $(j \neq k)$, the matrix $\mathcal{C}_{jk} \equiv B_{jk}|j\rangle\langle k| + B_{kj}|k\rangle\langle j|$ is Hermitian (because $B^*_{kj} = B_{jk}$) and of rank 2. Further, construct the diagonal matrix $\mathcal{D}$ with entries B_{jj}. We thus have $\mathcal{B} = \sum_{k=1}^{N}\sum_{j=1,j<k}^{N-1} \mathcal{C}_{jk} + \mathcal{D}$, which is the required construction. ■

The basic idea is that, because of linearity, Choi's method works even when the spectral decomposition of $\mathcal{B}$ is replaced by the spectral decomposition of elements $\mathcal{B}_a$ in its Hermitian decomposition. Each such element induces a 'submap'. Suppose $\mathcal{B}_a$ is nonpositive for some a. Let $r_a \equiv \text{rank}\,(\mathcal{B}_a) \leq 4$, and further, let the r_a OSDR operators derived from it by the method described in Theorem 2 be denoted $k_a^{(j_a)}$ $(j_a = 1, 2, \cdots, r_a)$. Then the submap induced by $\mathcal{B}_a$, namely $\mathcal{E}_{\mathcal{B}_a} : \rho \mapsto \sum_{j_a} c_a^{(j_a)} k_a^{(j_a)} \rho \left(k_a^{(j_a)}\right)^\dagger$, has at least one $c_a^{(j_a)} < 0$. The result is a Kraus-like representation, except that some terms will appear with a negative sign. The parent map will the sum of such submaps. By this construction, it will have a representation in terms of the sum or difference of Kraus-like terms $k_a^{(j_a)} \rho \left(k_a^{(j_a)}\right)^\dagger$.

4.3 Non-positive submaps and NCP maps: An example

We note that the representation of an NCP map as the difference of two CP maps is a special case of representation (4.2) in which we set $\mathcal{B} = \mathcal{B}_{(+)} + \mathcal{B}_{(-)}$, where $\mathcal{B}_{(+)} \geq 0$ and $\mathcal{B}_{(-)} \leq 0$. Now $\mathcal{B}_{(\pm)}$ may itself not be Abel-Galois tractable, in which case the method of Theorem 2 can be used to break it up into tractable pieces.

Further, an OSD representation of a CP map may contain negative Kraus terms. As an example of a such non-positive decomposition of a CP map, consider such a representation of the generalized amplitude damping channel (GAD) [1]. The Choi matrix corresponding to GAD channel with elements $\sqrt{p}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{pmatrix}$, $\sqrt{p}\begin{pmatrix} 0 & 0 \\ \sqrt{\lambda} & 0 \end{pmatrix}$, $\sqrt{1-p}\begin{pmatrix} \sqrt{1-\lambda} & 0 \\ 0 & 1 \end{pmatrix}$, and $\sqrt{1-p}\begin{pmatrix} 0 & \sqrt{\lambda} \\ 0 & 0 \end{pmatrix}$ is

$$\mathcal{B} = (\mathcal{I} \otimes \mathcal{E}^{\text{GAD}})\Phi = \begin{pmatrix} 1-\lambda+p\lambda & 0 & 0 & \sqrt{1-\lambda} \\ 0 & p\lambda & 0 & 0 \\ 0 & 0 & (1-p)\lambda & 0 \\ \sqrt{1-\lambda} & 0 & 0 & 1-p\lambda \end{pmatrix}, \tag{4.8}$$

where $\Phi \equiv (|00\rangle\langle 00| + |11\rangle\langle 11|)$ and $0 \leq p, \lambda \leq 1$. A Hermitian decomposition of $\mathcal{B}$ in

Eq. (4.8) is $\mathcal{B} = \mathcal{B}_{(+)} + \mathcal{B}_{(-)}$, such that $\mathcal{B}_{(+)} \geq 0$ while $\mathcal{B}_{(-)} \leq 0$, is

$$\begin{aligned} \mathcal{B}_{(+)} &= \begin{pmatrix} 1-\lambda+p\lambda+\frac{\sqrt{1-\lambda}}{2} & 0 & 0 & \frac{3\sqrt{1-\lambda}}{4} \\ 0 & p\lambda & 0 & 0 \\ 0 & 0 & (1-p)\lambda & 0 \\ \frac{3\sqrt{1-\lambda}}{4} & 0 & 0 & 1-p\lambda+\frac{\sqrt{1-\lambda}}{2} \end{pmatrix}, \\ \mathcal{B}_{(-)} &= \begin{pmatrix} \frac{-\sqrt{1-\lambda}}{2} & 0 & 0 & \frac{\sqrt{1-\lambda}}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\sqrt{1-\lambda}}{4} & 0 & 0 & -\frac{\sqrt{1-\lambda}}{2} \end{pmatrix}. \end{aligned} \tag{4.9}$$

In the manner of Eq. (4.5), the matrix $\mathcal{B}_{(+)}$ can be diagonalized, from which, per recipe (4.7), we can form the OSDR elements by 'folding' the eigenvectors, and thereby construct the submap induced by $\mathcal{B}_{(+)}$. This is given by $\mathcal{E}_{(+)} : \rho \mapsto \sum_{j_+=1}^{4} c_+^{j_+} A_+^{(j_+)} \rho A_+^{(j_+)\dagger}$, where

$$c_+^{(1)} = \frac{4+2\sqrt{1-\lambda}-2\lambda-\sqrt{a}}{4}; \quad A_+^{(1)} = \begin{pmatrix} -\frac{2\lambda(1-2p)+\sqrt{a}}{3\sqrt{1-\lambda}} & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.10}$$

$$\begin{aligned} c_+^{(2)} &= \frac{4+2\sqrt{1-\lambda}-2\lambda+\sqrt{a}}{4}; \quad A_+^{(2)} = \begin{pmatrix} -\frac{2\lambda(1-2p)-\sqrt{a}}{3\sqrt{1-\lambda}} & 0 \\ 0 & 1 \end{pmatrix}, \\ c_+^{(3)} &= (1-p)\lambda; \quad A_+^{(3)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ c_+^{(4)} &= p\lambda; \quad A_+^{(4)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \end{aligned} \tag{4.11}$$

where $a = 9(1-\lambda) + 4\lambda^2(1-2p)^2$. As with $\mathcal{B}_{(+)}$, in the manner of Eq. (4.5), the matrix $\mathcal{B}_{(-)}$ can be diagonalized, from which, per recipe (4.7), we can construct the submap induced by $\mathcal{B}_{(-)}$. This is given by $\mathcal{E}_{(-)} : \rho \mapsto \sum_{j=1}^{2} c_-^{(j_-)} A_-^{(j_-)} \rho A_-^{(j_-)\dagger}$, where

$$\begin{aligned} c_-^{(1)} &= -\frac{(1-\lambda)^{1/2}}{4}; \quad A_-^{(1)} = \mathbb{I}, \\ c_-^{(2)} &= -\frac{3(1-\lambda)^{1/2}}{4}; \quad A_-^{(2)} = \sigma_z, \end{aligned} \tag{4.12}$$

which correspond to the purely dephasing channel (in which the diagonal terms are unaffected and only off-diagonal terms are killed off.) The OSDR representation for the full GAD noise map is obtained by combining the above two submaps, yielding:

$$\mathcal{E}^{\mathrm{GAD}} : \rho \mapsto \sum_{j_+=1}^{4} c_+^{(j_+)} A_+^{(j_+)} \rho A_+^{(j_+)\dagger} + \sum_{j_-=1}^{2} c_-^{(j_-)} A_-^{j_-} \rho A_-^{(j_+)\dagger}, \tag{4.13}$$

where the elements $c_{\pm}^{(j\pm)}$ and $A_{\pm}^{(j\pm)}$ are taken from Eqs. (4.11) and (4.12). The form (4.13) allows us to interpret the GAD channel as a dissipative part minus a purely dephasing part. This example serves to underscore that the negative Kraus-like terms do not necessarily indicate that the map is not CP.

In the above example, we note that the positive and negative Kraus-like terms are, up to a constant factor, trace-preserving. The basic idea behind achieving this is that the Choi matrix can always be written as the difference of two positive operators: $\mathcal{B} = \mathcal{B}_1 - \mathcal{B}_2$. Let $\mathcal{B}_1 \equiv c_1\mathcal{B}'_1$ and $\mathcal{B}_2 \equiv c_2\mathcal{B}'_2$, where $\mathcal{B}'_j > 0$ and $\mathrm{Tr}\left(\mathcal{B}'_j\right) = 1$, i.e., each $\mathcal{B}'_j$ is a density operator. Thus each $\mathcal{B}'_j$ inividually yields a valid quantum channel, and we obtain the required construction.

4.4 Application to two-qubit noise

It is a frequent assumption that the noise $\mathcal{E}$ acting on a N-qubit system affects each qubit independently, in which case the noise is always tractable, because in this case $\mathcal{E} = \mathcal{E}^{(1)} \otimes \mathcal{E}^{(2)} \otimes \cdots \otimes \mathcal{E}^{(N)}$ (where $\mathcal{E}^{(j)}$ is the independent noise acting on the jth qubit) and each $\mathcal{E}^{(j)}$ is tractable (since the characteristic polynomial of the corresponding Choi matrix is at most of degree 4). This independent noise model, while reasonable, is not always accurate in practice, because the noise affecting two or more qubits can be correlated when they are sufficiently close to experience the same bath. When the noise is correlated, as it happens in the collective regime of the 2AD noise discussed here, the rank of $\mathcal{B}$ will exceed 4, and the noise is no longer guaranteed to be tractable.

In Ref. [19], the problem of two qubits interacting collectively with a bath via a dissipative interaction was studied. In the case of a vacuum bath, the model is exactly solvable. The degree of correlation in the noise is governed by the inter-qubit distance r such that the noise is in the collective regime when $r/L \to 0$, and in the independent regime when $r/L \gg 0$. Here L is the typical length scale defined by the bath mode.

In this Section, we derive the OSD representation for 2AD noise map, $\mathcal{E}^{2\mathrm{AD}}$. The Choi matrix in this case, $\left(\mathcal{I} \otimes \mathcal{E}^{2\mathrm{AD}}\right)(|\Phi\rangle\langle\Phi|)$, where $|\Phi\rangle \equiv |00\rangle|00\rangle + |01\rangle|01\rangle + |10\rangle|10\rangle +$

$|11\rangle|11\rangle$, is given by:

$$
\mathcal{B} = \begin{pmatrix}
A & 0 & 0 & 0 & 0 & J & 0 & 0 & 0 & 0 & M & 0 & 0 & 0 & 0 & L \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X & 0 & 0 \\
J^* & 0 & 0 & 0 & 0 & B & 0 & 0 & 0 & 0 & P & 0 & 0 & 0 & 0 & T \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E & 0 & 0 & 0 & 0 & 0 & Y & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
M^* & 0 & 0 & 0 & 0 & P^* & 0 & 0 & 0 & 0 & D & 0 & 0 & 0 & 0 & Q \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & H & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & X^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Y^* & 0 & 0 & 0 & 0 & 0 & G & 0 \\
L^* & 0 & 0 & 0 & 0 & T^* & 0 & 0 & 0 & 0 & Q^* & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \tag{4.14}
$$

where $A = e^{-2\Gamma t}$; $B = e^{-(\Gamma+\Gamma_{12})t}$; $C = \frac{\Gamma+\Gamma_{12}}{\Gamma-\Gamma_{12}}(1 - e^{-(\Gamma-\Gamma_{12})t})e^{-(\Gamma+\Gamma_{12})t}$; $D = e^{-(\Gamma-\Gamma_{12})t}$; $E = \frac{\Gamma-\Gamma_{12}}{\Gamma+\Gamma_{12}}(1 - e^{-(\Gamma+\Gamma_{12})t})e^{-(\Gamma-\Gamma_{12})t}$; $F = 1 - e^{-(\Gamma+\Gamma_{12})t}$; $G = 1 - e^{-(\Gamma-\Gamma_{12})t}$; $J = e^{-i(\omega_0-\Omega_{12})t}e^{(3\Gamma+\Gamma_{12})t/2}$; $L = e^{-i2\omega_0 t}e^{-\Gamma t}$; $M = e^{-i(\omega_0+\Omega_{12})t}e^{-(3\Gamma-\Gamma_{12})t/2}$; $P = e^{-i2\Omega_{12}t}e^{-\Gamma t}$; $Q = e^{-i(\omega_0-\Omega_{12})t}e^{-(\Gamma-\Gamma_{12})t/2}$; $T = e^{-i(\omega_0+\Omega_{12})t}e^{-(\Gamma+\Gamma_{12})t/2}$;

$$
\begin{aligned}
H &= \frac{\Gamma+\Gamma_{12}}{2\Gamma}\left[1 - \frac{2}{\Gamma-\Gamma_{12}}\left(\frac{\Gamma+\Gamma_{12}}{2}(1 - e^{-(\Gamma-\Gamma_{12}t}) + \frac{\Gamma-\Gamma_{12}}{2}\right)e^{-(\Gamma+\Gamma_{12})t}\right] \\
&+ \frac{\Gamma-\Gamma_{12}}{\Gamma+\Gamma_{12}}\left[(1 - e^{-(\Gamma-\Gamma_{12})t}) - \frac{\Gamma-\Gamma_{12}}{2\Gamma}(1 - e^{-2\Gamma t})\right],
\end{aligned} \tag{4.15}
$$

$$
\begin{aligned}
Y &= i\,\frac{\Gamma-\Gamma_{12}}{\Gamma^2+4\Omega_{12}^2}e^{-i(\omega_0-\Omega_{12})t}e^{-(\Gamma-\Gamma_{12})t/2}\left[2\Omega_{12}(1 - e^{-\Gamma t}\cos(2\Omega_{12}t)) - \Gamma e^{-\Gamma t}\sin(2\Omega_{12}t)\right] \\
&\quad - \frac{\Gamma-\Gamma_{12}}{\Gamma^2+4\Omega_{12}^2}e^{-i(\omega_0-\Omega_{12})t}e^{-(\Gamma-\Gamma_{12})t/2}\left[2\Omega_{12}e^{-\Gamma t}\sin(2\Omega_{12}t) + \Gamma(1 - e^{-\Gamma t}\cos(2\Omega_{12}t))\right], \\
X &= \frac{\Gamma+\Gamma_{12}}{\Gamma^2+4\Omega_{12}^2}e^{-i(\omega_0+\Omega_{12})t}e^{-(\Gamma+\Gamma_{12})t/2}\left[2\Omega_{12}e^{-\Gamma t}\sin(2\Omega_{12}t) + \Gamma(1 - e^{-\Gamma t}\cos(2\Omega_{12}t))\right] \\
&\quad + i\,\frac{\Gamma+\Gamma_{12}}{\Gamma^2+4\Omega_{12}^2}e^{-i(\omega_0+\Omega_{12})t}e^{-(\Gamma+\Gamma_{12})t/2}\left[2\Omega_{12}(1 - e^{-\Gamma t}\cos(2\Omega_{12}t)) - \Gamma e^{-\Gamma t}\sin(2\Omega_{12}t)\right],
\end{aligned} \tag{4.16}
$$

where $F = |F|e^{i\phi_F}, J = |J|e^{i\phi_J}, M = |M|e^{i\phi_M}, P = |P|e^{i\phi_P}, Q = |Q|e^{i\phi_Q}, X = |X|e^{i\phi_X}.Y = |Y|e^{i\phi_Y}$. The terms ω_0 and Ω_{12} are bath mode frequency parameters, while Γ and Γ_{12} are bath coupling paramters, details of which are given in Ref. [19].

In Eq. (4.14), rank($\mathcal{B}$) = 9, which exceeds 4, and an algebraic spectral decomposition does not exist. By Theorem 2, we can nevertheless construct an OSD representation by breaking up $\mathcal{E}^{2\mathrm{AD}}$ into tractable submaps. This partitioning of $\mathcal{E}^{2\mathrm{AD}}$ into submaps is not unique. We give a particular decomposition as a worked out illustration of a practical application of OSDR. To simplify the presentation, we note that the Choi matrix only has support in the span of the basis states $\mathfrak{B} \equiv \{|0000\rangle, |0100\rangle, |0101\rangle, |1000\rangle, |1010\rangle, |1100\rangle, |1101\rangle, |1110\rangle, |1111\rangle\}$. The following decomposition is given restricted to span($\mathfrak{B}$).

$$
\begin{aligned}
\mathcal{B} &= \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3 \\
&\equiv \begin{pmatrix}
A & 0 & 0 & J & 0 & 0 & 0 & 0 & 0 \\
0 & C & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
J^* & 0 & B & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & E & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & H & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & 0 & 0 & M & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & X & 0 & 0 \\
0 & 0 & 0 & 0 & P & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
M^* & 0 & P^* & 0 & D & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & X^* & 0 & 0 & 0 & 0 & F & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \\
&\quad + \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & L \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & Y & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & Y^* & 0 & 0 & 0 & G & 0 \\
L^* & 0 & T^* & 0 & Q^* & 0 & 0 & 0 & 1
\end{pmatrix} \\
&\equiv \sum_{j_1=1}^{5} \left|K_1^{(j_1)}\right\rangle \left\langle K_1^{(j_1)}\right| + \sum_{j_2=1}^{4} \left|K_2^{(j_2)}\right\rangle \left\langle K_2^{(j_2)}\right| + \sum_{j_3=1}^{4} \left|K_3^{(j_3)}\right\rangle \left\langle K_3^{(j_3)}\right|, \qquad (4.17)
\end{aligned}
$$

where we have absorbed the eigenvalues c_j into the eigenvectors, since all $\mathcal{B}_a$ above are positive, as are the eigenvalues.

Thus, in Eq. (4.17), we first decompose $\mathcal{B}$ into three Hermitian parts $\mathcal{B}_a$, and then diagonalize each, according to step (4.5). From each of the sub-matrices, per recipe (4.7), we can form Kraus operators, and construct the submap induced by each of these pieces. This is done below. In particular, from the spectral decomposition of the matrix $\mathcal{B}_1$, we obtain the following OSDR operators (using the above renormalized eigenvectors):

$$
\begin{aligned}
A_1^{(1)} &= \sqrt{\frac{\frac{1}{2}\left(A+B-\sqrt{(A-B)^2+4J^2}\right)}{\left(\frac{-A+B+\sqrt{(A-B)^2+4J^2}}{2J}\right)^2+1}}\begin{pmatrix} -\frac{e^{i\phi_J}\left(-A+B+\sqrt{(A-B)^2+4J^2}\right)}{2J} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \\
A_1^{(2)} &= \sqrt{\frac{\frac{1}{2}\left(A+B+\sqrt{(A-B)^2+4J^2}\right)}{\left(\frac{A-B+\sqrt{(A-B)^2+4J^2}}{2J}\right)^2+1}}\begin{pmatrix} \frac{e^{i\phi_J}\left(A-B+\sqrt{(A-B)^2+4J^2}\right)}{2J} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \\
A_1^{(3)} &= \sqrt{C}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad A_1^{(4)} = \sqrt{E}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \\
A_1^{(5)} &= \sqrt{H}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
\tag{4.18}
$$

One can verify that the initial state

$$
\rho \equiv \begin{pmatrix} \rho_{ee} & \rho_{es} & \rho_{ea} & \rho_{eg} \\ \rho_{se} & \rho_{ss} & \rho_{sa} & \rho_{sg} \\ \rho_{ae} & \rho_{as} & \rho_{aa} & \rho_{ag} \\ \rho_{ge} & \rho_{gs} & \rho_{ga} & \rho_{gg} \end{pmatrix}, \tag{4.19}
$$

under the submap $\mathcal{E}_{(1)}$, defined by the OSDR operators in Eqs. (4.18), transforms to

$$
\rho_1' = \sum_{j_1=1}^{5} k_1^{(j_1)} \rho \left(k_1^{(j_1)}\right)^\dagger = \begin{pmatrix} A\rho_{ee} & Je^{-i\phi_J}\rho_{es} & 0 & 0 \\ Je^{i\phi_J}\rho_{se} & B\rho_{ss}+C\rho_{ee} & 0 & 0 \\ 0 & 0 & E\rho_{ee} & 0 \\ 0 & 0 & 0 & H\rho_{ee} \end{pmatrix}. \tag{4.20}
$$

From $\mathcal{B}_2$, we obtain the OSDR operators:

$$
\begin{aligned}
A_2^{(1)} &= \sqrt{\frac{\frac{1}{2}\left(D-\sqrt{D^2+4(M^2+P^2)}\right)}{\left(\frac{2M}{-D+\sqrt{D^2+4(M^2+P^2)}}\right)^2+\left(\frac{2P}{-D+\sqrt{D^2+4(M^2+P^2)}}\right)^2+1}} \times \\
&\begin{pmatrix} \frac{-2e^{-i\phi_M}M}{-D+\sqrt{D^2+4(M^2+P^2)}} & 0 & 0 & 0 \\ 0 & \frac{-2e^{-i\phi_P}P}{-D+\sqrt{D^2+4(M^2+P^2)}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \\
A_2^{(2)} &= \sqrt{\frac{\frac{1}{2}\left(D+\sqrt{D^2+4(M^2+P^2)}\right)}{\left(\frac{2M}{D+\sqrt{D^2+4(M^2+P^2)}}\right)^2+\left(\frac{2P}{D+\sqrt{D^2+4(M^2+P^2)}}\right)^2+1}} \times \\
&\begin{pmatrix} \frac{2e^{-i\phi_M}M}{D+\sqrt{D^2+4(M^2+P^2)}} & 0 & 0 & 0 \\ 0 & \frac{2e^{-i\phi_P}P}{D+\sqrt{D^2+4(M^2+P^2)}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \\
A_2^{(3)} &= \sqrt{\frac{\frac{1}{2}\left(F-\sqrt{F^2+4X^2}\right)}{\left(\frac{F+\sqrt{F^2+4X^2}}{2X}\right)^2+1}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{e^{-i\phi_X}\left(F+\sqrt{F^2+4X^2}\right)}{2X} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \\
A_2^{(4)} &= \sqrt{\frac{\frac{1}{2}\left(F+\sqrt{F^2+4X^2}\right)}{\left(\frac{-F+\sqrt{F^2+4X^2}}{2X}\right)^2+1}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{e^{-i\phi_X}\left(-F+\sqrt{F^2+4X^2}\right)}{2X} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\end{aligned}
\tag{4.21}
$$

Finally, from $\mathcal{B}_3$ we obtain

$$A_3^{(1)} = \sqrt{\frac{\frac{1}{2}\left(1-\sqrt{1+4(L^2+Q^2+T^2)}\right)}{\left(\frac{2L}{-1+\sqrt{1+4(L^2+Q^2+T^2)}}\right)^2+\left(\frac{2Q}{-1+\sqrt{1+4(L^2+Q^2+T^2)}}\right)^2+\left(\frac{2T}{-1+\sqrt{1+4(L^2+Q^2+T^2)}}\right)^2+1}} \times$$
$$\begin{pmatrix} \frac{-2e^{-i\phi_L}L}{-1+\sqrt{1+4(L^2+T^2+Q^2)}} & 0 & 0 & 0 \\ 0 & \frac{-2e^{-i\phi_T}T}{-1+\sqrt{1+4(L^2+T^2+Q^2)}} & 0 & 0 \\ 0 & 0 & \frac{-2e^{-i\phi_Q}Q}{-1+\sqrt{1+4(L^2+T^2+Q^2)}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

$$A_3^{(2)} = \sqrt{\frac{\frac{1}{2}\left(1+\sqrt{1+4(L^2+Q^2+T^2)}\right)}{\left(\frac{2L}{1+\sqrt{1+4(L^2+Q^2+T^2)}}\right)^2+\left(\frac{2Q}{1+\sqrt{1+4(L^2+Q^2+T^2)}}\right)^2+\left(\frac{2T}{-1+\sqrt{1+4(L^2+Q^2+T^2)}}\right)^2+1}} \times$$
$$\begin{pmatrix} \frac{2e^{-i\phi_L}L}{1+\sqrt{1+4(L^2+T^2+Q^2)}} & 0 & 0 & 0 \\ 0 & \frac{2e^{-i\phi_T}T}{1+\sqrt{1+4(L^2+T^2+Q^2)}} & 0 & 0 \\ 0 & 0 & \frac{2e^{-i\phi_Q}Q}{1+\sqrt{1+4(L^2+T^2+Q^2)}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

$$A_3^{(3)} = \sqrt{\frac{\frac{1}{2}\left(G-\sqrt{G^2+4^2}\right)}{\left(\frac{G+\sqrt{G^2+4Y^2}}{2y}\right)^2+1}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{e^{-i\phi_Y}\left(G+\sqrt{G^2+4Y^2}\right)}{2Y} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix};$$

$$A_3^{(4)} = \sqrt{\frac{\frac{1}{2}\left(G+\sqrt{G^2+4^2}\right)}{\left(\frac{-G+\sqrt{G^2+4Y^2}}{2y}\right)^2+1}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{e^{-i\phi_Y}\left(-G+\sqrt{G^2+4Y^2}\right)}{2Y} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{4.22}$$

In the above expressions, $\phi_L = -2\omega_0 t, \phi_M = -(\omega_0 + \Omega_{12})t, \phi_P = -2\Omega_{12}t, \phi_T = -(\omega_0 + \Omega_{12})t, \phi_X = \mathrm{Arg}(X), \phi_Q = -(\omega_0 - \Omega_{12})t, \phi_Y = \mathrm{Arg}(Y)$.

We then obtain the full evolution in the OSD representation combining the operators in Eqs. (4.18), (4.21) and (4.22), to be given by:

$$\rho \to \mathcal{E}^{2\mathrm{AD}}(\rho) = \sum_{a=1}^{3}\sum_{j_a=1}^{r_a} A_a^{(j_a)}\rho A_a^{(j_a)\dagger}, \tag{4.23}$$

where j_1 ranges from 1 to 5, while j_2 and j_3 range from 1 to 4. where we employ 13 OSDR operators. If the channel were tractable, then 9 Kraus operators would have sufficed.

Given the enormous freedom in decomposing $\mathcal{B}$, and constructing OSDR elements, it is important to identify decompositions that are particularly useful. Here an optimal decomposition would be one that minimizes the number of analytic Kraus-like elements. For example, the rank-9 matrix in Eq. (4.14) can be decomposed into 1 diagonal + 8 off-diagonal pieces that yield $9 + 2 \times 8 = 25$ Kraus-like elements according to the recipe at the end of Theorem 2. However, by means of decomposition (4.17), we obtain a reduced number of 13 Kraus-like operators. The optimal decomposition would in general depend on the details of particular characteristic equations of matrices obtained under the decomposition.

4.5 Extension to general linear maps

A linear map transforming bounded operators in an input Hilbert space $\mathcal{H}_N$ of dimension N to bounded operators in an output Hilbert space $\mathcal{H}_M$ of dimension M, is given by [113]

$$\rho \longrightarrow \mathcal{E}^{\mathrm{L}}(\rho) = \sum_j A_j \rho A_j'^{\dagger}, \tag{4.24}$$

where $\{A_j\}$ ($\{A_j'\}$) are $M \times N$ ($N \times M$) matrices satisfying the completeness condition $\sum_j A_j'^{\dagger} A_j = \mathbb{I}$. Our method of decomposition of a Hermitian map into tractable submaps can be readily extended to linear maps (4.24). Here $|\Phi\rangle = |1,1\rangle + |2,2\rangle + \cdots + |M,M\rangle$, and the Choi matrix is given by $\sum_{j,k} |j\rangle\langle k| \otimes \mathcal{E}^L(|j\rangle\langle k|)$, a square matrix of size $MN \times MN$ (i.e., $M \times M$ blocks of size $N \times N$). We do this essentially by representing a possibly intractable linear $MN \times MN$ matrix $\mathcal{B}$ as a sum of simpler matrices corresponding to the linear submaps.

The proof of Theorem 2 goes through here again, except that the spectral decomposition of $(\mathcal{I} \otimes \mathcal{E})[\tilde{\Phi}]$ of individual Hermitian submaps is replaced by singular value decomposition (SVD) of individual linear submaps. Given an $M \times N$ matrix G, SVD is the factorization

$$G = V \Delta U^{\dagger}, \tag{4.25}$$

where Δ is a diagonal positive semi-definite $M \times N$ matrix, V (U) is a $M \times M$ ($N \times N$) matrix.

In Eq. (4.5), we have instead for the r.h.s $\sum_{a=1} U_a D_a V_a^{\dagger} \equiv \sum_a \left(\sum_{j_a} c_a^{(j_a)} |j_a\rangle\langle j_a'| \right)$ where U_a, V_a are unitaries and D_a a diagonal matrix with singular values $c_a^{(j_a)} \geq 0$. Now divide the column (row) vector $|j_a\rangle$ ($\langle j_a'|$) into n segments of length m. In the columnar (row) case, define a $M \times N$ ($N \times M$) matrix $A_a^{(j_a)}$ ($(A_a^{(j_a)\prime})^{\dagger}$) whose lth column (row) is the lth segment. Then, in $|j_a\rangle\langle j_a'|$, the (j,k)th block will be $A_a^{(j_a)}|j\rangle\langle(|A_a^{(j_a)\prime})^{\dagger}$, and the rest of the proof of Theorem 2 follows.

4.6 Conclusions and discussions

When a quantum noise model is exactly solvable, it may still not be possible to algebraically derive a canonical Kraus representation, on account of Abel-Galois irreducibility. We introduced the OSDR method that circumvents this obstacle, by representing the intractable channel as a linear combination of simpler channels that are tractable. The method yields a set of OSDR operators, which are Kraus operators, but may sometimes appear with a negative sign. The price to pay is that the number of these operators will in general exceed the channel rank. Our method is applicable to CP and NCP maps, and more generally, to an arbitrary linear noise. We applies our method to derive the OSD representation for a 2-qubit amplitude-damping channel, whose associated Choi matrix, is of rank 9.

Two sets of Kraus operators give the same quantum operation iff they are related by a unitary transformation [5,8]. This is equivalent to requiring that they generate the same Choi matrix. Similarly, two OSDR representations give the same quantum operation iff they correspond to the same Choi matrix. By the Choi isomorphism, CP maps are isomorphic to positive operators. Thus, departure from complete positivity is verified by determining that the Choi matrix has no negative eigenvalues. In our first example, the negative part is a bit-flip channel up to a factor. We can thus regard the decomposition as representing the difference between a non-Pauli channel (one whose process matrix is not diagonal in the Pauli representation) and a Pauli channel (the bit flip channel), yielding generalized amplitude damping, which is a non-Pauli channel. We have now included this material in the discussions.