

Chapter 3

Application of noise to cryptography

3.1 Introduction

In 1984, Bennett and Brassard first introduced a protocol [90], usually called BB84, for quantum key distribution (QKD). In QKD two remote legitimate users (Alice and Bob) can establish an unconditionally secure key through transmission of qubits. Since the pioneering work of Bennett and Brassard several protocols for different cryptographic tasks have been proposed. While most of the initial works on quantum cryptography [90, 91, 92] were concentrated on QKD, eventually this effort was extended to other ‘post-coldwar’ cryptographic tasks. For example, in 1999, Hillery [93] proposed a protocol for quantum secret sharing (QSS). In the same year, Shimizu and Imoto [94] proposed a protocol for direct secure quantum communication using entangled photon pairs. Protocols for deterministic secure quantum communication (DSQC) were later proposed [95, 96, 97], in which the receiver can read out the secret message only after the transmission of at least one bit of additional classical information for each qubit. A set of protocols exist which do not require exchange of classical information. Such protocols are generally referred to as protocols for “quantum secure direct communication” (QSDC) [98].

DSQC and QSDC protocols are reducible to secure QKD protocols in the sense that the former equipped with a source of quantum randomness, yield the latter. A conventional QKD protocol generates an unconditionally secure key by quantum means but then uses classical cryptographic resources to encode the message. No such classical means are required in DSQC and QSDC. In the recent past, these facts encouraged several groups to study DSQC and QSDC protocols in detail, see for example [98] (and references therein).

Since the works of Shimizu and Imoto [94], and Hillery *et al.* [93], several protocols for DSQC and QSS have been proposed. The unconditional security of the protocols is achieved by using different quantum resources. For example, DSQC schemes are proposed, a) with and without maximally entangled state (Ref. [99] and references therein), b) using teleportation [100], c) using entanglement swapping [101], d) using rearrange-

ment of the order of particles, etc. We are specifically interested in DSQC protocols based on rearrangement of the orders of particles. To be precise, here we aim to provide a DSQC protocol for a specific task/problem, which may be visualized as follows: Charlie, Alice and Bob are three employees of a company. Charlie is the Director of the company and Alice is a *custodian* of all the files of that company. Now Alice wishes to securely share some of the information with Bob, say a file, which may be considered as a string of classical bits. For the present work, we have chosen it to be classical information, though it can be easily generalized to quantum information. But to do so, Alice requires permission from Charlie, who, though permitting Alice to do so, has imposed a restriction that Bob can read the file only when Charlie permits him to do so. This is a problem of practical importance because Alice may transmit her quantum-encoded information to Bob in advance (say, because the quantum channel is available only then), but Charlie may wish to give his permission at a later time, determined by, say, certain administrative constraints.

In this Chapter¹ we introduce a protocol of DSQC to solve the above mentioned problem by using rearrangement of order of particles [103, 104] and dense coding [105]. Since a secret is shared, the protocol may be considered as a protocol of secret sharing. This can also be visualized as an application of *controlled dense coding*, in which the controller is Charlie, who determines how much classical information is delivered to Bob after Alice sends him all her dense coding qubits. Since the problem discussed here is of much practical importance, we made the protocol more realistic and relevant by considering the channel to be noisy. By doing so we generalize the idea of secret sharing through noisy quantum channel proposed in Ref. [106].

The remaining part of the paper is organized as follows: In the next section we describe a protocol that realizes a cryptographic switch by using a pure entangled state (Bell state) in an ideal scenario under certain restrictions on the channel between Alice and Bob. In Section III, the effect of noise on the protocol is discussed with specific attention to the squeezed generalized amplitude damping (SGAD) channel. In Section IV, we modify the protocol for the more general case where no restriction is imposed on the channel between Alice and Bob. Finally, Section V is dedicated to conclusions and discussions.

3.1.1 Cryptographic Switch

Let us first propose a solution to the above mentioned problem in an ideal scenario: Here Alice is semi-honest [107] and Bob can receive qubits from Alice but cannot communicate qubits to Alice (for example, because he lacks the requisite license or resources for state preparation). Thus the channel between Alice and Bob is one-way and is also assumed

¹The results included in this chapter have been published in [102]

to be noiseless (restrictions that we will relax later). By semi-honest, we mean that Alice strictly follows the protocol and does not disclose the contents of files to Bob before obtaining Charlie's permission. Further, she does not create a quantum channel between herself and Bob through which to send the information. But after obtaining her share of entanglement from Charlie she may try to help Bob to bypass the control of Charlie and to obtain the information before Charlie permits Bob to do so.

The use of the semi-honest model is well-justified in classical and quantum communication. In particular, semi-honest models are very often used in modern classical cryptography, especially in the context of data mining [108]. In the recent past, the notion of a semi-honest user has also been adopted in the field of multi-party secure quantum communication; various secure quantum communication protocols have been proposed [109], where one or more of the participants are considered semi-honest.

We do not require Bob to be semi-honest as even a dishonest Bob cannot obtain any information as long as Alice respects the protocol. Here it would be apt to note that in the context of secure multiparty communication, the semi-honest and malicious models are generally used as security models. In a semi-honest model, a protocol is considered secure against a collusion of participants (Alice and Bob in our case) if by accumulating their data, these participants cannot gain more information than what they can from the input and output of the protocol alone [109]. In contrast to this in a malicious model, participants can deviate from the original protocol (say Alice and Bob can use Lucamarini-Mancini 2005 (LM05) [97] protocol to share the information without any involvement of Charlie). The protocols proposed here and similar protocols are not secure under a malicious model.

In this ideal situation, our protocol works as follows:

1. After receiving Alice's request, Charlie prepares n Bell states (not all the same) and sends the first qubits of all the Bell states to Alice and the second qubits to Bob. Charlie does not disclose which Bell state, he has prepared.
2. After receiving the qubits from Charlie, Alice understands that she has been permitted to send the information to Bob.
3. Alice uses dense coding to encode two bits of classical information on each qubit and transmits her qubits to Bob.
4. When Charlie plans to allow Bob to know the secret information communicated to him, he discloses the Bell state he had prepared.
5. Since the initial Bell state is known, by measuring his qubits in the Bell basis, Bob obtains the information encoded by Alice.

Bob can perform Step 5 (i.e., measurement in the Bell basis) before Step 4 but he will not obtain any meaningful information without the knowledge of the initial state. A Bell-state measurement can reveal the state prepared by Charlie if both the qubits are in the possession of Alice or Bob. By the assumption that the channel between Alice and Bob is one-way, Bob cannot send his qubits to Alice for a Bell-state measurement. Alice does not send her qubits to Bob as she is assumed to be a semi-honest party, who strictly follows the protocol. Her semi-honesty is motivated by the fact that, while she may wish to potentially cheat Charlie, she wants her communication to Bob to be secure both in the sense of being protected from the rest of the world (in the usual QKD sense) as well as being undetected by Charlie.

To be precise, for protection from the rest of the world (Eve) during transmission of qubits from Alice to Bob, Alice may randomly insert some additional (decoy) photons, which are randomly prepared in states $|0\rangle, |1\rangle, |+\rangle$ or $|-\rangle$. After receiving an authenticated acknowledgement of receipt from Bob, she (Alice) discloses the position of the decoy photons. Bob randomly measures them in the computational basis or diagonal basis and announces the result. Alice uses the result to compute the error rate. They proceed with the protocol run only if the computed error rate is smaller than a tolerable limit. A non-vanishing error rate is not necessarily bad, in particular for Charlie's control, because it makes it more disadvantageous for Alice and Bob to determine Charlie's state through multiple use of the quantum channel.

Our model can be considered as a three-party quantum secret sharing scenario, in which Charlie prepares a Bell state, of which he transmits one half to Alice and the other half to Bob. From this viewpoint, the latter two receive an ensemble of Bell-states:

$$\rho_{AB} \equiv \sum_{j,k} a_{j,k} |B_{j,k}\rangle \langle B_{j,k}|, \quad (3.1)$$

j denotes the parity bit and k the phase bit: $|B_{0,0/1}\rangle \equiv \frac{1}{2}(|00\rangle \pm |11\rangle)$ and $|B_{1,0/1}\rangle = \frac{1}{2}(|01\rangle \pm |10\rangle)$. Alice encodes two bits on her qubit using the four Pauli operators of the superdense coding protocol [105], and sends the qubit to Bob via a possibly insecure channel.

In the noiseless case, Bob measures the two qubits in his possession to obtain the state that corresponds to Alice's encoding. However, Bob can decode the full information only if Charlie shares the full classical *key* information c that would make the initial entangled state pure. More generally (as detailed below), Bob recovers Alice's transmitted bits depending on the key information obtained from Charlie. Thus Charlie acts as a *cryptographic switch* who can determine the level of information Alice sends to Bob *after* the full transmission of her qubit.

In Ref. [106], a related protocol is considered, in which Charlie prepares a GHZ state, from which he sends a qubit to Alice and one to Bob. After Alice's encoding

and transmission to Bob, Charlie measures his qubit in the X basis, thereby collapsing the Alice-Bob state to one of the two Bell states $|B_{0,0}\rangle$ or $|B_{0,1}\rangle$. Only when Charlie reveals his 1 bit of key information does Alice's dense coding protocol succeed. Note that Alice's and Charlie's operations commute, so that we might as well consider that Charlie prepares one of these two Bell states, transmitting them to Alice and Bob. We can consider a family of protocols in which the key information c varies continuously as $0 \leq c \leq c_{\max} = 2$. Our protocol is characterized by $c_{\max} = 2$ and that of Ref. [106] by $c_{\max} = 1$, since we allow for Charlie to prepare any of the four Bell states. Further, unlike in the above protocol, ours does not require the preparation of expensive three-particle states like a GHZ state.

The present protocol differs in two other significant ways: (a) Charlie can reveal a continuous amount of information upto 2 bits, rather than being confined to 0 and 1 bit; (b) we consider a general (and more prevalent) dissipative noise acting on the quantum channel, instead of the purely dephasing noise considered in Ref. [106].

The physical significance of a continuous-valued key is that it corresponds to an arbitrary probability distribution over the Bell states. For example, Charlie may choose to reveal that the parity-0 Bell states are twice more like than parity-1 states and that Bell states of equal parity are equally likely. This corresponds to a probability distribution of $(\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6})$, i.e., an entropy of about 1.92 bits, implying that Charlie reveals $c = 0.08$ bits.

Suppose Bob's measurement yields the state $|B_{0,0}\rangle$. If he comes to know that Charlie had transmitted the state $|B_{0,1}\rangle$, then he knows Alice must have encoded Z , since $|B_{0,0}\rangle = (Z \otimes I)|B_{0,1}\rangle$. More generally, this idea can be compactly represented as:

$$B_{j,k} \xrightarrow{P_{ab}} B_{j \oplus a, k \oplus b}, \quad (j, k, a, b = 0, 1) \quad (3.2)$$

where the l.h.s is state $B_{j,k}$ prepared by Charlie, which is transformed under Alice's action P_{ab} to the state received by Bob, given in the r.h.s. The sign \oplus denotes mod 2 addition, and P_{ab} ($a, b = 0, 1$) represents, sequentially, the Pauli I , Z , X and Y operators.

The problem can be treated as a communication situation in which Alice is signaling Bob by means of Bell states. Then the maximum information Bob can extract from the pair of qubits is the Holevo quantity \mathbf{H} [11]. In our protocol, where $c = 2$, Bob can extract Alice's 2 bits of information.

There are a number of ways to implement c , and it is assumed that Alice, Bob and Charlie agree upon one such convention at the start of the protocol. Here we present one such method. The mixed state provided by Charlie is assumed to be a Werner state, and we parametrize the amount of key information Charlie reveals by means of a single variable ψ . The joint state of Alice and Bob, given by Eq. (3.1), is assumed to have the

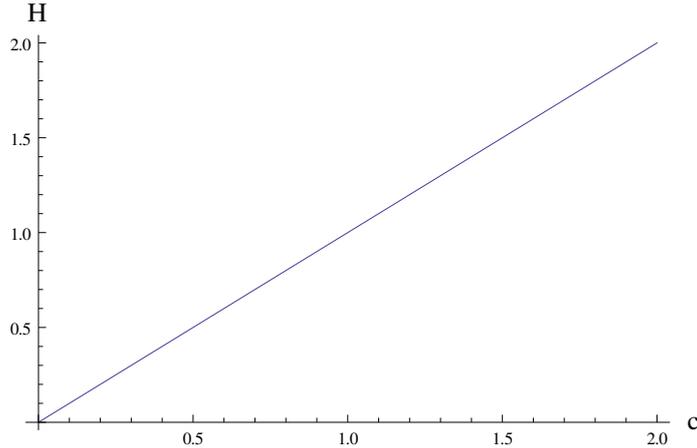


Figure 3.1: Information recovered by Bob, quantified by the Holevo quantity \mathbf{H} , as a function of the key information c communicated by Charlie, in the noiseless case.

form

$$\rho_{AB}^{(0,0)}(\psi) = a\Pi_{0,0} + b \sum_{j,k \neq 00} \Pi_{j,k}, \quad (3.3)$$

where $\Pi_{j,k}$ is projector to the Bell state $B_{j,k}$, $a = 0.25 + 0.75 \sin(\psi)$, $b = 0.25(1 - \sin(\psi))$ and $b \equiv \frac{1-a}{3}$. After transmitting the qubits to Alice and Bob, Charlie announces the value of angle ψ over a public channel. In so doing, the amount of information provided by him is $c = 2 - H(a, b, b, b)$ bits, where $H(\cdot)$ is Shannon entropy. If $c = 2$ ($a = 1$), then Bob knows Charlie had sent out a $|B_{0,0}\rangle$ state, and can work out Alice's encoded information via Eq. (3.2). Similarly other Werner states are possible. The maximum information Bob can extract from this ensemble is the Holevo quantity \mathbf{H} for the ensemble (3.3). Fig. (3.1) shows how Bob's information increases with key information in the noiseless case.

3.1.2 Noise considerations

In Ref. [106], the authors studied the effect of phase damping noise on the $c_{\max} = 1$ bit protocol. Here we consider noise to be a squeezed generalized amplitude damping channel [13] acting on Alice's qubit transmission. More realistically, we expect noise also to affect Charlie's transmissions to Alice and Bob, but as these added complications offer no new theoretical insight, we ignore them in this work. The action of a squeezed generalized amplitude damping channel is given by the set of Kraus operators $\{J'_+, J'_-, K'_+, K'_-\}$ was listed in Eq. (2.12) of Sec. 2.3.1 in Chapter 2.

In the asymptotic limit $t \rightarrow \infty$, $p_2(\infty) = N/(2N+1)$, $\nu(\infty) = \alpha(\infty) = 1$ and $\mu(\infty) = 0$ and the channel, which is contractive, maps any input state to $p_1(\infty)|0\rangle\langle 0| + p_2(\infty)|1\rangle\langle 1|$.

In Fig. (3.2), the variation of Bob's recovered information, quantified by the Holevo quantity \mathbf{H} , as a function of bath squeezing r , and Charlie's information c , is depicted. The Holevo quantity \mathbf{H} increases with c , but not as much as in the noiseless case (Fig.

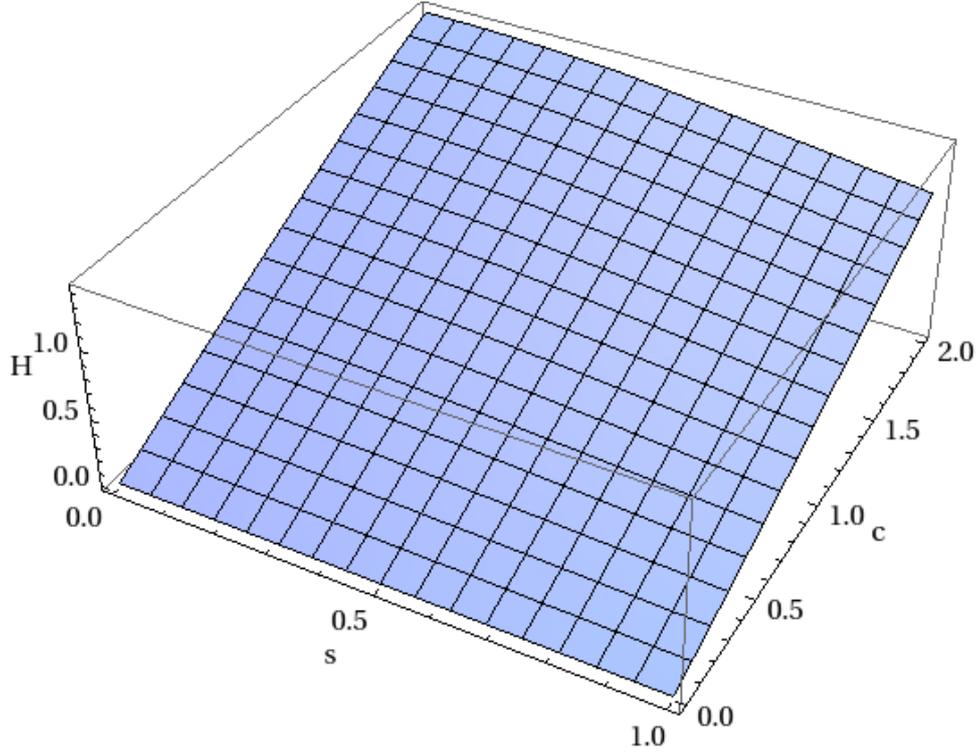


Figure 3.2: Information recovered by Bob, quantified by the Holevo quantity \mathbf{H} , as a function of the squeezing parameter s , coming from the SGAD Channel, and key information c communicated by Charlie. The time of evolution $t = 0.5$, while temperature $T = 0.1$ (in units where $\hbar \equiv k_B \equiv 1$).

(3.1)): because of the randomness introduced by the noise. Also, for a given information level c , \mathbf{H} decreases with increase in squeezing s .

The relationship between the Holevo quantity and the parameterizing angle ψ defined in Eq. (3.3) is shown in Fig. (3.3). For all cases shown the Holevo quantity \mathbf{H} increases with the angle ψ , with $\psi = \pi/2$ corresponding to a pure Bell state. Increase in angle ψ corresponds to increase in the purity of the ensemble (3.3) and an increase in \mathbf{H} . As expected, noise causes a reduction in the Holevo quantity. The maximum difference in \mathbf{H} , between the noiseless and noisy case, is seen to occur at $\psi = \pi/2$, that is, when the state is pure the reduction due to noise is maximum.

Fig. (3.4) shows the effect, on the Holevo quantity, of squeezing as a function of time, in particular, demonstrating its favorable influence in certain regimes. As seen from the figure, for sufficiently early times, squeezing fights thermal effects to cause an increase in the recovered information.

The significance of noise is that Alice and Bob may consume some of the Bell pairs to determine the noise level, and decide whether it is too high to permit secure information transfer, assuming conservatively that all the noise is due to Eve.

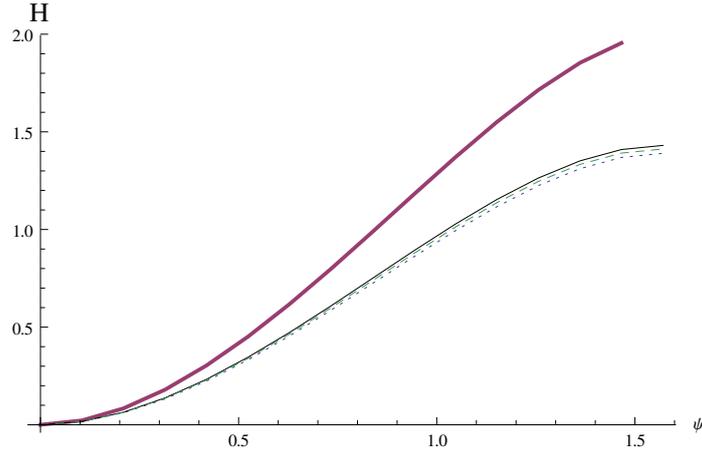


Figure 3.3: Information recovered by Bob, quantified by the Holevo quantity \mathbf{H} , as a function of the parameterizing angle ψ (3.3). The thick bold line corresponds to the noiseless case, the thin bold line to the case of noise with zero bath squeezing $s = 0$, while the large-dashed and small-dashed lines correspond to noise with squeezing s equal to -0.2 and 0.3 , respectively. The time of evolution $t = 0.5$, while the temperature $T = 0.1$ (in the units where $\hbar \equiv k_B \equiv 1$).

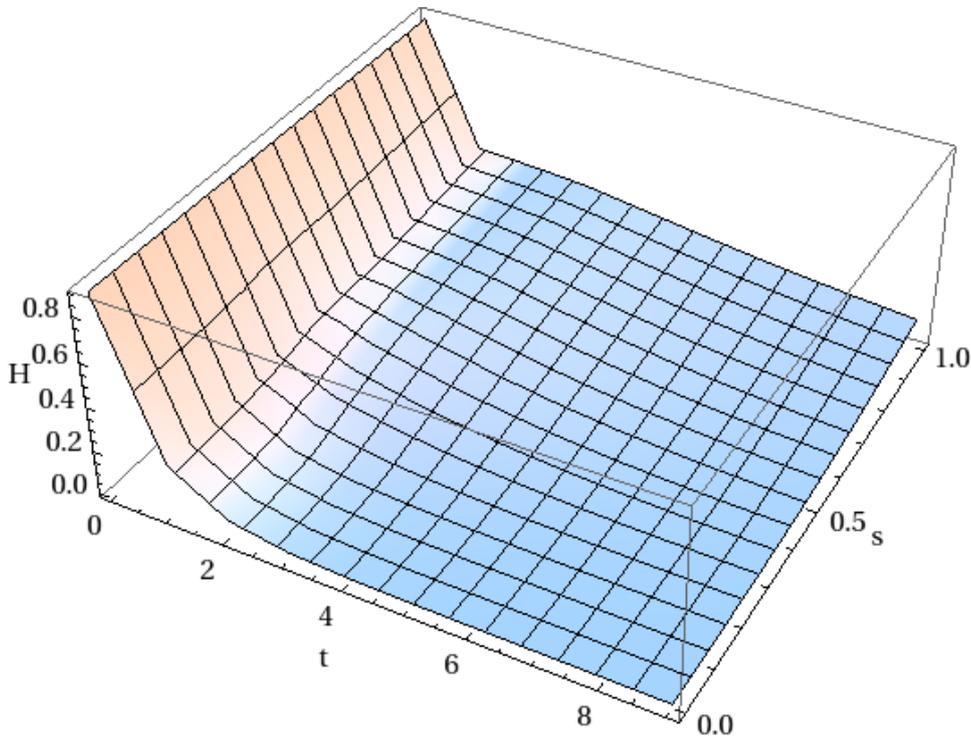


Figure 3.4: Information recovered by Bob, quantified by the Holevo quantity \mathbf{H} , as a function of the SGAD channel parameters r (squeezing) and t (time of evolution), assuming Charlie communicates one bit of information. We note that, for sufficiently early times, squeezing fights thermal effects (here $T = 0.1$, in the units where $\hbar \equiv k_B \equiv 1$) to cause an increase in the recovered information.

3.1.3 Cryptographic switch generalized to a two-way channel

Here we withdraw the restriction imposed on the channel (preventing quantum communication from Bob to Alice). We modify the proposed protocol as follows to remain secure:

1. After receiving Alice's request, Charlie prepares n Bell states (not all the same). He uses the Bell states to prepare an ordered sequence with all the first qubits as, $P_A = [p_1(t_A), p_2(t_A), \dots, p_n(t_A)]$, and another ordered sequence $P_B = [p_1(t_B), p_2(t_B), \dots, p_n(t_B)]$ with all the second qubits, resulting in an ordered sequence where the subscript $1, 2, \dots, n$ denotes the order of a particle pair $p_i = \{t_A^i, t_B^i\}$, which is in the Bell state.
2. Charlie scrambles the second qubits: that is, he disturbs the order of the qubits in P_B to create a new sequence $P'_B = [p'_1(t_B), p'_2(t_B), \dots, p'_n(t_B)]$ and sends it to Bob. The actual order is known to Charlie only. This ordering of information is part of the *extended key* of the modified protocol.
3. After receiving the qubits from Charlie, Alice understands that she has been permitted to send the information to Bob. Since the sequence with Alice and Bob are different, even if Alice or Bob obtain access to both P_A and P'_B , they will not be able to find out the Bell states prepared by Charlie. Thus any kind of collusion between Alice and Bob would fail.
4. Alice uses dense coding to encode two bits of classical information on each qubit and transmits her qubits to Bob.
5. When Charlie plans to allow Bob to know the secret information communicated to Bob by Alice, then Charlie discloses the Bell states which he had prepared and the exact sequence.
6. Since the initial Bell states and exact sequence is known, Bob now measures his qubits in Bell basis and obtains the information encoded by Alice.

3.1.4 Discussion and conclusions

The proposed protocol allows the secret sharing of classical information in presence of a noisy quantum channel. In the protocol Alice shares secret classical information with Bob but a degree of control is retained by Charlie. Thus the role of Charlie may be viewed as that of a switch, which controls the channel between Alice and Bob. The protocol may also be viewed as a protocol of controlled secret sharing or a protocol of controlled dense coding. In the sequences P_A and P'_B , as with Alice (which we described earlier), Charlie

can also insert some decoy photons and use them as check bits to ensure that there is no eavesdropping during his communication with Alice and Bob.

In Section II we assumed the channel to be one-way and in Section III we considered the effect of noise. The one-way channel assumption is a reasonable assumption when the channel is noisy. This is because the rate of success of an effort in which they illegally try to ascertain the Bell state sent by Charlie, for example by Alice sending her qubit to Bob first and Bob measuring the Bell state prepared by Charlie and re-sending the qubit to Alice for dense-coding operation, will be very small. This is so because it would require particles of the Bell state to travel thrice through the noisy channel. High noise level opens the danger of an eavesdropper that neither Alice nor Bob would desire. This potentially provides an instance where noisy channel communication is useful.

The restriction on the channel is removed in Section IV and we have seen that our modified protocol works in a very general scenario. Since it is easy to experimentally generate Bell states, the proposed cryptographic switch may be realized experimentally and can be used for many practical problems of a similar kind, say, in which the director of an organization wishes to keep control over the time and amount of information to be disclosed to an employee of the company. The custodian of the files (Alice) must not be worse than semi-honest, as she could otherwise create her own classical/quantum channel and communicate directly with Bob. But there is always a potential chance that Charlie can detect such communication. Further, it is reasonable to assume that as a subordinate in an organization, she lacks the resources to create or purchase the entanglement required. Further, the protocol is valid even when Charlie is semi-honest, the fully honest Charlie being one who does not get any part of the information sent by Alice to Bob.

We have visualized the problem of cryptographic switch as a practical problem in a particular situation. Many analogous examples exist, where similar situations arise. For example, one may think that the owner of a company (Charlie) has asked his semi-honest assistant (Alice) to send details of all his shares to a stock exchange broker, Bob, to sell it in the stock market. But Charlie wishes to keep an eye on stock fluctuations and to permit Bob to sell his shares only at some suitable time. Many similar examples can be given. The idea of a cryptographic switch is expected to be useful in various analogous situations of practical relevance, the idea being that the main information is priorly sent using a secure quantum channel, and ‘switched on’ by a small classical key, when convenient.