Chapter 2

Dissipative and Non-dissipative Single-Qubit Channels: Dynamics and Geometry

2.1 Introduction

Open quantum systems are ubiquitous in the sense that any system can be thought of as being surrounded by its environment (reservoir or bath) which influences its dynamics. They provide a natural route for discussing damping and dephasing. If the system and environment start out in product state, then the evolution of the state ρ can be described by the quantum process $\rho' = \mathcal{E}(\rho)$. It can be given an operator sum representation or Kraus representation [8, 11, 9] as described in Sec. 1.2 :

$$\mathcal{E}(\rho) = \sum_j K_j \rho K_j^\dagger, \tag{2.1}$$

where $\sum_j K_j^\dagger K_j = \mathbb{I}$ and $\mathbb{I}$ is the $d \times d$ identity operator. It may be noted that the converse problem, that of deducing the underlying Lindbladian process that generates a given completely positive (CP) map on the density operator, is computationally hard. Complexity theoretically, it is known to be NP-hard [61].

A result now familiar in quantum information theory is the isomorphism between the trace-preserving, CP maps on a d-dimensional system (qudit) and the $d^4 - d^2$ dimensional space of two-qudit density operators ρ which are maximally mixed on one of the particles [62, 63]. One way to obtain the state from the channel is to apply the latter on one half of a maximal two-qudit entangled state. The resulting state is called the Choi matrix. In the converse direction, a unique qudit channel can be associated with each such Choi matrix via the notion of gate teleportation [64]. It can be shown that the Kraus operators for the qudit channel can be derived by diagonalizing the Choi matrix [7, 65], obtained also by constructing the dynamical map for the transformation [66, 17].

In this work[1], we derive the Kraus operators for the squeezed generalized amplitude damping (SGAD) channel via the Choi matrix method, and establish its unitary equivalence to a previous derivation [13], where the connection to the amplitude damping (AD) and generalized amplitude damping (GAD) channels was manifest. The channel-state isomorphism is used to study and contrast amplitude damping channels and generalized depolarizing channels geometrically. Understanding the geometry of 1-qubit channels is important as it can simplify the study of other problems, such as channel capacity [68]. Finally the contrastive roles of temperature and squeezing in the case of the SGAD channel are noted.

2.2 Some physically motivated single-qubit channels

As mentioned earlier in Sec. 1.2, depending upon the system-reservoir ($S-R$) interaction, open systems can be broadly classified into two categories, viz., quantum non-dissipative or dissipative. A particular type of non-dissipative $S-R$ interaction may be achieved when the Hamiltonian H_S of the system commutes with the Hamiltonian H_{SR} describing the system-reservoir interaction, i.e., H_{SR} is a constant of the motion generated by H_S [28, 29, 30, 20, 69]. This results in pure dephasing without dissipation. Investigation into pure dephasing scenarios was originally motivated by the problem of the detection of gravitational waves [70, 71]. A dissipative open system would be when H_S and H_{SR} do not commute resulting in dephasing along with damping [6]. Impressive progress has been made on the experimental front in the manipulation of quantum states of matter towards quantum information processing and quantum communication. Myatt *et al.* [72] and Turchette *et al.* [73] performed a series of experiments in which they engineered both the pure dephasing as well as dissipative type of evolutions. In these experiments single-qubit channels, both of pure dephasing as well as dissipative type, were realized in real systems, Be ions, taking into account the ambient environment conditions. Since the channels studied below include these experimentally realizable channels, the results of our paper are of direct relevance to such systems. In another experiment [74, 75], a QND scheme of measurement was characterized using only linear optics devices. An experimental investigation of the dynamics of different kinds of bipartite correlations, in an all-optical setup was made in [76]. In [77], an interesting experiment was presented, in which dissipation induces entanglement between two atomic objects. Here we briefly discuss the two processes, QND as well as dissipative as applicable to single qubit channels. In this work we model the reservoir by a squeezed thermal bath. An advantage is that the decay rate of quantum coherence can be suppressed leading to preservation of nonclassical

[1] The results included in this chapter have been published in [67]

effects [20, 22, 23]. A squeezed reservoir may be constructed on the basis of establishment of squeezed light field [78]. Experiments probing the squeezed-light-atom system have been carried out in Refs. [79, 80]. All the results pertaining to the usual thermal bath can be obtained by setting the bath squeezing parameters to zero.

2.2.1 QND Channel

Setting $n, m = 1, 2$ in the Eq. (1.35), we obtain the evolution equation for a qubit, interacting with its environment by a coupling of the energy-preserving QND type where the environment is a bosonic bath of harmonic oscillators initially in a squeezed thermal state. The solution of Eq. (1.35) is [20]

$$\rho_{nm}(t) = \left[e^{-\frac{i}{\hbar}(\epsilon_n-\epsilon_m)t} e^{-i(\epsilon_n^2-\epsilon_m^2)\eta(t)} e^{-\frac{1}{2}(\epsilon_n-\epsilon_m)^2\gamma(t)}\right] \rho^{nm}(0), \tag{2.2}$$

apparently the population of the qubit do not change in the evolution. The channel corresponding to the evolution generated by Eq. (1.35) is called the phase flip channel [20, 81]. More generally, phase flip channels are a subset of Pauli or generalized depolarizing channels, which are unital, i.e, they map the identity matrix to itself.

2.2.2 Dissipative Channel

Consider a qubit interacting with a squeezed thermal bath in the weak Born-Markov, rotating wave approximation. The system Hamiltonian is given by $H_S = (\hbar\omega/2)\sigma_z$. The system interacts with the reservoir via the atomic dipole operator which in the interaction picture is given as $\vec{D}(t) = \vec{d}\sigma_- e^{-i\omega t} + \vec{d}^* \sigma_+ e^{i\omega t}$ where $\vec{d}$ is the transition matrix elements of the dipole operator. The master equation depicting the evolution of the reduced density matrix operator of the system S interacting with a squeezed thermal bath expressed in a manifestly Lindblad form is [6, 13, 81]

$$\frac{d}{dt}\rho(t) = \sum_{j=1}^{2} \left(2R_j\rho R_j^\dagger - R_j^\dagger R_j\rho - \rho R_j^\dagger R_j\right), \tag{2.3}$$

where $R_1 = [\gamma_0(N+1)/2]^{1/2}R$, $R_2 = [\gamma_0 N/2]^{1/2}R^\dagger$ and $R = \sigma_- \cosh(r) + e^{i\Phi}\sigma_+ \sinh(r)$. $\gamma_0 = (4\omega^3|\vec{d}|^2)/(3\hbar c^3)$, is the spontaneous emission rate and σ_+, σ_- are the standard raising and lowering operators. Also $N = N_{\rm th}[\cosh^2(s) + \sinh^2(s)] + \sinh^2(s)$, where $N_{\rm th} = 1/(e^{\hbar\omega/k_BT} - 1)$ is the Planck distribution giving the number of thermal photons at the frequency ω; s and Φ are bath squeezing parameters. Eq. (2.3) guarantees that the evolution of the density operator is CP. If $T = 0$, then R_2 vanishes, and a single Lindblad operator suffices. The general map generated by the Eq. (2.3) is the SGAD channel [13], which generalizes the notion of the AD and GAD channels [11]. These amplitude damping channels are non-unital and contractive, mapping any initial state to a unique asymptotic state.

2.3 Kraus representation of the SGAD channel

The action of the SGAD channel on the qubit ρ changes the corresponding Bloch vector Eq. (1.9) as [13]:

$$\begin{aligned} \langle \sigma_x(t) \rangle &= A\langle \sigma_x(0) \rangle - B\langle \sigma_y(0) \rangle, \\ \langle \sigma_y(t) \rangle &= G\langle \sigma_x(0) \rangle - B\langle \sigma_y(0) \rangle, \\ \langle \sigma_z(t) \rangle &= H\langle \sigma_z(0) \rangle - Y, \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} A &= [1 + \frac{1}{2}(e^{\gamma_0 a t} - 1)(1 + \cos(\Phi))]e^{\frac{-\gamma_0(2N+1+a)t}{2}}, \quad B = \sin(\Phi)\sinh(\frac{\gamma_0 a t}{2})e^{\frac{-\gamma_0(2N+1)t}{2}}, \\ G &= [1 + \frac{1}{2}(e^{\gamma_0 a t} - 1)(1 - \cos(\Phi))]e^{\frac{-\gamma_0(2N+1+a)t}{2}}, \quad H = e^{-\gamma_0(2N+1)t}, \\ Y &= \frac{(1 - e^{-\gamma_0(2N+1)t})}{2N+1}. \end{aligned} \tag{2.5}$$

In Eq. (2.5), $a = \sinh(2s)[2N_{th} + 1]$ and rest of the terms are defined in Sections 2.2.2.

Consider the maximally entangled (unnormalized) state $|\tilde{\psi}\rangle = |00\rangle + |11\rangle$. We construct the Choi matrix of the SGAD channel as per its action on ρ described by Eq. (2.4) :

$$C_{\mathcal{E}} \equiv (I \otimes \mathcal{E})|\tilde{\psi}\rangle\langle\tilde{\psi}| = \begin{pmatrix} \frac{1+H-Y}{2} & 0 & 0 & \frac{A+G}{2} \\ 0 & \frac{1-H+Y}{2} & \frac{A-G}{2} - iB & 0 \\ 0 & \frac{A-G}{2} + iB & \frac{1-H-Y}{2} & 0 \\ \frac{A+G}{2} & 0 & 0 & \frac{1+H+Y}{2} \end{pmatrix}. \tag{2.6}$$

Choi matrix corresponding to a map is equivalent to the B-map introduced in [9]. Spectrally decomposing this, suppose we obtain

$$C_{\mathcal{E}} = \sum_{j=0}^{3} |\xi_j\rangle\langle\xi_j|, \tag{2.7}$$

where $|\xi_j\rangle$ are the eigenvectors normalized to the value of the eigenvalue. Then by Choi's theorem [7, 65], each $|\xi_j\rangle$ yields a Kraus operator obtained by folding the d^2 (i.e., 4) entries of the eigenvector in to $d \times d$ (2×2) matrix, essentially by taking each sequential d-element segment of $|\xi_j\rangle$, writing it as a column, and then juxtaposing these columns to form the matrix.

The eigenvectors corresponding to non-vanishing eigenvalues are found to be

$$\begin{aligned}
|\xi_0\rangle &= (0, i\frac{(\sqrt{1-H-\Psi})(\Psi+Y)}{2B+i(G-A)}, \sqrt{1-H-\Psi}, 0) \\
|\xi_1\rangle &= (0, i\frac{(\sqrt{1-H+\Psi})(\Psi-Y)}{2B+i(G-A)}, \sqrt{1-H+\Psi}, 0) \\
|\xi_3\rangle &= (\frac{-\sqrt{1+H-\eta}(Y+\eta)}{A+G}, 0, 0, \sqrt{1+H-\eta}) \\
|\xi_4\rangle &= (\frac{-\sqrt{1+H+\eta}(Y-\eta)}{A+G}, 0, 0, \sqrt{1+H+\eta}).
\end{aligned} \tag{2.8}$$

From above eigenvectors we obtain Kraus representation for the SGAD channel with elements

$$\begin{aligned}
J_{\pm} &= \frac{1}{M_{\pm}} \begin{pmatrix} 0 & \sqrt{1-H\mp\Psi} \\ i\frac{(\sqrt{1-H\mp\Psi})(\Psi\pm Y)}{2B+i(G-A)} & 0 \end{pmatrix}, \\
K_{\pm} &= \frac{1}{N_{\pm}} \begin{pmatrix} \frac{-\sqrt{1+H\mp\eta}(Y\pm\eta)}{A+G} & 0 \\ 0 & \sqrt{1+H\mp\eta} \end{pmatrix},
\end{aligned} \tag{2.9}$$

where $\Psi = \sqrt{(A-G)^2+4B^2+Y^2}$, $\eta = \sqrt{(A+G)^2+Y^2}$, and $M_{\pm} = \sqrt{2}\sqrt{1+\left|\frac{\mp Y+\Psi}{2B+i(G-A)}\right|^2}$, $N_{\pm} = \sqrt{2}\sqrt{1+\left|\frac{Y\pm\eta}{A+G}\right|^2}$.

2.3.1 Unitary equivalence of Kraus representations of the SGAD channel

A superoperator $\mathcal{E}$ acting on a system, due to interaction with the environment, that starts in a product state with the system, is given by $\rho \longrightarrow \mathcal{E}(\rho) = \sum_k \langle e_k|U(\rho \otimes |0\rangle\langle 0|)U^{\dagger}|e_k\rangle$, where U is the unitary operator representing the free evolution of the system, reservoir, as well as the interaction between the two, $|0\rangle$ is the environment's initial state, and $\{|e_k\rangle\}$ is a basis for the environment. The environment and the system are assumed to start in a product state. The $\langle e_k|U|0\rangle \equiv K_j$ are the Kraus operators, described in Eq. (2.1). It can be shown that any transformation that can be cast in Kraus representation is a CP map [11].

There are infinitely many Kraus operator representations even within the same representation basis of the system, depending on the choice of tracing basis $\{|e_k\rangle\}$ of the environment. Each of these sets of Kraus operators is unitarily related to the other: let $K_k = \langle e_k|U|0\rangle$ and $K'_k = \langle e'_k|U|0\rangle$. Define unitary operation V such that $\langle e'_k| = \langle e_k|V^{\dagger}$, and hence $K'_k = \langle e_k|V^\dagger U|0\rangle$. Now $V|e_k\rangle = \sum_j \alpha_{j,k}|e_j\rangle$ and thus

$$K'_k = \langle e_k|V^\dagger U|0\rangle = \sum_j \alpha^*_{j,k}\langle e_j|U|0\rangle = \sum_j \alpha^*_{j,k}K_j. \tag{2.10}$$

The above can be represented as a matrix-valued vector equation

$$\vec{K'_k} = V^\dagger \vec{K}_k. \tag{2.11}$$

The Kraus operators for the noise process, generated by the SGAD channel, derived in Ref. [13], using an ansatz based on a standard operator-sum representation [11] are

$$\begin{aligned} J'_+ &\equiv \sqrt{p}\begin{bmatrix} \sqrt{1-\alpha'} & 0 \\ 0 & 1 \end{bmatrix}, & J'_- &\equiv \sqrt{p}\begin{bmatrix} 0 & 0 \\ \sqrt{\alpha'} & 0 \end{bmatrix}, \\ K'_+ &\equiv \sqrt{1-p}\begin{bmatrix} \sqrt{1-\mu'} & 0 \\ 0 & \sqrt{1-\nu'} \end{bmatrix}, & K'_- &\equiv \sqrt{1-p}\begin{bmatrix} 0 & \sqrt{\nu'} \\ \sqrt{\mu}e^{-i\Phi} & 0 \end{bmatrix}, \end{aligned} \tag{2.12}$$

where $0 \le p \le 1$ [11, 13]. In Eq.(2.12),

$$\begin{aligned} p &= 1 - \frac{1}{(A'+B'-C'-1)^2 - 4D'} \times \\ & \big[A'^2B' + C'^2 + A'(B'^2 - C' - B'(1+C') - D') - (1+B')D' - C'(B'+D'-1) \\ &\pm 2\sqrt{D'(B'-A'B'+(A'-1)C'+D')(A'-A'B'+(B'-1)C'+D')}\big], \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} A' &= \frac{2N+1}{2N}\frac{\sinh^2(\gamma_0 at/2)}{\sinh(\gamma_0(2N+1)t/2)}\exp\left(-\gamma_0(2N+1)t/2\right), \\ B' &= \frac{N}{2N+1}(1-\exp(-\gamma_0(2N+1)t)), \\ C' &= A' + B' + \exp(-\gamma_0(2N+1)t), \\ D' &= \cosh^2(\gamma_0 at/2)\exp(-\gamma_0(2N+1)t). \end{aligned} \tag{2.14}$$

Also, in Eq.(2.12),

$$\begin{aligned} \nu' &= \frac{N}{(1-p)(2N+1)}(1-e^{-\gamma_0(2N+1)t}), \\ \mu' &= \frac{2N+1}{2(1-p)N}\frac{\sinh^2(\gamma_0 at/2)}{\sinh(\gamma_0(2N+1)t/2)}\exp\left(-\frac{\gamma_0}{2}(2N+1)t\right), \\ \alpha' &= \frac{1}{p}\left(1 - p_2[\mu(t)+\nu(t)] - e^{-\gamma_0(2N+1)t}\right). \end{aligned} \tag{2.15}$$

As illustrated by the Eq. (2.11), a necessary and sufficient condition for the equivalence of two different Kraus operator representations is that they are related by a unitary transformation. We demonstrate this for the SGAD channel by finding the unitary transformation connecting the Kraus operators derived via Choi formalism, Eqs. (2.9), and those derived in Ref. [13], described in Eq. (2.12). Writing

$$\begin{pmatrix} J'_+ \\ J'_- \\ K'_+ \\ K'_- \end{pmatrix} = U \begin{pmatrix} J_+ \\ J_- \\ K_+ \\ K_- \end{pmatrix}, \tag{2.16}$$

we find that

$$U = \begin{pmatrix} 0 & \Upsilon_+ & 0 & \Upsilon''_+ \\ 0 & \Upsilon_- & 0 & \Upsilon''_- \\ \Upsilon'_+ & 0 & \tilde{\Upsilon}_+ & 0 \\ \Upsilon'_- & 0 & \tilde{\Upsilon}_- & 0 \end{pmatrix}, \tag{2.17}$$

where

$$\begin{aligned} \Upsilon_\pm &= \frac{\sqrt{1-H\mp\Psi}}{M_\pm\sqrt{p\,\alpha'}}\left(\frac{\pm i(\Psi\mp Y)}{2B+i(G-A)} - \sqrt{\frac{\mu'}{\nu'}}e^{-i\Phi}\right), \\ \Upsilon'_\pm &= \frac{\sqrt{1+H\mp\eta}}{N_\pm\sqrt{p}(\sqrt{1-\mu'}-\sqrt{1-\alpha'}\sqrt{1-\nu'})}\left(\sqrt{1-\mu'} - \frac{\sqrt{1-\nu'}(Y\pm\eta)}{A+G}\right), \\ \Upsilon''_\pm &= \frac{\sqrt{1-H\mp\Psi}}{M_\pm\sqrt{(1-p)\nu'}}, \\ \tilde{\Upsilon}_\pm &= -\frac{\sqrt{1+H\mp\eta}}{N_\pm\sqrt{(1-p)}\left(\sqrt{1-\mu'}-\sqrt{1-\alpha'}\sqrt{1-\nu'}\right)}\left(\sqrt{1-\alpha'} - \frac{(Y\pm\eta)}{A+G}\right). \end{aligned} \tag{2.18}$$

Also, the terms p, μ', ν' and α', are as described in Appenndix A. It may be checked that $UU^\dagger = U^\dagger U = \mathcal{I}$. Let $\vec{\mathcal{A}} = (\hat{A}_i)$ and $\vec{\mathcal{B}} = (\hat{B}_i)$ where $\hat{A}_i$ and $\hat{B}_i$ are $d\times d$ matrices (here Kraus operators) and $i = 1, 2, \dots, d^2$. Consider the transformation of the matrix-valued inner product between $\vec{\mathcal{A}}$ and $\vec{\mathcal{B}}$.

$$\begin{aligned} \vec{\mathcal{A}}'\cdot\vec{\mathcal{B}}' &\equiv \sum_i (\hat{A}'_i)^\dagger \hat{B}'_i \\ &= \sum_{ijk} (U_{ij}\hat{A}_j)^\dagger U_{ik}\hat{B}_k \\ &= \sum_{ijk} \hat{A}^\dagger_j U^*_{ji} U_{ik}\hat{B}_k \\ &= \sum_{jk} \hat{A}^\dagger_j \delta_{jk}\hat{B}_k \\ &= \sum_j \hat{A}^\dagger_j \hat{B}_j \\ &\equiv \vec{\mathcal{A}}\cdot\vec{\mathcal{B}}. \end{aligned} \tag{2.19}$$

As a corollary, the Hilbert-Schmidt product of any two Kraus 'vectors', $\sum_j \mathrm{Tr}(\hat{A}_j^\dagger \hat{B}_j)$ is preserved.

Consider the vector obtained by reading off a fixed element, say element of index lm, namely, $(K_j)_{lm}$, for each Kraus operator. Eq. (2.11) can be thought of as applying a unitary transformation to d^2 (not necessarily independent) such vectors. Thus one can define a host of other norms that are preserved under this transformation. Any channel parameter (such as gate fidelity, etc.) must be a function of such a generalized norm in order to be unitarily invariant under the transformation Eq. (2.11) and thus be a valid measure to characterize channel performance. For example, the quantity $\sum_j |\mathrm{Tr}(K_j)|^2$ is another acceptable norm.

The Kraus operators (2.9) obtained by the Choi method satisfy the orthogonality condition $\mathrm{Tr}(K_j^\dagger K_k) = 0$ for $j \neq k$, which is not a unitarily invariant condition. In particular, the Kraus operators for the SGAD channel obtained in Ref. [13] lack this form.

2.3.2 Dissipativeness of the SGAD channel

The SGAD channel derived here is typical of dissipative channels [20], which are characterized by the non-commutativity of the interaction Hamiltonian H_{SR} and the system Hamiltonian H_S. By contrast, the QND case, where these two do commute, is marked by pure phase damping and no dissipation, i.e., populations remain unchanged.

The QND condition $[H_S, H_{SR}] = 0$ implies the commutativity of the Kraus operators and quantum states used for communication (the signal states), assumed to be eigenstates of H_S. Let $|e\rangle$ be the initial state of the environment (which may be generalized to a separable mixed state) and $\{|e_j\rangle\}$ an environmental basis.

For arbitrary non-dissipative interaction, we take: $H_{SR} = \sum_k \alpha_k |k\rangle\langle k| \otimes \hat{P}$, where $\{|k\rangle\}$ is a basis for the first particle and $\hat{P}$ an environmental observable. Given $H = H_S + H_R + H_{SR}$, with $H_S = \sum_k \lambda_k |k\rangle\langle k|$ and $H_R = f(\hat{P})$, we have:

$$\begin{aligned} K_j &= \langle e_j|e^{iHt}|e\rangle \\ &= \langle e_j|e^{i\sum_k(\lambda_k|k\rangle\langle k| + \alpha_k|k\rangle\langle k|\otimes\hat{P}t)}|e'\rangle \\ &= \langle e_j|\sum_k |k\rangle\langle k| \otimes e^{i(\lambda_k+\alpha_k\hat{P}t)}|e'\rangle \\ &= \sum_k |k\rangle\langle k|\beta_k^{(j)}, \end{aligned} \tag{2.20}$$

where $\beta_k^{(j)} \equiv \langle e_j|e^{i(\lambda_k+\alpha_k\hat{P}t)}|e'\rangle$ and $|e'\rangle = e^{if(\hat{P})}|e\rangle$. If the eigenstates of the system Hamiltonian, denoted $\{|k\rangle\}$, are taken to be the signal states, then the statement that $[|k\rangle\langle k|, K_j] = 0$ is equivalent to $[H_S, H_{SR}] = 0$. If the signal states do not commute with Kraus operators, then $[H_S, H_{SR}] \neq 0$, and the system-environmental interaction must be dissipative. This is a unitarily invariant feature since the condition $[H_S, H_{SR}] = 0$ is independent of the tracing basis used to determine the Kraus operators.

A matrix A is *normal* if it has the property $AA^\dagger = A^\dagger A$. Normalcy is a necessary and sufficient for the diagonalization of any matrix A. Two operators commute if and only if they are normal and share all eigenvectors. It is easily checked that in general the Kraus operators Eq. (2.9) lack normalcy. Thus these Kraus operators do not commute with any basis, implying that their dissipative behavior is generic. This is in contrast to QND interaction, where there is a preferred basis (the so-called pointer basis) for which there is no dissipation, and populations are robust.

2.4 Geometric structure of channels

Given any set of points $x_i \in S$, if the convex combination $x = \sum_i \mu_i x_i \in S$, where $\mu_i \geq 0$ and $\sum_i \mu_i = 1$, then the set S is *convex*. A point x is said to be pure or extreme if it cannot be expressed as a (non-trivial) convex combination two or more points. The smallest convex set H that contains a given set S is the *convex hull* of S. The convex hull of a given finite number of pure points is a *convex polytope*. Geometrically, a polytope can be visualized as an object or tile with flat sides. In the space of dimension n, the convex hull of $n+1$ points that are not confined to a $n-1$ dimensional subspace is an n-simplex, Ξ_n. The *dimension* of a given convex set S is the largest integer n, such that $\Xi_n \in S$.

2.4.1 Channel rank

Given a map $\mathcal{E}$ that maps the algebra of $m \times m$ complex matrices to another matrix algebra, we may define the rank of the channel as that of the matrix associated with $\mathcal{E}$ [82]. Here, by virtue of the Choi isomorphism, one may associate a rank with the channel, identified with that of the corresponding Choi matrix. For the SGAD channel, the eigenvalues of the Choi matrix are given by

$$\begin{aligned} e_\pm &= \frac{1}{2}\left(1 - H \pm \sqrt{(A-G)^2 + 4B^2 + Y^2}\right), \\ f_\pm &= \frac{1}{2}\left(1 + H \pm \sqrt{(A+G)^2 + Y^2}\right), \end{aligned} \tag{2.21}$$

where A, B, G, H, Y are as described in Eq. (2.5). Clearly, $e_+ \geq e_-$ and $f_+ \geq f_-$. The trivial case corresponds to the unitary channel, wherein channel parameters $T = s = 0$, and $f_+ = 1$ with all other eigenvalues equal to zero. Let us consider a nontrivial noise where $e_- = 0$ for a given channel. From Eq. (2.21), it follows that $1 - H = \sqrt{(A-G)^2 + 4B^2 + Y^2}$. This, as can be seen from Eq. (2.4), is equivalent to $s = T = 0$, which in turn implies that $1 + H = \sqrt{(A+G)^2 + Y^2}$ and therefore that $f_- = 0$.

Conversely, it can be shown that $f_- = 0 \Longrightarrow e_- = 0$. One way to understand this is to note that Eq. (2.3) that generates the SGAD channel $\mathcal{E}$ has only one Lindblad operator when $T = 0$, and two when $T > 0$.

We thus find that e_- and f_- simultaneously vanish (in the case of unitary and amplitude damping channels with vanishing T and s) or both are non-vanishing (for more general channels). For the SGAD channel for a qubit, we thus find that the rank is either 2 or 4. This of course is not a general quantum feature, and noise channels for qubits exist with odd rank greater than 1. An example of a rank 3 channel is the Pauli channel with Kraus operator elements I, σ_x and σ_y with weights p, q and r, where $p + q + r = 1$. For this the Choi matrix is given by:

$$\frac{1}{2}\begin{pmatrix} p & 0 & 0 & p \\ 0 & q+r & q-r & 0 \\ 0 & q-r & q+r & 0 \\ p & 0 & 0 & p \end{pmatrix}, \tag{2.22}$$

which is manifestly of rank 3 (in that precisely 3 rows are linearly independent).

2.4.2 Pauli and depolarizing channels

Under the above isomorphism, the set of unitaries on a qudit maps to pure states in V, the set of two-qudit states isomorphic to CP maps on a single qudit. The general state of a two-qubit density operator is given by:

$$\rho = \frac{1}{4}\left(I \otimes I + \sum_j r_j \sigma_j \otimes I_2 + s_j I_2 \otimes \sigma_j + \sum_{j,k} t_{j,k} \sigma_j \otimes \sigma_k\right), \tag{2.23}$$

where r_j, s_j and the tensor $t_{j,k}$ are generally complex numbers subject to requirement $\rho = \rho^\dagger$ and $Tr(\rho) = 1$. Letting $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle$, we have

$$|\psi\rangle\langle\psi| = \frac{1}{4}\left(I \otimes I + \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z\right) = (\mathcal{E}_\mathcal{I} \otimes I)|\psi\rangle\psi| \equiv \mathcal{I}, \tag{2.24}$$

where $\mathcal{E}_\mathcal{I}$ is the trivial noise, corresponding to the identity operator. Under the Choi isomorphism the corresponding state is therefore $\mathcal{I}$.

In this Section, for two-qubit states which have only the $I \otimes I$ and the $t_{j,j}$ components non-vanishing, we will represent ρ by its *signature*, the list of these components multiplied by 4. Thus, $\mathcal{I} \equiv \overline{(1, 1, -1, 1)}$. More generally, the class of states we consider in this subsection have the signature $\overline{(1, a, b, c)}$, and are characterized by the (quadratic) mixedness $\frac{d}{d-1}(1 - \text{Tr}\rho^2) = 1 - \frac{|a|^2+|b|^2+|c|^2}{3}$.

Consider the phase flip quantum channel represented by the set of Kraus operators

$[\sqrt{\alpha}I, \sqrt{(1-\alpha)}Z]$, where Z stands for the Pauli operator σ_z and α is a real positive number such that $0 \leq \alpha \leq 1$. The state isomorphic to the channel corresponding to application of Z is $\mathcal{Z} \equiv (\mathcal{E}_{\mathcal{Z}} \otimes I)|\psi\rangle\langle\psi| = \overline{(1,-1,1,1)}$. Thus the phase flip channel is given by the 1-simplex, $\hat{\mathcal{F}}$:

$$\alpha\mathcal{I} + (1-\alpha)\mathcal{Z} = \overline{(1,0,0,1)} + (2\alpha-1)\overline{(0,1,-1,0)}. \quad (2.25)$$

It is closely related to the phase damping channel, given by the set of Kraus-operators: $\left[\sqrt{\beta}I, \sqrt{(1-\beta)}P_0, \sqrt{(1-\beta)}P_1\right]$, where $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$ are projectors and β is a real positive number such that $0 \leq \beta \leq 1$. By the Choi isomorphism, they correspond to states: $\mathcal{I}$ and $\mathcal{Z}_{\mathcal{P}} \equiv (\mathcal{E}_{\mathcal{P}} \otimes I)|\psi\rangle\langle\psi| = \overline{(1,0,0,1)}$. The phase damping channel is given by the 1-simplex

$$\beta\mathcal{I} + (1-\beta)\mathcal{Z}_{\mathcal{P}} = \overline{(1,0,0,1)} + \beta\overline{(0,1,-1,0)}. \quad (2.26)$$

Comparing this with Eq. (2.25), it is seen that the phase damping channel is strictly a subset of the phase flip channel. In particular, the phase damping channel extreme point obtained with $\beta = 0$ corresponds to the mixed point of the phase flip channel given by $\alpha = \frac{1}{2}$. The former corresponds to the latter in the range $\alpha \in [\frac{1}{2}, 0]$, where they are related according to $\beta = 2\alpha - 1$.

The generalized depolarizing or Pauli channels have Pauli operators (apart from a factor) as their Kraus operators, i.e., $\left[\sqrt{\alpha}I, \sqrt{\beta}\sigma_x, \sqrt{\gamma}\sigma_y, \sqrt{\delta}\sigma_z\right]$, where $\alpha, \beta, \gamma, \delta \geq 0$ are real numbers satisfying $\alpha+\beta+\gamma+\delta = 1$. We define $\mathcal{X} \equiv (\mathcal{E}_{\mathcal{X}} \otimes I)|\psi\rangle\langle\psi| = \overline{(1,1,1,-1)}$ and $\mathcal{Y} \equiv (\mathcal{E}_{\mathcal{Y}} \otimes I)|\psi\rangle\langle\psi| = \overline{(1,-1,-1,-1)}$. Thus every Pauli channel is a member of the polytope given by four pure points $\mathcal{I}, \mathcal{X}, \mathcal{Y}, \mathcal{Z}$, as

$$v = \alpha\mathcal{I} + \beta\mathcal{X} + \gamma\mathcal{Y} + \delta\mathcal{Z} = \overline{(1, \alpha+\beta-\gamma-\delta, -\alpha+\beta-\gamma+\delta, \alpha-\beta-\gamma+\delta)}. \quad (2.27)$$

It follows from the properties of vector spaces that if $\mathcal{I}, \mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ are mutually orthogonal, then the decomposition (2.27) is indeed unique.

It is readily seen that six inner products between these elements, given by the Hilbert-Schmidt product $\mathrm{Tr}(\mathcal{I}\mathcal{X})$, $\mathrm{Tr}(\mathcal{Y}\mathcal{X})$, etc., indeed vanish. Thus the set of all Pauli channels, the polytope $\hat{\mathcal{P}}$, is a 3-simplex (a tetrahedron) embedded within V. The phase flip channel $\hat{\mathcal{F}}$ corresponds to a proper subset of $\hat{\mathcal{P}}$, in particular, the edge $(\mathcal{I}, \mathcal{Z})$ of the latter, and the volume of phase damping channels in this set is $\frac{1}{2}$. This structure has been studied using affine maps on Bloch sphere in Ref. [83], where it was shown that the fraction of the channels that can be simulated with a one-qubit environment is $\frac{3}{8}$.

The elements of the important, depolarizing channel are characterized by the action:

$$\rho \mapsto p\rho + (1-p)\frac{I}{2}. \quad (2.28)$$

Noting that since for any ρ, $\frac{I}{2} = \frac{1}{4}(\rho + \sigma_x\rho\sigma_x + \sigma_y\rho\sigma_y + \sigma_z\rho\sigma_z)$, Eq. (2.28) is seen to have a Kraus representation $\left[\sqrt{\frac{1+3p}{4}}I, \sqrt{\frac{3(1-p)}{4}}\sigma_x, \sqrt{\frac{3(1-p)}{4}}\sigma_y, \sqrt{\frac{3(1-p)}{4}}\sigma_z\right]$. The Choi matrix for this process has the convex structure:

$$\begin{aligned}\mathcal{V} &= p\mathcal{I} + \frac{(1-p)}{4}(\mathcal{I} + \mathcal{X} + \mathcal{Y} + \mathcal{Z}) \; (0 \leq p \leq 1) \\ &= \overline{(1, p, -p, p)}, \end{aligned} \tag{2.29}$$

which is just the two-qubit Werner state $pI \otimes I + (1-p)|\psi\rangle\langle\psi|$. It follows from Eq. (2.29) that $\mathcal{V} = \alpha\mathcal{I} + (1\alpha)\mathcal{D}$ with, $(\frac{1}{4} \leq p \leq 1)$, where $\mathcal{D} \equiv (\mathcal{X} + \mathcal{Y} + \mathcal{Z})/3$ and $\alpha = (3p+1)/4$. The set $\hat{\mathcal{D}}$ of all depolarizing channels forms a 1-simplex embedded within $\hat{\mathcal{P}}$, suspended from $\mathcal{I}$ towards the $\mathcal{XYZ}$ base of the tetrahedron, but terminating above the base at the point $\frac{1}{4}(\mathcal{I} + \mathcal{X} + \mathcal{Y} + \mathcal{Z})$.

The twirling operation [62] on states is defined by:

$$\mathcal{T}(\rho) = \frac{1}{4\pi}\int d\phi d\theta U(\theta,\phi) \otimes U^*(\theta,\phi)\rho U^\dagger(\theta,\phi) \otimes U^{*\dagger}(\theta,\phi)\sin\theta, \tag{2.30}$$

where the invariant Haar measure is employed for averaging over the group of uniformly distributed local unitaries $U(\theta,\phi)$ parameterized by θ and ϕ on space of qubits. While it leaves a singlet state invariant, it maps an arbitrary two-qubit state to a Werner state. Interpreted as an operation on maps, it maps any channel to the depolarizing channel. It can be shown to have the property that the fidelity $F = F(|\psi\rangle, (I \otimes \mathcal{E})|\psi\rangle\langle|\psi|)$ is preserved under the twirling operation. Under the Choi isomorphism, this is a contractive CP map collapsing arbitrary points in the above Pauli 3-simplex into points in the depolarizing simplex.

For the depolarizing channels, representing Werner family of states $\phi_D(p) \equiv \overline{(1, p-p, p)}$, one finds $F = \frac{3p+1}{4}$, while for the Pauli channel, represented by the state $\phi_P(\alpha,\beta,\gamma) \equiv \overline{(1, \alpha+\beta-\gamma-\delta, -\alpha+\beta-\gamma+\delta, \alpha-\beta-\gamma+\delta)}$, one finds $F = \alpha$, independent of β, γ, δ. From the property of preservation of F under twirling, it follows ϕ_P is twirled to $\phi_D = (1, \frac{4\alpha-1}{3}, -\frac{4\alpha-1}{3}, \frac{4\alpha-1}{3})$.

2.4.3 The SGAD channel

The geometry of the SGAD channels turns out to be quite different from that of the Pauli channels, illustrating the more complicated nature of these channels. To see this, we require the following result, which specifies a connection between channel geometry and the Choi matrix. In the following we use the notation that $\sigma_0 \equiv I_2$, $\sigma_1 \equiv \sigma_x$, $\sigma_2 \equiv \sigma_y$ and $\sigma_3 \equiv \sigma_z$.

Theorem 1 *Given a set ζ of channels represented by the Choi matrix $\mathcal{C}(x) = \sum_{j,k} \tau^3_{j,k=0}(x)\sigma_j \otimes \sigma_k$, where x represents collectively the physical parameters of the channel, some of the* *coefficients $\tau_{j,k}(x)$ may in general bear a functional relation with other coefficients. A necessary condition for the convexity of ζ is that all these relations must be linear.*

Proof. Consider elements $\mathcal{C}(x)$ and $\mathcal{C}(x')$ of the set. Given an arbitrary convex combination of two channels, parametrized by q, we seek x'' such that

$$\mathcal{C}(x'') = q\mathcal{C}(x) + (1-q)\mathcal{C}(x'), \tag{2.31}$$

where $0 \leq q \leq 1$. This implies that

$$\tau_{j,k}(x'') = q\tau_{j,k}(x) + (1-q)\tau_{j,k}(x'). \tag{2.32}$$

Suppose two distinct coefficients $\tau_{j,k}(x)$ and $\tau_{j',k'}(x)$ of a given $\mathcal{C}(x)$ are described by the functional relation

$$\tau_{j,k}(x) = f(\tau_{j',k'}(x)). \tag{2.33}$$

For such x'' to exist, the same functional form must be preserved under convex combinations, i.e., $\tau_{j,k}(x'') = f\left(\tau_{j',k'}(x'')\right)$, which entails that

$$q\tau_{j,k}(x) + (1-q)\tau_{j,k}(x') = f\left(q\tau_{j',k'}(x) + (1-q)\tau_{j',k'}(x')\right). \tag{2.34}$$

Since q is independent of channel parameters, this will be true in general only if f has the property that $f(\lambda_1\tau_{j,k}(x) + \lambda_2\tau_{j,k}(x')) = \lambda_1 f(\tau_{j,k}(x)) + \lambda_2 f(\tau_{j,k}(x'))$, i.e., f is linear on the elements $\tau_{j,k}$. If $\tau_{j,k}(x)$ and $\tau_{j',k'}(x)$ are independent, then so are the LHS and RHS of the Eq.(2.34), and no restriction applies. ■

Consider the phase damping channel. From Eq.(2.26) we find that $\tau_{XX} = -\tau_{YY}$, whereas $\tau_{ZZ} = 1$. Thus it is not forbidden by Theorem 1 to possess a convex structure. Applying Theorem 1 to the GAD channel, we find that it lacks convex structure, giving an understanding of why it is inherently more complicated than Pauli channels. We give below an illustration of this result demonstrating the non-convexity of the GAD channel, to give an intuition. The Choi matrix of the GAD channel given in Ref. [11]

$$\mathcal{C}_{GAD} = \frac{1}{4}\left(I \otimes I + \sqrt{1-\gamma}(\sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y) + (1-\gamma)\sigma_z \otimes \sigma_z\right), \tag{2.35}$$

where

$$\tau_{ZZ} = \tau_{XX}^2 = \tau_{YY}^2. \tag{2.36}$$

The convex combination $p\mathcal{C}_{\text{GAD}}(\alpha) + (1-p)\mathcal{C}_{\text{GAD}}(\beta)$ is

$$\begin{aligned}&\frac{1}{4}\Big(I \otimes I - \sigma_z \otimes I + +(p\sqrt{1-\alpha} + (1-p)\sqrt{1-\beta})(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y)\\ &\quad + (p(1-\alpha) + (1-p)(1-\beta))\sigma_z \otimes \sigma_z\Big),\end{aligned} \tag{2.37}$$

where

$$\tau_{XX} = p\sqrt{1-\alpha} + (1-p)\sqrt{1-\beta}. \tag{2.38}$$

For Eq.(2.37) to be a GAD channel $\mathcal{C}_{\rm GAD}(\gamma)$ for some noise parameter γ, comparing the two τ_{ZZ} components in Eq.(2.35) and Eq.(2.37), we require

$$1-\gamma = p(1-\alpha) + (1-p)(1-\beta). \tag{2.39}$$

But this would mean, from Eq. (2.36) that

$$\tau_{XX} = \sqrt{\tau_{ZZ}} = \sqrt{1-\gamma} = \sqrt{p(1-\alpha)+(1-p)(1-\beta)}, \tag{2.40}$$

which is in general not compatible with Eq.(2.38). More generally, we obtain the following result.

Corollary 1 *The set of SGAD channels is not convex.*

Proof. From Eq. (2.9), the Choi matrix corresponding to SGAD channel is found to be

$$\begin{aligned}
\mathcal{C}_{SGAD} \;=\; & \frac{1}{4}\Big(I\otimes I + [(1-p)(\nu-\mu)-p]\sigma_z\otimes I - [(1-p)\sqrt{1-\mu}]I\otimes\sigma_y \\
& +[p\sqrt{1-\alpha}+(1-p)(\sqrt{(1-\mu)(1-\nu)}+\sqrt{\mu\nu}\cos(\phi))]\sigma_x\otimes\sigma_x \\
& -[p\sqrt{1-\alpha}+(1-p)\sqrt{\mu\nu}\cos(\phi))]\sigma_y\otimes\sigma_y \\
& -\,[(1-p)\sqrt{\mu\nu}\sin(\phi)](\sigma_y\otimes\sigma_x+\sigma_x\otimes\sigma_y)+p(1-\alpha)\sigma_z\otimes\sigma_z\Big)\,.
\end{aligned} \tag{2.41}$$

Although some Choi coefficients bear a linear relation to others (e.g., $\tau_{XY}=\tau_{YX}$), in general, the relations between the coefficients are manifestly not linear. By Theorem 1, the non-convexity of the set of SGAD channels follows. ■

2.5 Protective role of environmental squeezing

In contrast to temperature, squeezing in the environment can play a beneficial role in protecting quantum information, even though the channel is dissipative. In this connection, we can visualize the effect of the SGAD channel in terms of performance parameters which, as noted earlier, should be unitarily invariant.

2.5.1 Mixedness and concurrence

Since all unitary operations are mapped to pure (maximally entangled) Choi matrices, mixedness of the latter implies decoherence in the channel. This suggests that the degree of decoherence can be quantified by the amount of mixedness of the Choi matrix, $C_{\mathcal{E}}$, Eq. (2.6), Ref. [84].

$$S = -\text{Tr}\left[C_{\mathcal{E}} \log_2 C_{\mathcal{E}}\right]. \tag{2.42}$$

Likewise, since the noise acting on one of the states will lead to a reduction in correlation between the two states, the degree of separableness of the Choi matrix also can be considered as a quantification of channel decoherence. An appropriate measure of entanglement is concurrence [85]:

$$\mathcal{L} = \max\left[0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\right] \tag{2.43}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues of $Q = \sqrt{\sqrt{C_{\mathcal{E}}}\tilde{C_{\mathcal{E}}}\sqrt{C_{\mathcal{E}}}}$, arranged in decreasing order, where $\tilde{C_{\mathcal{E}}} = (\sigma_y \otimes \sigma_y) C^*_{\mathcal{E}} (\sigma_y \otimes \sigma_y)$. '$*$' denotes the standard conjugation operation on matrices.

As a quick illustration, for a phase flip channel given by Kraus operator elements $\{\sqrt{p}I, \sqrt{1-p}Z\}$ $(0 \leq p \leq \frac{1}{2})$, it is easily seen that the von Neumann entropy of the Choi matrix is given by $H(p)$, the Shannon entropy, and the concurrence by $1 - 2p$. Thus we find that as the noise level is increased, so does the degree of mixedness and the separability of the Choi matrix. In Figures (2.1(a)) and (2.1(b)), we plot the von Neumann entropy and concurrence, respectively, of the Choi matrix subjected to a SGAD channel. As expected we find that they indicate an increase of decoherence under increase of T. However, there are regions where squeezing appears to suppress decoherence, as seen from the Figure 2.1(a), near $T = 1$.

2.5.2 Average gate fidelity and channel fidelity

Another parameter to characterize channel performance is the gate fidelity [86]

$$g = \max_{\rho} F(\rho, \mathcal{E}(\rho)), \tag{2.44}$$

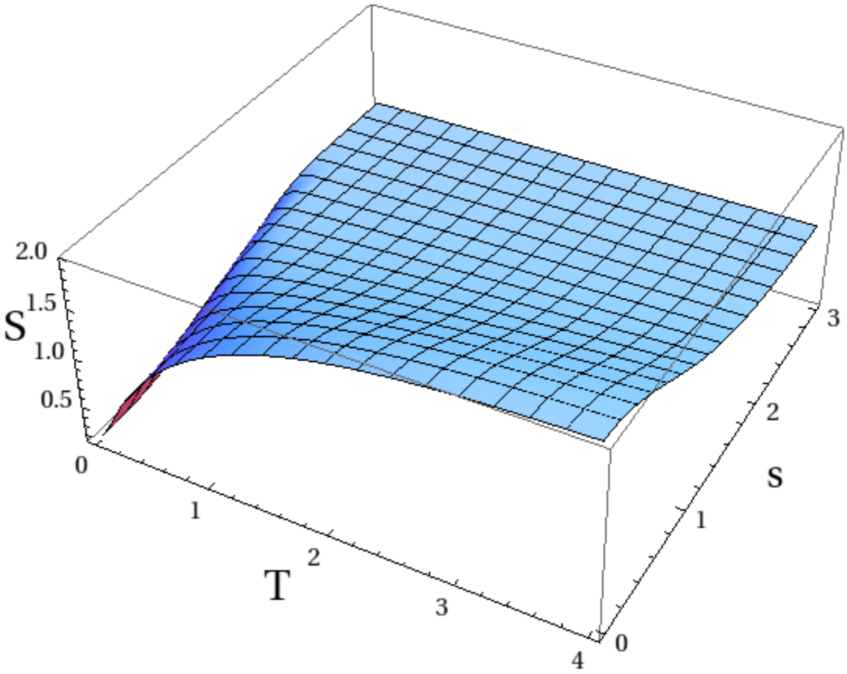
the fidelity of a state with its noisy version, maximized over all states, where fidelity $F(\rho, \sigma) = \sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}$. Intuitively it represents how well a (noisy) gate performs the operation it is supposed to implement. Another is the average gate fidelity [87, 88]:

$$g_{av} = \int_{\rho} F(\rho, \mathcal{E}(\rho))\mathcal{D}(\rho)\lceil\rho, \tag{2.45}$$

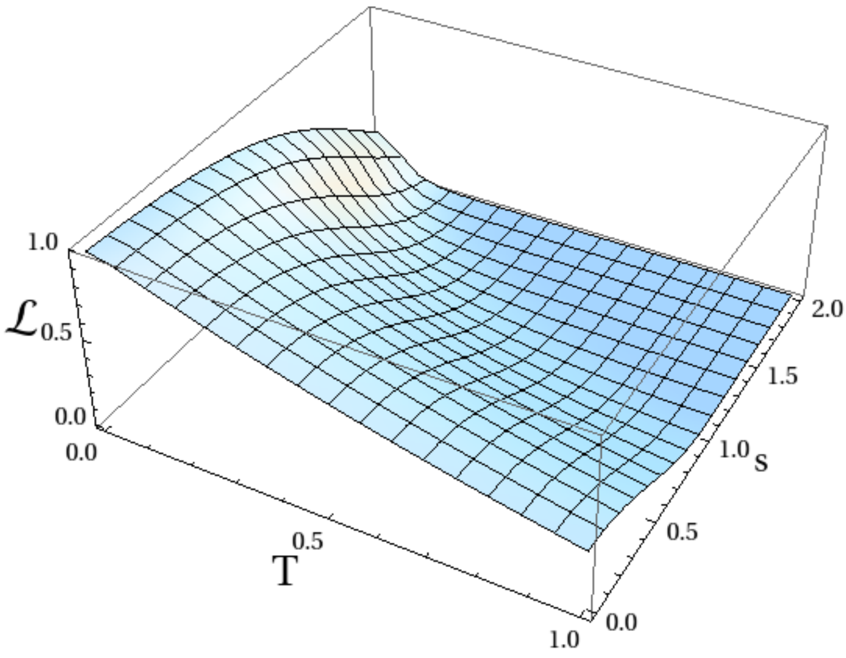
where $\mathcal{D}$ is the uniform measure over state space. A closed analytic expression exists for this [62, 88], given by

$$g_{\text{av}} = \frac{d + \sum_i |\text{Tr}(K_i)|^2}{d(d+1)}. \tag{2.46}$$

Another similar parameter is teleportation distance [64]. A plot of g_{av} for the SGAD channel is given in Figure (2.2(a)). While at low enough temperature, squeezing reduces



(a) Channel entropy (Eq. (2.42))



(b) Channel concurrence (Eq. (2.43))

Figure 2.1: Two possible quantifications of the decoherence due to the action of a SGAD channel, as function of temperature (T) and squeezing, parametrized by r, with time $t = 0.5$, frequency $\omega = 0.1$, bath parameter $\gamma_0 = 0.1$ and squeezing angle $\Phi = 0.3$ (in the units where $\hbar = k_B = 1$).

average gate fidelity $g_{\rm av}$, a range of temperature is seen to exist, in which squeezing causes an increase in the quantity.

To explain this we consider the output density matrix of the SGAD channel which can be written as

$$\rho^s(t) = \left(\begin{array}{cc} \frac{1}{2}(1+\tilde{A}) & \tilde{B}e^{-i\omega t} \\ \tilde{B}^* e^{i\omega t} & \frac{1}{2}(1-\tilde{A}) \end{array} \right), \qquad (2.47)$$

where

$$\begin{aligned} \tilde{A} &= e^{-\gamma_0(2N+1)t}\langle\sigma_z(0)\rangle - \frac{1}{(2N+1)}\left(1-e^{-\gamma_0(2N+1)t}\right), \\ \tilde{B} &= \left[1+\frac{1}{2}\left(e^{\gamma_0 at}-1\right)\right] e^{-\frac{\gamma_0}{2}(2N+1+a)t}\langle\sigma_-(0)\rangle + \sinh(\frac{\gamma_0 at}{2})e^{i\Phi-\frac{\gamma_0}{2}(2N+1)t}\langle\sigma_+(0)\rangle. \end{aligned} \qquad (2.48)$$

Choose $\langle\sigma_+(0)\rangle = \langle\sigma_-(0)\rangle \equiv x$, where x is assumed to be real, and $\Phi = 0$. Then, for sufficiently large time t, we find

$$\begin{aligned} \tilde{B} &\approx \frac{x}{2}e^{(\gamma_0/2)(a-2N)t} + \sinh(\gamma_0 at/2)e^{-\gamma_0 Nt} \\ &\approx xe^{(\gamma_0/2)(a-2N)t}. \end{aligned} \qquad (2.49)$$

If $r = 0$, we find $N = N_{\rm th}$ while $a = 0$ implying a suppression of off-diagonal terms, and hence loss of coherence.

On the other hand, for sufficiently large squeezing s, $N \approx \sinh^2(s)(2N_{\rm th}+1)$, so that $a/N \approx \sinh(2s)/\sinh^2(s) \approx 2$. Using this in Eq. (2.49), we find that $\tilde{B} \approx x$, meaning that full coherence is recovered. This effect is irrelevant to initial states for which $x = 0$, while it is maximal for initial states for which $x = \pm 1$. In the latter case, clearly the distinguishability of two initial states, prepared in the X basis, is helped by increasing squeezing.

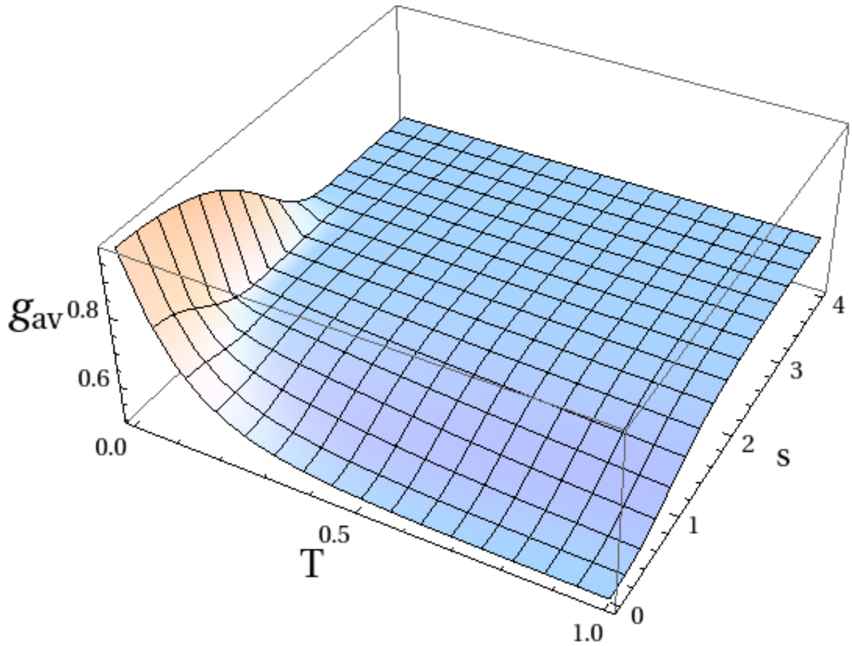
This behavior is captured by another channel performance parameter, introduced in Ref. [13], is the channel fidelity, which is a measure of how well a gate preserves the distinguishability of states:

$$\kappa \equiv \max_{\mathcal{B}} \mathbf{H}(\mathcal{B}, \mathcal{E}), \qquad (2.50)$$

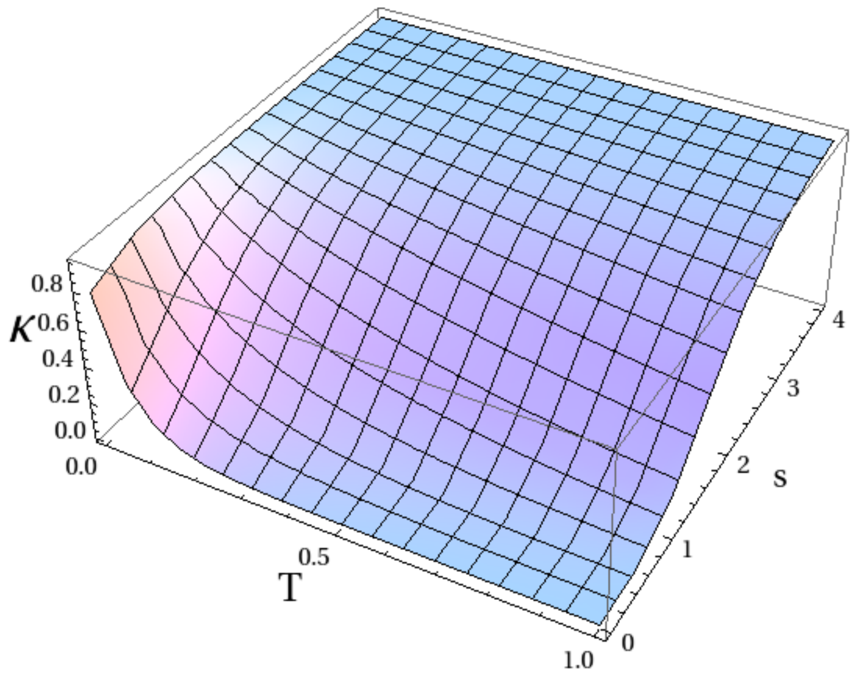
where $\mathbf{H}(\mathcal{B}, \mathcal{E})$ is the Holevo bound for a state prepared in basis elements of basis $\mathcal{B}$, subjected to noise $\mathcal{E}$. Clearly $\kappa \leq \mathcal{C}_1 \leq \mathcal{C}$, where $\mathcal{C}_1$ is the product state channel capacity, and $\mathcal{C}$, the channel capacity maximized over n-fold entanglement ($n \to \infty$) [89].

κ manifestly possesses unitary invariance because it is computed from the density operator directly, and is thus independent of the tracing basis used to obtain the Kraus operators. Currently no closed expression for channel fidelity is known to us. We expect that its behavior should be similar to that of gate fidelity, at least qualitatively.

What is remarkable is that even in regimes where noise increases, as indicated by



(a) Average gate fidelity (Eq. (2.46))



(b) Channel fidelity (Eq. (2.50))

Figure 2.2: SGAD channel properties, as function of temperature (T) and squeezing, parametrized by r, with time $t = 0.5$, frequency $\omega = 0.007$, bath parameter $\gamma_0 = 0.1$ and squeezing angle $\Phi = 0.3$ for (a) and $\Phi = 0.0$ for (b) (in the units where $\hbar = k_B = 1$). Both quantities show a counter-intuitive rise with squeezing.

Figures 2.1(a), 2.1(b) and 2.2(a)), the distinguishability of initially orthogonal states, as given by channel fidelity in Figure 2.2(b)), increases, by increasing squeezing at fixed temperature. We confirmed this behavior by employing the trace distance measure instead of channel fidelity, obtaining the same pattern.

2.6 Discussions and Conclusions

Single-qubit channels have been studied under the two broad classes of dissipative and non-dissipative channels, which are fairly exhaustive in real life situations. Two of the authors had earlier derived [13] an operator sum representation of the SGAD channel, by generalizing the GAD channel. A different derivation, that exploits the Choi channel-state isomorphism was presented here, along with the unitary operation relating it to the previous derivation.

The Choi method produces orthogonal (with respect to the Hilbert-Schmidt product) Kraus operators. This orthogonality is not a unitary invariant, which is shown to manifest as the non-orthogonality of the Kraus operators for the same channel, obtained by the ansatz method of [13]. The equivalence of non-commutativity of the system and interaction Hamiltonian (and hence dissipativeness of the interaction), with the non-commutativity of the symbol states and Kraus operators is shown. The Choi matrix approach to characterize single-qubit channel geometry, adopted here, has, to our knowledge, not been used before for this purpose. Instead, we only find an affine map approach, where channels are represented by points in a three-dimensional parameter space. By contrast, our method requires the use of a four-dimensional (Hilbert) space.

There is a rich structure to be explored by the isomorphism. As a small part of larger work that may be undertaken here, we characterize the difference in the geometry and rank of these channel classes. The degree of decoherence of the qubit channel is quantified according to the amount of mixedness, as quantified by the von Neumann entropy, or separability, quantified by the absence of concurrence, of the Choi matrix.

Whereas the generalized depolarizing channels possess a convex structure and form a 3-simplex, the family of AD channels lacks such a structure, and are thus more complicated to study. Further, where the rank of generalized depolarizing channels can be any positive integer up to 4, that of amplitude damping ones is either 2 or 4. Various channel performance parameters can be used to bring out the different influences of temperature and squeezing in dissipative channels. In particular, a noise range in terms of r and T was identified where initially orthogonal states prepared in a suitable basis can become more distinguishable in spite of decohering. This happens only when squeezing is increased, rather than temperature. The experimental realization of the single-qubit QND and GAD channels, as mentioned in Section II, puts our work in perspective.