Chapter 7

Ambiguous stabilizer codes and quantum noise characterization

7.1 Introduction

A quantum error correcting code determines a subspace $\mathcal{C}$ such that allowed errors can be corrected by a fixed recovery operation. When the recovered state is only required to error-free up to a logical Pauli operation within code space $\mathcal{C}$, we obtain an ambiguous stabilizer code (ASC). This new type of stabilizer code introduced by us, generalizes the concept of a degenerate code, which is the special case where the only residual logical operation after recovery is the trivial one. An ASC cannot be used for error correction, or even, strictly speaking, for error detection. The motivation for introducing ASCs is the characterization of quantum dynamics. In comparison to QECCD, the present method using ASCs requires a smaller size of quantum states. This can be helpful from an experimental perspective, inspite of the cost of increased number of operations. We call this method for process tomography as 'quantum ASC-based characterization of dynamics' (QASCD).

The chapter is organized as follows. In Sec. 7.2 we define and provide the features of ASCs via stabilizer formalism. We also show how the error correcting conditions Eq. (1.54) are modified to suit the ASCs. Further in Sec. 7.2.3 we provide procedures to construct ASCs and in Sec. 7.3 study their group theoretic properties. Next in Sec. 7.4 we describe the QASCD, and in Sec. 7.5 introduce various 4-qubit ASCs and work out an example using them.

7.2 Ambiguous stabilizer codes

7.2.1 Definition and basic features

A 2^k-dimensional subspace $\mathcal{C}'$ of n qubits, together with an allowed set $\mathbb{E}$ of Pauli error basis elements, is *ambiguous* when one or more errors cannot be distinguished via syndrome measurements on the logical state. The indistinguishable errors need not require the same recovery operations. Thus ambiguity generalizes the concept of degeneracy. Ambiguity can be represented by partitioning $\mathbb{E}$ into *ambiguous sets* $A^{(p)} \equiv \{E_1^{(p)}, E_2^{(p)}, \cdots, E_{\gamma(p)}^{(p)}\}$ of mutually indistinguishable Pauli errors. The collection of all ambiguous sets is the ambiguous class $\mathcal{A} = \{A^{(1)}, A^{(2)}, \cdots, A^{(\sigma)}\}$. The order of ambiguity of the code is σ, while the degree of ambiguity is $\gamma \equiv \max_p |A^{(p)}|$. Any set of up to s known errors drawn from distinct ambiguous sets $A^{(p)}$ can be detected, and if the errors are known, they can be corrected.

In an ambiguous set $A^{(p)}$, every element $E_m^{(p)}$ generates the same erroneous subspace, with projector

$$\Pi^{(p)} \equiv \sum_{j=0}^{2^k-1} E_m^{(p)}|j_L\rangle\langle j_L|E_m^{(p)}, \tag{7.1}$$

but individual code words are not necessarily mapped to the same erroneous code word. Eq. (7.1) entails that the action of two ambiguous errors $E_n^{(p)}$ and $E_m^{(p)}$ are related by

$$E_n^{(p)}|j_L\rangle = NE_m^{(p)}|j_L\rangle, \tag{7.2}$$

where $N \in \mathcal{N}$. To see this, note that $NE_m^{(p)}|j_L\rangle = \pm E_m^{(p)}N|j_L\rangle$ for any pair of Pauli operators. Now,

$$\sum_{j=0}^{2^k-1} E_n^{(p)}|j_L\rangle\langle j_L|E_n^{(p)} = \sum_{j=0}^{2^k-1} E_m^{(p)}N|j_L\rangle\langle j_L|NE_m^{(p)}. \tag{7.3}$$

For the RHSs of Eqs. (7.1) and (7.3) to be equal, clearly N must be a logical Pauli operation. Now, from Eq. (7.2), we have $N = E_n^{(p)}E_m^{(p)}$. If $E_m^{(p)}|j_L\rangle = N'E_n^{(p)}|j_L\rangle$, then $N' = E_m^{(p)}E_n^{(p)}$. Thus, $N^\dagger = N'$. If, and only if, $[E_m^{(p)}, E_n^{(p)}] = 0$, then $N^\dagger = N$, and so $N = N'$.

However, projectors to distinct unambiguous erroneous subspaces are orthogonal:

$$\Pi^{(p)}\Pi^{(q)} = 0, \tag{7.4}$$

if $p \neq q$.

7.2.2 Ambiguously detectable errors

Ambiguous errors $E_m^{(p)}$ and $E_n^{(p)}$ that are linked in Eq. (7.2) with $N = I_L$, where I_L is the logical Pauli identity operator, require the same recovery operation. Ambiguous errors related by non-trivial logical Pauli operations will require distinct recovery operations. Thus, an ambiguous code cannot be used for quantum error correction. Any error not ambiguous with no-error (i.e., having +1 for all syndromes) can be detected, but errors ambiguous with no-error cannot be detected. Thus ambiguous codes are weaker than quantum error detecting codes.

For ASCs, the error correcting conditions (1.54) of Sec. 1.3.3 in Chapter 1 becomes the *ambiguous error detecting conditions*:

$$p \neq q \;\Rightarrow\; E_m^{(p)} E_n^{(q)} \;\notin\; \mathcal{N} \tag{7.5a}$$

$$p = q \;\Rightarrow\; E_m^{(p)} E_n^{(q)} \;\in\; \mathcal{N} \tag{7.5b}$$

Eq. (7.5a) implies that quantum error correction can be implemented for any collection of *known* errors which belong to distinct ambiguous sets. Eq. (7.5b) implies that any pair of errors belonging to the same ambiguous set will produce the same syndrome, and thus be indistinguishable. In particular, if $E_m^{(p)} E_n^{(p)} \in \mathcal{S}$, then $\langle\psi_L|E_m^{(p)} E_n^{(p)}|\psi_L\rangle = \langle\psi_L|\psi_L\rangle$, meaning that the two errors are mutually degenerate, and the ambiguity is harmless. On the other hand, if $E_m^{(p)} E_n^{(p)} \in \mathcal{N} - \mathcal{S}$, then the erroneous code words they produce are related by non-trivial logical Pauli operations Eq. (7.2). Therefore, if one implements a fixed recovery operation, as in a QECC, in the case of ASC, this will in general produce a mixture of states within the code space $\mathcal{C}'$, which are logical Pauli rotated versions of each other.

One may ask whether in Eq. (7.5), the implications (7.5a) and (7.5b) may be replaced by $E_m^{(p)} E_n^{(q)} \notin \mathcal{N} - \mathcal{S}$ and $E_m^{(p)} E_n^{(q)} \in \mathcal{N} - \mathcal{S}$, respectively. However the first condition would mean that it is possible to have $E_m^{(p)} E_n^{(q)} \in \mathcal{S}$ even if $p \neq q$. If so, then we would have $\langle\psi_L|E_m^{(p)} E_n^{(q)}|\psi\rangle = \langle\psi_L|\psi_L\rangle = 1$, which contradicts Eq. (7.4). Therefore, the implication (7.5a) is the only required definition of detectability and (7.5b) consequently defines ambiguity. Degeneracy of two errors E_1, E_2 i.e., $E_1 E_2 \in \mathcal{S}$, can happen if and only if they belong to the same ambiguous set i.e., degeneracy is a special case of ambiguity where $N = I_L$ in Eq. (7.2).

In $\mathcal{A}$, each ambiguous set $A^{(p)}$ corresponds to the same error syndrome, so that order $\sigma \leq 2^{n-k}$. By definition, the set $A^{(0)}$ will contain the element I and, by virtue of Eq. (7.5b), only elements of the normalizer $\mathcal{N}$. The remaining sets $A^{(1)}, A^{(2)}, \cdots$ will contain Pauli operators not present in $\mathcal{N}$, since they will fail to commute with at least one stabilizer generator. For unambiguous (and non-degenerate) recovery using a linear QECC, the dimension of the code space and the volume $|\mathbb{E}|$ must satisfy the quantum

Hamming bound, $2^k|\mathbb{E}| \leq 2^n$, or

$$\log(\mathbb{E}) \leq n - k. \tag{7.6}$$

A QECC that saturates Eq. (7.6) is called *perfect*. The 5-qubit code of Ref. [37] is such an example. Any ambiguity will thus cause a perfect code to violate the Hamming bound, while for an imperfect QECC, a sufficiently large degree of ambiguity would be required to violate Ineq. (7.6).

7.2.3 Constructing ASC's

The simplest way to produce an ASC is by *error overloading* a stabilizer code. This entails allowing additional errors in violation of condition (1.54), such that instead condition (7.5) holds. For example, consider the (perfect) 5-qubit code of Ref. [37]

$$\begin{aligned} |0_L\rangle_5 &= \frac{1}{2\sqrt{2}}(-|00000\rangle + |0111\rangle - |10011\rangle + |11100\rangle \\ &\quad +|00110\rangle + |01001\rangle + |10101\rangle + |11010\rangle) \\ |1_L\rangle_5 &= \frac{1}{2\sqrt{2}}(-|11111\rangle + |10000\rangle + |01100\rangle - |00011\rangle \\ &\quad +|11001\rangle + |10110\rangle - |01010\rangle - |00101\rangle)\,, \end{aligned} \tag{7.7}$$

which corrects an arbitrary single-qubit error on any qubit. The code space is stabilized by generators $IXXYY, IYYXX, XIYZY, YXYIZ$. They can each take values ± 1, thereby determining 16 syndromes, corresponding to the 16 allowed errors $\mathbb{E} \equiv \{I, X_i, Y_i, Z_i\}$ where $i = 1, \cdots, 5$. By allowing any more errors into the error set $\mathbb{E}$, we introduce ambiguity. In Table 7.1, we present a partial listing of the ambiguous class $\mathcal{A}$ for this code. In all, it has $1 + \binom{5}{1} \cdot 3 + \binom{5}{2} \cdot 3^2 = 106$ arbitrary 1-qubit and 2-qubit errors, of which 49 are displayed. The errors are partitioned into their respective ambiguous sets, labelled by the corresponding error syndrome. Set $A^{(0)}$ has only 1 element, I, since all other elements of $\mathcal{N}$ have a Hamming weight greater than 2.

Another way to create a ASC from a QECC is by *syndrome coarse-graining*: dropping one or more syndrome measurements. For example consider not to measure the last stabilizer of the QECC (6.4). From the first row of the Table 7.1 it can be seen that $|\mathcal{A}| = 8$, $A^{(0)} = \{I, X_1\}$ corresponding to syndrome $(+ + +)$, $A^{(1)} = \{Y_1, Z_1\}$ corresponding to syndrome $(+ + -)$, and so on. The order of ambiguity is halved and the degree of ambiguity is doubled.

A final method to obtain an ASC begins by constructing a stabilizer code that corrects arbitrary errors on m known coordinates. An ASC may then be obtained by allowing noise

$++++$	$+++-$	$++-+$	$++--$	$+-++$	$+-+-$	$+--+$	$+---$
I	X_1 Y_2Y_3 X_3Y_4	Y_1 Z_2Z_3 Y_3X_4	Z_1 X_2X_3 Z_3Z_4	X_2 Z_1X_3 Y_3Z_4	Y_5 X_1X_2 Z_2Y_3	Y_4 Y_1X_2 Y_2Z_3	X_3 Z_1X_2 Z_2X_4
$-+++$	$-++-$	$-+-+$	$-+--$	$--++$	$--+-$	$---+$	$----$
Y_3 X_1Y_2 X_2Z_4	Y_2 X_1Y_3 Z_3Y_4	X_4 Z_1Y_2 Z_2X_3	X_5 Y_1Y_2 X_2Z_3	Z_4 X_1Z_2 X_2Y_3	Z_2 Y_1Z_3 X_3X_4	Z_5 Z_1Z_2 X_1Z_3	Z_3 Y_1Z_2 Y_2Y_4

Table 7.1: Ambiguous class (partial listing) for the ASC obtained by error-overloading the code (6.4), to allow arbitrary errors on any two qubits. Each error syndrome labels an ambiguous set. The first error row in each column corresponds to arbitrary single-qubit errors allowed in the original QECC. Addition of the two-qubit errors (second and third rows of the table) to the list turns the QECC into an ASC. In all, there are 106 elements in the ambiguous class, with $|A^{(0)}| = 1$ and $|A^{(p)}| = 7$ for $p = 1, 2, \cdots, 15$. Thus the degree of ambiguity is 7. An example of a full ambiguous set, corresponding to the syndrome $+++-$ has four more elements $E_3^{(1)} \equiv X_4X_5, E_4^{(1)} \equiv Z_3Z_5, E_5^{(1)} \equiv X_2Y_5, E_6^{(1)} \equiv Z_2Z_4$. The normalizers between $E_0^{(1)} \equiv X_1$ and other elements in the set are $Z_L \equiv X_1Y_2Y_3$, $-Y_L \equiv X_1X_3Y_4$, $Z_L \equiv X_1X_4X_5$, $-X_L \equiv X_1Z_3Z_5$, $-Y_L \equiv X_1X_2Y_5$ and $-X_L \equiv X_1Z_2Z_4$. Any set of sixteen elements, with one drawn from each ambiguous set will satisfy condition Eq. (7.5a), while any pair of errors within a column satisfy Eq. (7.5b) and thus are ambiguous. Further note that the product of ambiguous errors linked by the same logical Pauli are mutually degenerate (e.g., $E_4^{(1)}E_6^{(1)} \in \mathcal{S}$), while those linked by different logical Pauli operators are not (e.g., $E_4^{(1)}E_5^{(1)} \in \mathcal{N} - \mathcal{S}$).

to act on m' known coordinates, where $m' > m$. A detailed description of this method and its application to the characterization of quantum dynamics [140] are considered below.

7.3 Ambiguous group

An arbitrary error on l qubits can be expressed as a linear combination of 4^l Pauli operators. Suppose these l-qubits form a subsystem of a $[[n, k]]$ QECC. Setting $|\mathbb{E}| := 4^l$ in Ineq. (7.6) we find:

$$l \leq \lfloor \frac{n-k}{2} \rfloor \tag{7.8}$$

This means that a 5-qubit code can correct all possible errors on at most 2 fixed coordinates. An example of a perfect code of this kind will be presented later. We thus obtain a $[[n, k]]$ ASC by allowing m noisy coordinates, where $m > l$ in Ineq. (7.8). The order σ of the code is just the number of syndromes, 2^{n-k}, while the degree of ambiguity $\gamma = 4^m/2^{n-k} = 2^{2m-n+k}$.

Suppose we are given a $[[n, k]]$ ASC with errors allowed on m known coordinates. It is worth noting here that the set of errors (including the factors $\pm 1, \pm i$) forms a group, i.e., $\mathbb{E} = \mathcal{P}_m$. Further more, the subset of $\mathcal{P}_m$ that is ambiguous with I_m (the trivial error on the m qubits) constitutes a group, the *ambiguous group*, $\mathfrak{B}$, as shown below.

Theorem 3 *Given a* $[[n, k]]$ *ASC with* $\mathbb{E} = \mathcal{P}_m$*, the subset* $\mathfrak{B}$ *of allowed errors that correspond to the no-error syndrome forms a normal group.*

Proof. Note that if $B_j, B_k \in \mathfrak{B}$, then $I_m B_j = B_j$ and $I_m B_k = B_k$ both commute with all stabilizers, by virtue of Eq. (7.5b). (Note that this doesn't imply that $[B_j, B_k] = 0$.) For any element $G \in \mathcal{G}$, then $[B_j B_k, G] = B_j B_k G - G B_j B_k = 0$, meaning that $B_j B_k \in \mathfrak{B}$. This guarantees closure of the set. By definition, I_m is an element of this set, and a Pauli operator is its own inverse. Thus all required group properties are satisfied. ■

For an ASC obtained in this way, the ambiguous class $\mathcal{A}$ has a simple structure. It corresponds to a partition of $\mathcal{P}_m$, determined by the quotient group

$$\mathcal{Q} \equiv \frac{\mathcal{P}_m}{\mathfrak{B}}. \tag{7.9}$$

This means that any element E in $\mathcal{P}_m$ is either in $\mathfrak{B}$ or can be expressed as the product of an element of $\mathfrak{B}$ and an element not in $\mathfrak{B}$.

Example of a $[[3, 1]]$ ASC

A $[[3, 1]]$ perfect QECC that unambiguously corrects errors on the first qubit is:

$$\begin{aligned} |0_L\rangle_3 &= \frac{1}{2}(|001\rangle + |010\rangle + |100\rangle + |111\rangle) \\ |1_L\rangle_3 &= \frac{1}{2}(|110\rangle - |101\rangle + |011\rangle - |000\rangle), \end{aligned} \tag{7.10}$$

whose stabilizer generators are given by the set $\mathcal{G}_3 \equiv \{XIX, YYZ\}$. The stabilizer is thus the set of four elements, $\mathcal{S}_3 = i^4 \times 2^{\mathcal{G}} \equiv i^4 \times \{I, XIX, YYZ, ZYY\}$, where the pre-factor to the set indicates possible factors $\pm 1, \pm i$. The normalizer $\mathcal{N}_3$ is the set of all elements of $\mathcal{P}_3$ that commute with the elements of $\mathcal{S}_3$. (We note that a Pauli operator P commutes with every element of $\mathcal{S}_3$ iff P commutes with every of $\mathcal{G}_3$.)

For QECC (6.14), the full normalizer $\mathcal{N}_3$ is given in Table 7.2. The subset $\mathcal{S}_3$ corresponds to the identity logical Pauli operation I_L, while the elements of $\mathcal{N}_3 - \mathcal{S}_3$ correspond to non-trivial logical Pauli operations, as tabulated in the columns of Table 7.2.

We create an ASC for the code words (6.14) by allowing errors, in addition to the first coordinate, also on the second coordinate. There are four elements in Table 7.2 that have no non-trivial operator on the last qubit, i.e., they are elements of $\mathcal{P}_2 \otimes \mathbb{I}_3$, where $\mathbb{I}_3$ is the identity operator on the third qubit. They are $\{III, XZI, IYI, XXI\}$, which constitute the ambiguous group $\mathfrak{B}_3$. The partitioning of $\mathcal{A}$ for ASC (6.14) with $\mathbb{E} = \mathcal{P}_2$

I_L	$-X_L$	Y_L	Z_L
III	XZI	IYI	XXI
XIX	IZX	XYX	IXX
YYZ	YXY	YIZ	YZY
ZYY	ZXZ	ZIY	ZZZ

Table 7.2: Normalizer for the [[3, 1]] stabilizer code (6.14) and their equivalence to logical operations. All elements commute with the elements of $\mathcal{S}_3$, while their commutation properties amongst themselves reflect the logical operation they represent. Thus, an element in the column Y_L commutes with all elements in the same column and the column I_L, but will anti-commute with every element in the columns -X_L and Z_L. On the other hand, the elements in the column I_L, which are precisely those of $\mathcal{S}_3$, commute with every other element in the normalizer.

$++$	$+-$	$-+$	$--$	Normalizer
I	X_1	Y_1	Z_1	I_L
Y_2	X_1Y_2	Y_1Y_2	Z_1Y_2	Y_L
X_1X_2	X_2	Z_1X_2	Y_1X_2	Z_L
X_1Z_2	Z_2	Z_1Z_2	Y_1Z_2	$-X_L$

Table 7.3: Ambiguous class $\mathcal{A}_3$ for errors on the first 2 qubits of 3-qubit code code (6.14), depicting the quotient group (7.11). The first column is the ambiguous group $\mathfrak{B}_3$, drawn from Table 7.2. The remaining three columns are its cosets $X_1\mathfrak{B}_3, Y_1\mathfrak{B}_3$ and $Z_1\mathfrak{B}_3$, which represent ambiguous sets. The last column lists the normalizer element with respect to first element in the column, in the sense of Eq. (7.2). For example, the error Z_1Z_2 takes the encoded state to the same error space as Y_1 (denoted $Y_1\mathcal{C}'$), but then $Z_1Z_2|j_L\rangle = -X_LY_1|j_L\rangle$.

can be represented by the quotient group:

$$\mathcal{Q}_3 \equiv \frac{\mathcal{P}_2}{\mathfrak{B}_3} \tag{7.11}$$

This is depicted in Table 7.3.

7.4 Application to noise characterization

We recall that, if ρ represents the quantum state of the system at time $t = 0$, then it evolves under the action of the noise to

$$\mathcal{E}(\rho) \sum_{m,n} \chi_{m,n} E_m \rho E_n^\dagger. \tag{7.12}$$

Here we employ ASCs to determine the process matrix $\chi_{m,n}$. A feature of QECCD is that it makes use of the properties of the QECCs to allow CQD to run concurrently with quantum computation, provided the allowed noise forms a group. Here, we extend

QECCD by replacing the use of QECCs with that of ASCs. The purpose of invoking ambiguity–indeed the principal motivation behind the construction of ambiguous codes– is to be able to use smaller code words, thereby improving experimental feasibility. Of course, this would entail, as detailed below, that more state preparations involving other ASCs are required to unambiguously determine the process matrix. Thus there is a trade-off between spatial resources (length of code words) and temporal resources (number of ASCs). We call this ambiguous extension of QECCD as 'quantum ASC-based characterization of dynamics' (QASCD).

7.4.1 Noise characterization and QEC codes.

In QECCD, the basic idea is that the syndrome obtained from the stabilizer measurement is used to correct the noisy state, while the experimental probabilities of syndromes will characterize the noisy quantum channel. While direct syndrome measurements yield the diagonal terms of the process matrix, for off-diagonal terms preprocessing via suitable unitaries is required. For the purpose of noise characterization, the code qubits are divided into two parts; (a) the qubits on which the elements of $\mathbb{E}$ act non-trivially; (b) the remaining qubits.

The former qubits constitute the principal system **P**, whose unknown dynamics is to be determined. The latter qubits constitute the CQD ancilla **A**, and are assumed to be clean, i.e., noiseless. Suppose the full system $\mathbf{P}+\mathbf{A}$ is in the state

$$|\psi_L\rangle \equiv \sum_{j=0}^{2^k-1} \alpha_j |j_L\rangle, \tag{7.13}$$

where $\{|j_L\rangle\}$ denotes a logical basis for the code space of a $[[p+q,k]]$ ASC (which encodes k qubits into $n \equiv p+q$ qubits) such that allowed errors in the p known coordinates of **P** can be ambiguously detected. Herebelow, we introduce a protocol to determine the process matrix. The protocol has a quantum part and a classical part. The quantum part involves using state preparations and syndrome measurements of different ASCs to determine $\chi_{m,n}$ ambiguously. The classical part involves post-processing to disambiguate the $\chi_{m,n}$ data.

7.4.2 Ambiguously determining the diagonal terms $\chi_{j,j}$ and ambiguous coherence terms $\chi_{j,k}$

Let Q be an ASC that can detect noise $\mathcal{E}$, with associated process matrix χ. Let E_{α_j} $(j = 0,1,2,\cdots,\gamma-1)$ be the elements of an ambiguous set in Q, with E_x denoting any one of these α_j's. It is convenient to employ the notation $\left|j_L^{(\alpha)}\right\rangle \equiv E_\alpha|j_L\rangle$. The probability that one of these ambiguous errors occur:

$$
\begin{aligned}
\xi\left(\bigwedge_j \alpha_j\right) &= \mathrm{Tr}\left(\mathcal{E}\left(|\psi_L\rangle\langle\psi_L|\right)\left[\sum_{j=0}^{2^k-1}|j_L^x\rangle\langle j_L^x|\right]\right) \\
&= \sum_{j=0}^{2^k-1}\langle j_L^x|\Big[\cdots+\chi_{\alpha_1,\alpha_1}|\psi_L^{(\alpha_1)}\rangle\langle\psi_L^{(\alpha_1)}|+\chi_{\alpha_1,\alpha_2}|\psi_L^{(\alpha_1)}\rangle\langle\psi_L^{(\alpha_2)}| \\
&\quad+\chi_{\alpha_2,\alpha_1}|\psi_L^{(\alpha_2)}\rangle\langle\psi_L^{(\alpha_1)}|+\chi_{\alpha_2,\alpha_2}|\psi_L^{(\alpha_2)}\rangle\langle\psi_L^{(\alpha_2)}|+\cdots\Big]|j_L^x\rangle \\
&= \cdots+\chi_{\alpha_1,\alpha_1}+\chi_{\alpha_1,\alpha_2}\langle\psi_L^{(\alpha_2)}|\psi_L^{(\alpha_1)}\rangle+\chi_{\alpha_2,\alpha_1}\langle\psi_L^{(\alpha_1)}|\psi_L^{(\alpha_2)}\rangle \\
&\quad+\chi_{\alpha_2,\alpha_2}+\cdots \\
&= \cdots+\chi_{\alpha_1,\alpha_1}+\chi_{\alpha_1,\alpha_2}\langle\psi_L|N_{1,2}|\psi_L\rangle+\chi_{\alpha_2,\alpha_1}\langle\psi_L|N_{2,1}|\psi_L\rangle \\
&\quad+\chi_{\alpha_2,\alpha_2}+\cdots \\
&= \sum_j \chi_{\alpha_j,\alpha_j}+2\sum_{j\neq k}\mathrm{Re}\left(\chi_{\alpha_j,\alpha_k}\langle N_{j,k}\rangle_L\right), \qquad (7.14)
\end{aligned}
$$

where $N_{m,n} \equiv E_m E_n$.

Let $D \equiv 2^p$, the dimension of $\mathbf{P}$. In an unambiguous code, the D^2 diagonal terms of χ would appear as probabilities of syndrome measurements [140]. Now, however, in any measurement outcome probability, only blocks of size γ can be disambiguated from syndrome measurements. Additionally, the diagonal terms must be disambiguated from $\binom{\gamma}{2}$ cross-terms from within ambiguous sets. Part of the problem is solved by inputting different initial states, by exploiting the fact that the χ terms have factors given by expectation values of different normalizer elements (logical Pauli operations). However, the problem of disambiguation would still remain *within* each such logical Pauli class. This can be sorted out using suitably chosen unitaries to pre-process the state before measurement. This is discussed in Section 6.2.3. For accessing coherence terms of χ across ambiguous sets, we can either use the same technique, but with a different type of unitary pre-processing, or use different ASCs. Finally, the method as described here would only give either the real or imaginary part of any cross-term. Another unitary pre-processor would be required, similar to that introduced by us in Ref. [140], to 'toggle' real and imaginary parts.

As an example of result (7.14), for the data in Table (7.3), the probability to obtain the outcome "++" is:

$$\begin{aligned}
p(++) &= \chi_{II,II} + \chi_{IY,IY} + \chi_{XX,XX} + \chi_{XZ,XZ} + 2\times[\mathrm{Re}(\chi_{II,IY})\langle Y_L\rangle + \mathrm{Re}(\chi_{II,XX})\langle Z_L\rangle \\
&- \mathrm{Re}(\chi_{II,XZ})\langle X_L\rangle + \mathrm{Im}(\chi_{IY,XX})\langle X_L\rangle + \mathrm{Im}(\chi_{IY,XZ})\langle Z_L\rangle - \mathrm{Im}(\chi_{XX,XZ})\langle Y_L\rangle], \\
p(+-) &= \chi_{XI,XI} + \chi_{XY,XY} + \chi_{IX,IX} + \chi_{IZ,IZ} + 2\times[\mathrm{Re}(\chi_{XI,XY})\langle Y_L\rangle + \mathrm{Re}(\chi_{XI,IX})\langle Z_L\rangle \\
&- \mathrm{Re}(\chi_{XI,IZ})\langle X_L\rangle + \mathrm{Im}(\chi_{XY,IX})\langle X_L\rangle + \mathrm{Im}(\chi_{XY,IZ})\langle Z_L\rangle - \mathrm{Im}(\chi_{IX,IZ})\langle Y_L\rangle], \\
p(-+) &= \chi_{YI,YI} + \chi_{YY,YY} + \chi_{ZX,ZX} + \chi_{ZZ,ZZ} + 2\times[\mathrm{Re}(\chi_{YI,YY})\langle Y_L\rangle + \mathrm{Re}(\chi_{YI,ZX})\langle Z_L\rangle \\
&- \mathrm{Re}(\chi_{YI,ZZ})\langle X_L\rangle + \mathrm{Im}(\chi_{YY,ZX})\langle X_L\rangle + \mathrm{Im}(\chi_{YY,ZZ})\langle Z_L\rangle - \mathrm{Im}(\chi_{ZX,ZZ})\langle Y_L\rangle], \\
p(--) &= \chi_{ZI,ZI} + \chi_{ZY,ZY} + \chi_{YX,YX} + \chi_{YZ,YZ} + 2\times[\mathrm{Re}(\chi_{ZI,ZY})\langle Y_L\rangle + \mathrm{Re}(\chi_{ZI,YX})\langle Z_L\rangle \\
&- \mathrm{Re}(\chi_{ZI,YZ})\langle X_L\rangle + \mathrm{Im}(\chi_{ZY,YX})\langle X_L\rangle + \mathrm{Im}(\chi_{ZY,YZ})\langle Z_L\rangle - \mathrm{Im}(\chi_{YX,YZ})\langle Y_L\rangle] \qquad (7.15)
\end{aligned}$$

By choosing input $|0\rangle_L$, one finds $P(++) = \chi_{II,II} + \chi_{IY,IY} + \chi_{XX,XX} + \chi_{XZ,XZ} + \mathrm{Re}(\chi_{II,XX}) + \mathrm{Im}(\chi_{IY,XZ}) \equiv c + \mathrm{Re}(\chi_{II,XX}) + \mathrm{Im}(\chi_{IY,XZ})$. By choosing input $|+\rangle_L \equiv \frac{1}{\sqrt{2}}(|0\rangle_L + |1\rangle_L)$, one finds $P(++) = c + \mathrm{Re}(\chi_{II,XZ}) + \mathrm{Im}(\chi_{IY,XX})$. By choosing input $|\uparrow\rangle_L \equiv \frac{1}{\sqrt{2}}(|0\rangle_L + i|1\rangle_L)$, one finds $P(++) = c + \mathrm{Re}(\chi_{II,IY}) - \mathrm{Im}(\chi_{XX,XZ})$. We thus have four unknowns, given by c (the diagonal contributions), and the coefficients of $\langle X_L\rangle$, $\langle Y_L\rangle$ and $\langle Z_L\rangle$. One more input, say $\cos(\theta)|0_L\rangle + \sin(\theta)|1_L\rangle$ will suffice to determine these 4 quantities. It will suffice to determine c. More generally, 4^k (the number of logical Pauli operations) preparations are needed to solve for c. When c is extracted for each outcome, then each code gives $D^2/\gamma = 2^{n-k}$ equations.

The cross-terms for ambiguous errors can be dealt with in other ASCs, where they correspond to cross-terms that are unambiguous (described below) or using a pre-processing unitary $U(a,b)$ of the type described earlier.

At least $\gamma+1$ ASCs will be required to disambiguate all D^2 diagonal variables. To see this, suppose we begin with γ ASCs. The D^2 equations corresponding to their outcome probability will correspond to an adjacency matrix, wherein the D^2/γ rows corresponding to each code will sum to a *unit row*, i.e., one with 1's in all columns. Thus there are (at least) $\gamma - 1$ constraints among the D^2 equations. Adding one more code will introduce D^2/γ equations and one more constraint i.e., $2^{n-k} - 1$ constraints. If there are no other constraints in the first D^2 rows, and if $2^{n-k} - 1 \geq \gamma - 1$, i.e., $n - k \geq p$, then the remaining required linearly independent equations can be found from the last code. Thus, in general, with γ-fold full degeneracy, the necessary number of preparations is $\gamma + 1$.

7.4.3 Ambiguously determining coherence terms $\chi_{j,k}$.

The method described in the preceding section can determine only the diagonal terms $\chi_{j,j}$ and off-diagonal terms of pairs of ambiguous errors. To derive off-diagonal terms for unambiguous errors, we need to preprocess the full system by applying a suitable unitary

U, as detailed in the following subsection. However, even this may allow one to access only the real or imaginary part of some off-diagonal terms. To access the other part of the these off-diagonal terms, one would require pre-processing with unitary UT^+, prior to stabilizer measurement. The construction of T^+ is described later below.

Preprocessing with U.

The kind of U we consider will be in one of two forms. In the first form, $U = \frac{1}{\sqrt{2}}(E_a + E_b)$, in case $[E_a, E_b] \neq 0$. Here U is also Hermitian. The second kind unitary is given by $U = \frac{1}{\sqrt{2}}(E_a + iE_b)$, in case $[E_a, E_b] = 0$. We require E_a and E_b to be mutually unambiguous, for otherwise this method reduces to that of the preceding section.

First let us consider case of $U(a,b) = U^\dagger(a,b)$. Let $g_{A_j}E_j = E_A E_{\alpha_j}$, where the E_{α_j}'s constitute an ambiguous set, and $g_{A_j} \in \{\pm 1, \pm i\}$ is the *Pauli factor*. Similarly, let $g_{B_j}E_j = E_b E_{\beta_j}$, where the E_{β_j}'s constitute an ambiguous set, and $g_{B_j} \in \{\pm 1, \pm i\}$ is a Pauli factor.

When $U(a,b)$ is applied to the noisy logical state, and an outcome x has been observed, then one of the E_j must have been detected, and thus the only contributing terms of $\mathcal{E}(\rho_L)$ will be those restricted to $|\psi_L^{\alpha_j}\rangle$ and $|\psi_L^{\beta_j}\rangle$. Denoting by $\Pi_\mathcal{C}$ the projector to the code space $\mathcal{C}$ of the ASC, the probability to observe x when $U(a,b)$ has been applied is:

$$\xi(a,b,x) \equiv \mathrm{Tr}\left(U\left[\mathcal{E}(|\Psi_L\rangle\langle\Psi_L|)\right]U^\dagger\left(E_x\Pi_\mathcal{C}E_x\right)\right) \tag{7.16}$$

The terms within the square bracket in Eq. (7.16) that would make a contribution to the probability of obtaining ambiguous outcome E_x are:

$$\begin{aligned}\cdots &+ \chi_{\alpha_1,\alpha_1}|\psi_L^{(\alpha_1)}\rangle\langle\psi_L^{(\alpha_1)}| + \chi_{\alpha_1,\alpha_2}|\psi_L^{(\alpha_1)}\rangle\langle\psi_L^{(\alpha_2)}| \\ &+ \chi_{\alpha_2,\alpha_1}|\psi_L^{(\alpha_2)}\rangle\langle\psi_L^{(\alpha_1)}| + \chi_{\alpha_2,\alpha_2}|\psi_L^{(\alpha_2)}\rangle\langle\psi_L^{(\alpha_2)}| + \cdots \\ &+ \chi_{\alpha_1,\beta_1}|\psi_L^{(\alpha_1)}\rangle\langle\psi_L^{(\beta_1)}| + \chi_{\beta_1,\alpha_1}|\psi_L^{(\beta_1)}\rangle\langle\psi_L^{(\alpha_1)}| + \cdots \\ &+ \chi_{\alpha_1,\beta_2}|\psi_L^{(\alpha_1)}\rangle\langle\psi_L^{(\beta_2)}| + \chi_{\beta_2,\alpha_1}|\psi_L^{(\beta_2)}\rangle\langle\psi_L^{(\alpha_1)}| \\ &+ \cdots\end{aligned} \tag{7.17}$$

When the expression in Eq. (7.17) is left- and right-multiplied by $U(a,b)$, then the only resulting terms that contribute to the lhs of Eq. (7.16) are:

$$\begin{aligned}&\cdots + \chi_{\alpha_1,\alpha_1}|\psi_L^{(1)}\rangle\langle\psi_L^{(1)}| + \chi_{\alpha_1,\alpha_2}g_{A_1}g_{A_2}^*|\psi_L^{(1)}\rangle\langle\psi_L^{(2)}| \\ &+ \chi_{\alpha_2,\alpha_1}g_{A_2}g_{A_1}^*|\psi_L^{(2)}\rangle\langle\psi_L^{(1)}| + \chi_{\alpha_2,\alpha_2}|\psi_L^{(2)}\rangle\langle\psi_L^{(2)}| + \cdots \\ &+ \chi_{\alpha_1,\beta_1}g_{A_1}g_{B_1}^*|\psi_L^{(1)}\rangle\langle\psi_L^{(1)}| + \chi_{\beta_1,\alpha_1}g_{B_1}g_{A_1}^*|\psi_L^{(1)}\rangle\langle\psi_L^{(1)}| + \cdots \\ &+ \chi_{\alpha_1,\beta_2}g_{A_1}g_{B_2}^*|\psi_L^{(1)}\rangle\langle\psi_L^{(2)}| + \chi_{\beta_2,\alpha_1}g_{B_2}g_{A_1}^*|\psi_L^{(2)}\rangle\langle\psi_L^{(1)}| \\ &+ \cdots\end{aligned} \tag{7.18}$$

The contribution of the first term in Eq. (7.18) to the probability in Eq. (7.16) would be:

$$\begin{aligned}\epsilon_{\alpha_1,\alpha_1} &\equiv \chi_{\alpha_1,\alpha_1}\sum_{j=1}^{2^k}\langle j_L^{(x)}|\psi_L^{(1)}\rangle\langle\psi_L^{(1)}|j_L^{(x)}\rangle \\ &= \chi_{\alpha_1,\alpha_1}. \end{aligned} \tag{7.19}$$

since the traced quantity has support only in the erroneous code space $E_x\mathcal{C}'$. Analogously, the contribution of the fourth term in Eq. (7.18) to Eq. (7.16) would be $\epsilon_{\alpha_2,\alpha_2} = \chi_{\alpha_2,\alpha_2}$. In like fashion, the contribution of the fifth and sixth terms in Eq. (7.18) to Eq. (7.16) would be $\epsilon_{\alpha_1,\beta_1} = \chi_{\alpha_1,\beta_1} g_A g_B^*$ and $\epsilon_{\beta_1,\alpha_1} = \chi_{\beta_1,\alpha_1} g_B g_A^*$.

The contribution of the second term in Eq. (7.18) to the probability in Eq. (7.16) would be:

$$\begin{aligned}\epsilon_{\alpha_1,\alpha_2} &\equiv \chi_{\alpha_1,\alpha_2} g_{A_1} g_{A_2}^* \sum_{j=1}^{2^k}\langle j_L^{(x)}|\psi_L^{(1)}\rangle\langle\psi_L^{(2)}|j_L^{(x)}\rangle \\ &= \chi_{\alpha_1,\alpha_2} g_{A_1} g_{A_2}^* \langle\psi_L^{(2)}|\psi_L^{(1)}\rangle \\ &= \chi_{\alpha_1,\alpha_2} g_{A_1} g_{A_2}^* \langle N_{12}\rangle_L, \end{aligned} \tag{7.20}$$

where N_{21} is the normalizer element that propagates error E_{A_2} to E_{A_1}. The contribution of the third term in Eq. (7.18) to the probability in Eq. (7.16) would be, analogously to Eq. (7.20), namely, $\epsilon_{\alpha_2,\alpha_1} = \chi_{\alpha_2,\alpha_1} g_{A_2} g_{A_1}^* \langle N_{21}\rangle_L$. In like fashion, the contribution of the seventh and eighth terms in Eq. (7.18) to Eq. (7.16) would be $\epsilon_{\alpha_1,\beta_2} = \chi_{\alpha_1,\beta_2} g_{A_1} g_{B_2}^* \langle N_{12}\rangle_L$ and $\epsilon_{\beta_2,\alpha_1} = \chi_{\beta_2,\alpha_1} g_{B_2} g_{A_1}^* \langle N_{21}\rangle_L$.

Putting together all these $\epsilon_{\alpha_j,\alpha_k}, \epsilon_{\alpha_j,\beta_k}$, etc., terms into Eq. (7.16), we obtain:

$$\begin{aligned}\xi(a,b,x) &= \sum_{j=1}^{\gamma}\left[\frac{1}{2}\left(\chi_{\alpha_j,\alpha_j}+\chi_{\beta_j,\beta_j}\right) + \mathrm{Re}\left(g_A g_B^* \chi_{\alpha_j,\beta_j}\right)\right] \\ &+ \sum_{j<k}\Big[\mathrm{Re}\left(\chi_{\alpha_j,\alpha_k} g_{A_j} g_{A_k}^* \langle N_{j,k}\rangle_L\right) \\ &\quad + \mathrm{Re}\left(\chi_{\beta_j,\beta_k} g_{B_j} g_{B_k}^* \langle N_{j,k}\rangle_L\right) \\ &\quad + \mathrm{Re}\left(\chi_{\alpha_j,\beta_k} g_{A_j} g_{B_k}^* \langle N_{j,k}\rangle_L\right)\Big]. \end{aligned} \tag{7.21}$$

If $[E_a, E_b] \neq 0$ then we set $U = \frac{E_a + iE_b}{\sqrt{2}}$. As a result, instead of Eq. (7.18), one gets:

$$\begin{aligned}
& \cdots + \chi_{\alpha_1,\alpha_1}|\psi_L^{(1)}\rangle\langle\psi_L^{(1)}| + \chi_{\alpha_1,\alpha_2} g_{A_1} g_{A_2}^* |\psi_L^{(1)}\rangle\langle\psi_L^{(2)}| \\
+ \; & \chi_{\alpha_2,\alpha_1} g_{A_2} g_{A_1}^* |\psi_L^{(2)}\rangle\langle\psi_L^{(1)}| + \chi_{\alpha_2,\alpha_2}|\psi_L^{(2)}\rangle\langle\psi_L^{(2)}| + \cdots \\
- \; & i\chi_{\alpha_1,\beta_1} g_{A_1} g_{B_1}^* |\psi_L^{(1)}\rangle\langle\psi_L^{(1)}| + i\chi_{\beta_1,\alpha_1} g_{B_1} g_{A_1}^* |\psi_L^{(1)}\rangle\langle\psi_L^{(1)}| + \cdots \\
- \; & i\chi_{\alpha_1,\beta_2} g_{A_1} g_{B_2}^* |\psi_L^{(1)}\rangle\langle\psi_L^{(2)}| + i\chi_{\beta_2,\alpha_1} g_{B_2} g_{A_1}^* |\psi_L^{(2)}\rangle\langle\psi_L^{(1)}| \\
+ \; & \cdots \qquad (7.22)
\end{aligned}$$

Consequently, one obtains in place of Eq. (7.21):

$$\begin{aligned}
\xi(a,b,x) &= \frac{1}{2}\left(\sum_{j=1}^{\gamma} \chi_{\alpha_j,\alpha_j} + \chi_{\beta_j,\beta_j} + \chi_{\alpha_j,\beta_j} + \chi_{\beta_j,\alpha_j}\right) \\
&+ \sum_{j<k} \left[\mathrm{Re}\left(\chi_{\alpha_j,\alpha_k} g_{A_j} g_{A_k}^* \langle N_{j,k}\rangle_L\right)\right. \\
&\quad + \mathrm{Re}\left(\chi_{\beta_j,\beta_k} g_{B_j} g_{B_k}^* \langle N_{j,k}\rangle_L\right) \\
&\quad \left. + \;\mathrm{Im}\left(\chi_{\alpha_j,\beta_k} g_{A_j} g_{B_k}^* \langle N_{j,k}\rangle_L\right)\right]. \qquad (7.23)
\end{aligned}$$

It is worth noting that in Eqs. (7.21) or (7.23), in the terms that contain Pauli factors, the matter of whether the real or imaginary part of the process element of the process matrix contributes to the measured probability, depends on whether the Pauli factors are of same type (real/imaginary). This is solved as in the case of QECCD by applying toggling operation T^+ described in Sec. 6.2.4 of Chapter 6.

7.5 Illustration with 4-qubit ambiguous codes

One can construct the ASC from dropping one or more qubits from QECC keeping the $\mathcal{S}$ unchanged, so that it satisfies the error correction condition Eq. (7.5). Consider the $[[4,1]]$ ambiguous code constructed by dropping the last qubit of $[[5,1]]$ QECC,

$$\begin{aligned}
|0_L\rangle_4 &= \frac{1}{2\sqrt{2}}\left(-|0000\rangle + |0010\rangle + |0101\rangle + |0111\rangle\right. \\
&\quad \left. -|1001\rangle + |1011\rangle + |1100\rangle + |1110\rangle\right) \\
|1_L\rangle_4 &= \frac{1}{2\sqrt{2}}\left(-|1111\rangle + |1101\rangle + |1010\rangle + |1000\rangle\right. \\
&\quad \left. -|0110\rangle + |0100\rangle + |0011\rangle + |0001\rangle\right), \qquad (7.24)
\end{aligned}$$

The stabilizers for this code are given by $XIIX, YIXY$ and $YYZZ$.

By computer search, we obtained a family of 4-qubit code by dropping one of qubits in the 5-qubit code of Ref. [37]. Because the code space has dimension 2^3, the two code words are stabilized by 3 linearly independent 4-qubit Pauli operators, which correspond to the measured syndrome. Eq. (7.25) presents three such codes which beat the Hamming bound by a factor of 2, and their corresponding syndrome operators are given in Table 7.4:

$$|0'_L\rangle_4 \;=\; H_{ZY}^{\otimes 4}|0_L\rangle_4;\; |0''_L\rangle_4 = H_{YX}^{\otimes 4}|0_L\rangle_4. \tag{7.25}$$

The logical 1 is given by $|1_L\rangle_4 = |\overline{0}_L\rangle_4$, where the overline in the ket indicates binary complement (flipping 0 and 1). Similarly, for $|1'_L\rangle_4$ and $|1''_L\rangle_4$. Here $H_{ZY} = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + i|0\rangle\langle 1| + i|1\rangle\langle 0| + |1\rangle\langle 1|)$, $H_{YX} = \frac{1}{2}((1+i)|0\rangle\langle 0| + (1+i)|0\rangle\langle 1| - (1-i)|1\rangle\langle 0| + (1-i)|1\rangle\langle 1|)$ and X, Y, Z, I are 1-qubit Pauli operators. The logical operations on the first code are $\overline{X} = XXXX$ and $\overline{Z} = ZZIZ$, while those for the other two codes are corresponding Hadamard-transformed operations.

Because of two-fold ambiguity, each syndrome corresponds to precisely two errors, and the pairing of errors depends on the QEC code chosen, as seen from the Table 7.4. The probability to obtain one of a pair of errors is the probability to obtain a syndrome δ, which is ambiguous for these two errors. For example, the probability that outcome $(+,-,-)$ in code $\mathcal{Q}$ is obtained yields the sum of the probabilities to obtain errors X_1 and XY, i.e., $\chi_{I,I} + \chi_{XY,XY}$. Thus, from the above conditions one obtains a system of simultaneous equations that must be solved to characterize the noise completely. Using Eqs. (7.14), (7.21) and (7.23), the statistics of syndrome measurements on QECs $\mathcal{Q}$, $\mathcal{Q}'$ and $\mathcal{Q}''$, $\chi_{m,n}$ for the 2-qubit noise $\mathcal{E}$ can be completely characterized. From the Table 7.4, one can construct the following 16 expressions which can characterize the diagonal elements of $\chi_{m,n}$ representing the "population of dynamics"

From $\mathcal{Q}$ we have

$$\begin{aligned}
&\chi_{I,I} + \chi_{Y2,Y2} = a_1,\;\; \chi_{X1,X1} + \chi_{XY,XY} = b_1,\\
&\chi_{X2,X2} + \chi_{Z2,Z2} = c_1, \chi_{XX,XX} + \chi_{XZ,XZ} = d_1,\\
&\chi_{YX,YX} + \chi_{YZ,YZ} = e_1,\;\; \chi_{Y1,Y1} + \chi_{YY,YY} = f_1,\\
&\chi_{ZZ,ZZ} + \chi_{Zx,ZX} = g_1,\;\; \chi_{Z1,Z1} + \chi_{ZY,ZY} = h_1
\end{aligned} \tag{7.26}$$

From $\mathcal{Q}'$ we have

$$\begin{aligned}
&\chi_{I,I} + \chi_{Z2,Z2} = a_2,\;\; \chi_{X1,X1} + \chi_{XZ,XZ} = b_2,\\
&\chi_{Y1,Y1} + \chi_{YZ,YZ} = c_2,\;\; \chi_{Z1,Z1} + \chi_{ZZ,ZZ} = d_2
\end{aligned} \tag{7.27}$$

From $\mathcal{Q}''$

$$\begin{aligned}
\chi_{I,I} + \chi_{X2,X2} = a_3,\;\; \chi_{X1,X1} + \chi_{XX,XX} = b_3,\\
\chi_{Y1,Y1} + \chi_{YX,YX} = c_3,\;\; \chi_{Z1,Z1} + \chi_{ZX,ZX} = d_3.
\end{aligned} \tag{7.28}$$

$\mathcal{Q}$	II Y_2	X_1 XY	X_2 Z_2	Y_1 YY	Z_1 ZY	XX XZ	YX YZ	ZX ZZ
$XIIX$	+	+	+	−	−	+	−	−
$YIXY$	+	−	+	+	−	−	+	−
$YYZZ$	+	−	−	+	−	+	−	+
$\mathcal{Q}'$	II Z_2	X_1 XZ	X_2 Y_2	Y_1 YZ	Z_1 ZZ	XX XY	YX YY	ZX ZY
$IZZX$	+	+	−	+	+	−	−	−
$XIIX$	+	+	+	−	−	+	−	−
$YZYZ$	+	−	−	+	−	+	−	+
$\mathcal{Q}''$	II X_2	X_1 XX	Y_1 YX	Y_2 Z_2	Z_1 ZX	XY XZ	YY YZ	ZY ZZ
$IXXZ$	+	+	+	−	+	−	−	−
$XIXZ$	+	+	−	+	−	+	−	−
$YXYX$	+	−	+	−	−	+	−	+

Table 7.4: The Hadamard operation H_{ZY} (H_{YX}) toggles errors Z and Y (errors Y and X) while keeping error X (Z) fixed, and the above syndromes are corresponding toggled versions of each other. The entry is "+" when $[S_j, E_j] = 0$ and "−" otherwise.

Here $\chi_{i,i}$ are probabilities of getting syndrome δ on codes $\mathcal{Q}$, $\mathcal{Q}'$ and $\mathcal{Q}''$. For this we need 3 state preparations and a measurement on each state.

Example

To demonstrate how the method of characterizing an arbitrary quantum noise using ambiguous QEC code works we consider noisy QEC code due to some artificial noise $\mathcal{E}_A$ to be

$$\begin{aligned}\mathcal{E}_A(\rho_L) \quad &= \delta\rho_L + \frac{1-\delta}{5}(X_1\rho_L X_1 + XZ\rho_L XZ + Y_2\rho_L Y_2 \\ &+X_2\rho_L X_2 + XX\rho_L XX) + \frac{1}{6}((a+ib)X1\rho_L X_2 \\ &+(c+id)\rho_L XX + (e+if)XZ\rho_L Y_2 + c.c)\end{aligned} \tag{7.29}$$

The diagonal terms of $\chi_{m,n}$ can be determined using the codes $\mathcal{Q}$, $\mathcal{Q}'$, $\mathcal{Q}''$.

Here for solving the off-diagonal terms using Eq. (6.7) only one code $\mathcal{Q}$ suffices as there are fewer errors in $\mathcal{E}_A$. The following set of linearly independent equations for off-diagonal terms are obtained by performing various unitary operations $U(a, b)$ followed by ambiguous syndrome measurements.

$$\xi(X_1, X_2, I/X_2) - \frac{1}{2}\sum_j \chi_{j,j} \equiv \mathcal{O}_1 = \text{Im}(X_1, X_2) - \text{Im}(I, XX) \quad (7.30)$$

$$\xi(X_1, X_2, I/Z_2) - \frac{1}{2}\sum_j \chi_{j,j} \equiv \mathcal{O}_2 = \text{Im}(X_1, X_2) - \text{Re}(XZ, Y_2) \quad (7.31)$$

$$\xi(I, XX, I/Y_2) - \frac{1}{2}\sum_j \chi_{j,j} \equiv \mathcal{O}_3 = \text{Im}(I, XX) + \text{Re}(XZ, Y_2) \quad (7.32)$$

$$(7.33)$$

where $j = X_1, X_2, XY, Z_2$. In Eq. (7.33), while the values of $\xi(a, b, x)$ are obtained during syndrome measurements on noisy QEC codes, that of $\sum_j \chi_{j,j}$ are obtained from solving Eq.(7.26,7.27,7.28). Solving the above set of equations,

$$\begin{aligned} \text{Im}(X_1, X_2) &= \frac{1}{2}(\mathcal{O}_1 + \mathcal{O}_2 + \mathcal{O}_3) = \frac{b}{6} \\ \text{Re}(XZ, Y_2) &= \frac{1}{2}(\mathcal{O}_1 - \mathcal{O}_2 + \mathcal{O}_3) = \frac{e}{6} \\ \text{Im}(I, XX) &= \frac{1}{2}(-\mathcal{O}_1 + \mathcal{O}_2 + \mathcal{O}_3) = \frac{d}{6} \end{aligned} \quad (7.34)$$

$$\mathcal{O}_1' = -\text{Re}(X_1, X_2) - \text{Re}(I, XX) \quad (7.35)$$

$$\mathcal{O}_2' = \text{Re}(X_1, X_2) + \text{Im}(XZ, Y_2) \quad (7.36)$$

$$\mathcal{O}_3' = \text{Re}(I, XX) + \text{Im}(XZ, Y_2) \quad (7.37)$$

$$(7.38)$$

A general observation points out that we have 8 variables in the system of linearly independent equation but only 6 such equations making the set to have no solution. For the set to be solvable we need two more linearly independent equations. This can be solved by preprocessing the noisy states with the toggling operation S. The two equations obtained this way are

The above two is obtained using the toggling $S(X_1, XY, Z_2..|X_2)$. Solving the above linearly independent equations we have

$$\begin{aligned} \text{Re}(X_1, X_2) &= \frac{1}{2}(-\mathcal{O}_1' + \mathcal{O}_2' - \mathcal{O}_3') = \frac{a}{6} \\ \text{Im}(XZ, Y_2) &= \frac{1}{2}(\mathcal{O}_1' + \mathcal{O}_2' + \mathcal{O}_3') = \frac{f}{6} \\ \text{Re}(I, XX) &= \frac{1}{2}(-\mathcal{O}_1' - \mathcal{O}_2' + \mathcal{O}_3') = \frac{c}{6} \end{aligned} \quad (7.39)$$

7.6 Discussion and conclusion

In this Chapter, we introduced a new class of stabilizer codes, namely ambiguous stabilizer codes (ASCs), for which the recovered state is only required to be error-free up to a logical Pauli operation within code space $\mathcal{C}$. This generalizes the concept of a degenerate code, which is the special case where the only residual logical operation after recovery is the trivial one. We discussed different procedures to construct ASCs.

We showed that arbitrary errors on m known coordinates of a $[[n, k]]$ ASC can be characterized as a quotient group over the set of all Pauli errors that are ambiguous with no-error. We also showed how these ASCs can be employed for CQD, which helps in reducing the size of quantum states and would thus be helpful from an experimental perspective. But the price to be paid was in terms of increased number of operations required for disambiguating the errors.