Chapter 5

The Unruh effect interpreted as a quantum noise channel

5.1 Introduction

Dechorence is an unavoidable phenomenon associated with open quantum systems, which occurs because of their interaction with the ambient environment. This causes the decay of quantum correlations, a resource essential for quantum information processing. A detailed understanding of dechorence, and more generally, quantum information in a relativistic setting would be relevant both from a fundamental perspective as well as to help future experiments involving relativistic observers.

The relativistic effect named after Unruh [115, 116, 117] predicts that the Minkowski vacuum as seen by an observer accelerating with constant proper acceleration a will appear as a warm gas emitting black-body radiation at the *Unruh temperature*

$$\tau = \frac{\hbar a}{2\pi k_B c}, \tag{5.1}$$

where c is the speed of light in vacuum, and k_B is Boltzmann's constant.

The Unruh effect produces a decoherence-like effect, earning it the moniker 'Unruh channel'. It degrades the quantum information shared between an inertial observer (Alice) and an accelerated observer (Rob), as seen in the latter's frame, in the case of bosonic or Dirac field modes [118, 119, 120]. It may be combined with other, non-relativistic noise, such as the bit flip or amplitude damping channels [121, 122]. Being conjugate degradable, the Unruh channel allows for its quantum capacity to be computed efficiently with a single-use of the channel [123]. A channel is degradable (conjugate degradable) if the environment can be simulated from the output (up to a complex conjugation). The capacity of the Unruh channel for various resources such as classical, quantum, public or private communication [124], and trade-off relations between such capacities, has been extensively studied [125, 126] as part of Hadamard channels in the bosonic case [127, 128], and as part of Grassmann channels in the fermionic case [129]. The Hadamard (Grassmann) class is induced by the two-mode squeezing operator for operators obeying bosonic (fermionic) algebra.

Differences between the two cases arise because of the different algebras and also because multi-mode fermions lack tensor product structure. The Unruh channel is known to render separable an initially maximally entangled state in the limit of infinite acceleration in the case of bosonic entanglement, but not so in the case of fermionic entanglement [119]. However, the channel capacities of both channels for quantum communication between the inertial and the accelerated observer are found to be qualitatively similar, in that both vanish asymptotically [129].

Linear maps describing the dynamic evolution of density operators of a quantum system initially entangled with another quantum system are not necessarily completely positive (CP) [130, 114], i.e., they can map a positive operator to one that is not positive, unless the map's domain is restricted. It is possible to represent such non-CP maps in the usual operator-sum or Kraus form, with some negative terms included, so that they are described as the difference of two CP maps. In this work[1], we first derive the operator-sum representation for one of two modes of a Dirac field described by relatively accelerated systems, for e.g., detectors, in a Minkowski space time. We then compare and contrast the Unruh channel with conventional noise arising from environmental decoherence, in order to obtain a physical interpretation of it.

The article is organized as follows: In the following Section we introduce the Unruh effect for a two-mode fermionic system. The degradation of quantum correlations caused by the effect are studied subsequently. We then characterize the Unruh effect as a quantum noise channel, comparing and contrasting it with the amplitude damping (AD) channel. Later we discuss some aspects of its physical interpretation, constructing an inverse Unruh map, which is NCP.

5.2 Unruh effect

Consider two observers, Alice (A) and Rob (R) sharing a maximally entangled state of two Dirac field modes, at a point in Minkowski spacetime, of the form

$$|\psi\rangle_{A,R} = \frac{|00\rangle_{A,R} + |11\rangle_{A,R}}{\sqrt{2}}, \tag{5.2}$$

where $|j\rangle$ denote Fock states. Let Rob move away from stationary Alice with a uniform proper acceleration a. The effect of constant proper acceleration is described by a Rindler

[1] The results included in this chapter are on arXiv.org [131]

spacetime, which manifests two causally disconnected regions I and II, where region I is accessible to Rob, and separated from region II by an event horizon.

From Rob's frame the Minkowski vacuum state is a two-mode squeezed state, while the excited state appears as a product state [119]:

$$\begin{aligned} |0\rangle_M &= \cos r|0\rangle_I|0\rangle_{II} + \sin r|1\rangle_I|1\rangle_{II}, \\ |1\rangle_M &= |1\rangle_I|0\rangle_{II}, \end{aligned} \tag{5.3}$$

where ω is a Dirac particle frequency while $\cos r = \frac{1}{\sqrt{e^{-\frac{2\pi\omega c}{a}}+1}}$ is one of the Bogoliubov coefficients, connecting the Minkowski and Rindler vacua. It follows that $\cos r \in [\frac{1}{\sqrt{2}}, 1]$ as a ranges from ∞ to 0. As a peculiarity of this effect, as noted in the Introduction, we observe that the states in the left hand side of Eq. (5.3) are single-mode states, while those in the right hand side are not.

Under representation (5.3), the state Eq. (5.2) becomes

$$|\psi\rangle_{A,I,II} = \frac{1}{\sqrt{2}}\left(|0\rangle_A(\cos r|0\rangle_I|0\rangle_{II} + \sin r|1\rangle_I|1\rangle_{II}) + |1\rangle_A|1\rangle_I|1\rangle_{II}\right). \tag{5.4}$$

Tracing out mode II, we obtain the density matrix:

$$\rho'_{A,R} = \frac{1}{2}\left[\cos^2(r)|00\rangle\langle 00| + \cos r(|00\rangle\langle 11| + |11\rangle\langle 00|) + \sin^2(r)|01\rangle\langle 01| + |11\rangle\langle 11|\right]. \tag{5.5}$$

The 'evolution' of Rob's qubit to a mixed state under the transformation $\mathcal{E}_U : \rho_R \rightarrow \rho'_R$ constitutes what we call the Unruh channel for a fermionic qubit.

5.3 Degradation of quantum information under Unruh channel

The nonclassicality of quantum information can be characterized in terms of nonlocality (the strongest condition), entanglement, teleportation fidelity or weaker nonclassicality measures like quantum discord or measurement induced disturbance. As seen in the accelerated reference frame, the Unruh effect degrades the quantumness of the state $\rho'_{A,R}$ according to each such measure, as discussed in the following subsections.

5.3.1 Bell inequalities

The violation of Bell-type inequality indicates that a given bipartite state ρ cannot be modelled using a deterministic local hidden variable (DLHV) theory. Suppose Alice holds a particle with dichotomic properties $A_1, A_2 \in \{\pm 1\}$, and Rob holds a correlated particle with similar properties B_1, B_2. If a hidden variable λ specifies the values of these four properties, then the quantity $A_1(B_1 - B_2) + A_2(B_1 + B_2) = \pm 2$. Averaging over any measure of λ, we obtain the the Clauser-Horne-Shimony-Holt (CHSH) inequality [132]

$$|\langle A_1 B_1\rangle - \langle A_1 B_2\rangle + \langle A_2 B_1\rangle + \langle A_2 B_2\rangle| \leq 2, \tag{5.6}$$

where $\langle \dots \rangle$ stands for the expectation value. Quantum mechanics is nonlocal in the sense that there are entangled states that violate the CHSH inequality [133, 134]. On the other hand, the violation may not be observed for a nonlocal state if the settings are not optimal. The Peres-Horodecki criterion [135, 136] helps here by giving a necessary and sufficient condition for a bipartite quantum state to be nonlocal. Consider the geometric representation for a general two-qubit mixed state

$$\rho = \frac{1}{4}[I_4 + (\hat{r}\cdot\vec{\sigma})\otimes I_2 + I_2 \otimes (\hat{s}\cdot\vec{\sigma}) + \sum_{i,j=1}^{3} \gamma_{ij}(\sigma_i \otimes \sigma_j)], \tag{5.7}$$

where I_4 and I_2 are identity in four and two dimensions, respectively. Let μ_1, μ_2 be two eigenvalues of $\Gamma^\dagger\Gamma$, where Γ is the matrix $\{\gamma_{ij}\}$. The CHSH inequality is violated for some pairs of settings of Alice and Rob if and only if

$$B(\rho) > 1, \tag{5.8}$$

where $B(\rho) = \max(\mu_i + \mu_j)$, [137]. For $\rho'_{A,R}$, $\Gamma^\dagger\Gamma = \text{Diag}(\cos^2 r, \cos^2 r, \cos^4 r)$ and hence $B(\rho) = 2\cos^2 r$. For $a \longrightarrow \infty$, $\cos^2 r \longrightarrow \frac{1}{2}$ and hence $B(\rho) \longrightarrow 1$. In Fig. 5.1, the quantity $\frac{B(\rho)}{2}$ (which indicates nonlocality if greater than $\frac{1}{2}$) is plotted as a function of Unruh acceleration. There it is seen that the state perceived by Rob becomes local around $a = 4.6$.

5.3.2 Concurrence

A weaker measure of quantum correlation is entanglement, for example as quantified by concurrence for the general mixed state ρ of two qubits [85] is given by

$$C = \max(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0), \tag{5.9}$$

where λ_i is the square root of the eigenvalues, in decreasing order, of the matrix $\sqrt{\rho}(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)\sqrt{\rho}$, where ρ^* is the complex conjugate of ρ, here $\rho'_{A,R}$, in the eigenbasis of $\sigma_z \otimes \sigma_z$. In Fig. 5.1, the black curve presents C as a function of Unruh acceleration. As $C > 0$ asymptotically in Figure 5.1, the Unruh channel is seen to be not entanglement-breaking. Since the quantum capacity of the Unruh channel, $Q(\mathcal{E}_U)$, is also known to vanish asymptotically, it appears that the behavior of $Q(\mathcal{E}_U)$ is closer to nonlocality than to entanglement, a point that merits further investigation.

5.3.3 Teleportation

From the perspective of application to quantum information processing, the quantumness of correlations in $\rho'_{A,R}$ may be quantified by the fidelity between the input and output states of quantum teleportation [138] using the EPR channel $\rho'_{A,R}$. Here it may be noted that $\rho'_{A,R}$ can also be a mixed state [139], and not necessarily pure, as required in the original teleportation protocol.

Suppose Alice and Rob share an initial mixed state $\rho'_{A,R}$ to be used as channel for teleportation, and Alice has an unknown state $|\psi\rangle$ of a third particle A', to be teleported to Rob. If $\mathcal{B}_k$ denotes the projectors to the Bell basis in the Hilbert space $H_{A,A'}$, then the probability to obtain a given Bell basis outcome k is $q_k = \mathrm{Tr}[(\mathcal{B}_k \otimes I)(|\psi\rangle\langle\psi| \otimes \rho'_{A,R})]$. Based on Alice's classical commutation of k to Rob, he applies a suitable local unitary V_k to obtain state ρ_k. If $|\psi\rangle$ is chosen uniformly from the Bloch sphere then the fidelity of the input and output states of the teleportation is given by

$$F = \int_S d_H\psi \sum_k q_k \mathrm{Tr}(\rho_k |\psi\rangle\langle\psi|), \tag{5.10}$$

where the integral uses a uniform Haar measure. Maximizing over all quadruples of such V_k gives [136]

$$F_{\max} = \frac{1}{2}\left(1 + \frac{1}{3}\left(\mathrm{Tr}\sqrt{\Gamma^\dagger\Gamma}\right)\right), \tag{5.11}$$

where μ_j are eigenvalues of $\Gamma^\dagger\Gamma$. The maximum fidelity attainable in the above teleportation protocol with out the use of entanglement is known to be $\frac{2}{3}$. Thus when $F_{\max} > \frac{2}{3}$, genuine teleportation occurs, and the state $\rho'_{A,R}$ is said to contain quantum correlations.

For $\rho'_{A,R}$, $F_{\max} = \frac{1}{2}\left(1 + \frac{1}{3}\left(2\cos r + \cos^2 r\right)\right)$. In Fig. 5.1, the large-dashed curve presents $F_{\max}$ as a function of Unruh acceleration.

5.3.4 Measurement induced disturbance

Measurement induced disturbance (QMID) quantifies the quantumness of the correlation between the quantum bipartite states shared amongst Alice and Rob. For the given $\rho'_{A,R}$, if ρ'_A and ρ'_R are the reduced density matrices, then the mutual information that quantifies the correlation between Alice and Rob is

$$I = S(\rho'_A) + S(\rho'_R) - S(\rho'_{A,R}), \tag{5.12}$$

where $S(.)$ represents von Neumann entropy. If $\rho'_A = \sum_i \lambda^i_A \Pi^i_A$ and $\rho'_R = \sum_j \lambda^j_R \Pi^j_R$ denotes the spectral decomposition of ρ'_A and ρ'_R, respectively, then the state $\rho'_{A,R}$ after

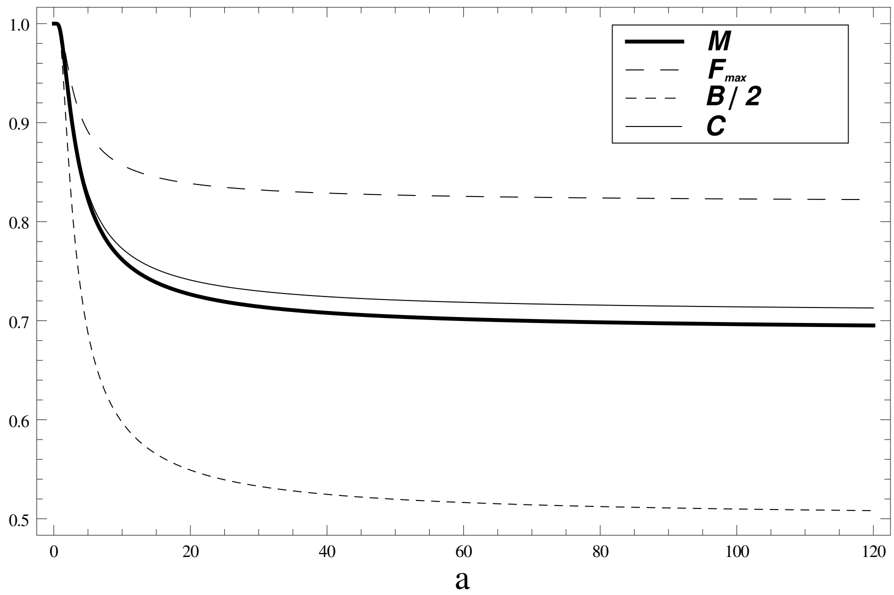

Figure 5.1: Degradation of QMID (M), teleportation fidelity ($F_{\max}$), Bell quantity ($B/2$) and concurrence (C) as a function of Unruh acceleration (a), for $\omega = 0.1$ (in units where $\hbar \equiv c \equiv 1$). The system becomes local ($B/2 < 1/2$) at $a \approx 4.6$, but stays nonclassical with respect to the other parameters ($C > 0, F > \frac{2}{3}, M > 0$)).

measuring in joint basis $\{\Pi_A, \Pi_R\}$ is

$$\Pi(\rho'_{A,R}) = \sum_{i,j} (\Pi_A^i \otimes \Pi_R^j)\rho'_{A,R}(\Pi_A^i \otimes \Pi_R^j). \tag{5.13}$$

Since there exists a local measurement strategy that leaves $\rho'_{A,R}$ unchanged, $\rho'_{A,R}$ can be considered as classical. Then, $I(\Pi(\rho'_{A,R}))$ quantifies the classical correlation. Thus the difference between $I(\rho'_{A,R})$ and $I(\Pi(\rho'_{A,R}))$ should quantify the quantumness of correlation between Alice and Rob. This difference known as QMID,

$$M(\rho'_{A,R}) = I(\rho'_{A,R}) - I(\Pi(\rho'_{A,R})) \tag{5.14}$$

is a measure of quantumness of the correlation. In Figure 5.1, we find that $M > 0$ throughout the range considered, implying that the system remains nonclassical, as expected.

5.4 Geometric characterization of the Unruh channel

Consider the maximally entangled two-mode state in which the second mode is Unruh accelerated. The resulting state, which is the state given in Eq. (5.5) is

$$\rho_U = \frac{1}{2}\begin{pmatrix} \cos^2 r & 0 & 0 & \cos r \\ 0 & \sin^2 r & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \cos r & 0 & 0 & 1 \end{pmatrix}. \tag{5.15}$$

Without the factor 1/2, ρ_U is the Choi matrix $\sum_{j,k} |j\rangle\langle k| \otimes \mathcal{E}_U(|j\rangle\langle k|)$ corresponding to Unruh channel $\mathcal{E}_U$. Spectrally decomposing this, suppose we obtain

$$\rho_U = \sum_{j=0}^{3} |\xi_j\rangle\langle\xi_j| \tag{5.16}$$

, where $|\xi_j\rangle$ are the eogenvectors normalized to the value of the eigenvalue. Then by Chois theorem [7, 65], each $|\xi_j\rangle$ yields a Kraus operator obtained by folding the d^2 (i.e., 4) entries of the eigenvector in to $d \times d$ (2×2) matrix, essentially by taking each sequential d-element segment of $|\xi_j\rangle$, writing it as a column, and then juxtaposing these columns to form the matrix as explained in Sec. 2.3 of Chapter 2.

The two eigenvectors corresponding to the two non-vanishing eigenvalues are found to be

$$\begin{aligned} |\xi_0\rangle &= (\cos r, 0, 0, 1) \\ |\xi_1\rangle &= (0, \sin r, 0, 0). \end{aligned} \tag{5.17}$$

From these, we have the Kraus representation for $\mathcal{E}_U$ as

$$K_1^U = \begin{pmatrix} \cos r & 0 \\ 0 & 1 \end{pmatrix}; \quad K_2^U = \begin{pmatrix} 0 & 0 \\ \sin r & 0 \end{pmatrix}, \tag{5.18}$$

whereby

$$\mathcal{E}_U(\rho) = \sum_{j=1,2} K_j^U \rho \left(K_j^U\right)^\dagger, \tag{5.19}$$

with the completeness condition

$$\sum_{j=1,2} \left(K_j^U\right)^\dagger K_j^U = \mathbb{I}. \tag{5.20}$$

While this is formally similar to the operator elements of an amplitude damping (AD) channel [11], there is a fundamental difference, as discussed in Section 5.6.

For an initial pure qubit state $\rho = |0\rangle\langle 0|\cos^2\frac{\theta}{2} + |0\rangle\langle 1|e^{i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} + |1\rangle\langle 0|e^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} + |1\rangle\langle 1|\sin^2\frac{\theta}{2}$, the action of the Unruh channel is

$$\mathcal{E}_U(\rho) = \begin{pmatrix} \cos^2 r\cos^2\frac{\theta}{2} & \cos r e^{i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ \cos r e^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin^2 r\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} \end{pmatrix}. \tag{5.21}$$

For infinite time and acceleration, setting $\cos^2 r = \frac{1}{2}$ in Eq. (5.21), the asymptotic state is

$$\rho_\infty = \begin{pmatrix} \frac{1}{2}\cos^2\frac{\theta}{2} & \frac{1}{\sqrt{2}}e^{i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ \frac{1}{\sqrt{2}}e^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} & \frac{1}{2}\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} \end{pmatrix}, \tag{5.22}$$

with Bloch vector

$$\hat{n}^\infty(\theta,\phi) \equiv (\hat{x},\hat{y},\hat{z}) = \left(\frac{\cos\phi\sin\theta}{\sqrt{2}}, \frac{\sin\phi\sin\theta}{\sqrt{2}}, -\sin^2\frac{\theta}{2}\right),$$

which shows that the Bloch sphere gets mapped to the inscribed solid object shown in Figure 5.2, whose south pole osculates with that of the Bloch sphere, while the north pole (corresponding to initial $\theta = 0$) is located midway between the Bloch sphere center and south pole: $\hat{n}^\infty(0,\phi) = (0,0,0)$ while $\hat{n}^\infty(\pi,\phi) = (0,0,-1)$. This is thus a kind of an interrupted AD channel. By virtue of linearity of the map, it follows that the maximally mixed state maps to the Bloch vector which is the average of the above two, being

$$\hat{n}^\infty(\mathbb{I}) = (0,0,-\frac{1}{2}). \tag{5.23}$$

Thus the channel is non-unital, with the new Bloch representation of the initially maximally mixed state being

$$\mathcal{E}_U^\infty(\mathbb{I}/2) = \begin{pmatrix} 1/4 & 0 \\ 0 & 3/4 \end{pmatrix}. \tag{5.24}$$

The geometry of the contracted, noisy version of the Bloch sphere can be inferred to be

$$R(\theta) = |\hat{n}^\infty(\theta,\phi) - \hat{n}^\infty(\mathbb{I})| = \frac{\sqrt{3-\cos 2\theta}}{2\sqrt{2}}. \tag{5.25}$$

From $\hat{n}^\infty(\theta,\phi)$, the area of the circular section of the oblate spheroid is $A(\theta) = \pi(\hat{x}^2 + \hat{y}^2) = \frac{\pi}{2}\sin^2\theta$. The corresponding volume is then, using the z-component in $\hat{n}^\infty(\theta,\phi)$, $V = \int_0^\pi A(\theta)\frac{\sin\theta}{2}d\theta = \frac{\pi}{3}$. Since the volume of the Bloch sphere is $V_0 \equiv \frac{4\pi}{3}$, it follows that the volume contraction factor of the Bloch sphere under the relativistic channel is $K \equiv \frac{1}{4}$. The eccentricity of the oblate sphere is given by

$$e = \sqrt{1 - \frac{a^2}{b^2}} = \frac{1}{\sqrt{2}}, \tag{5.26}$$

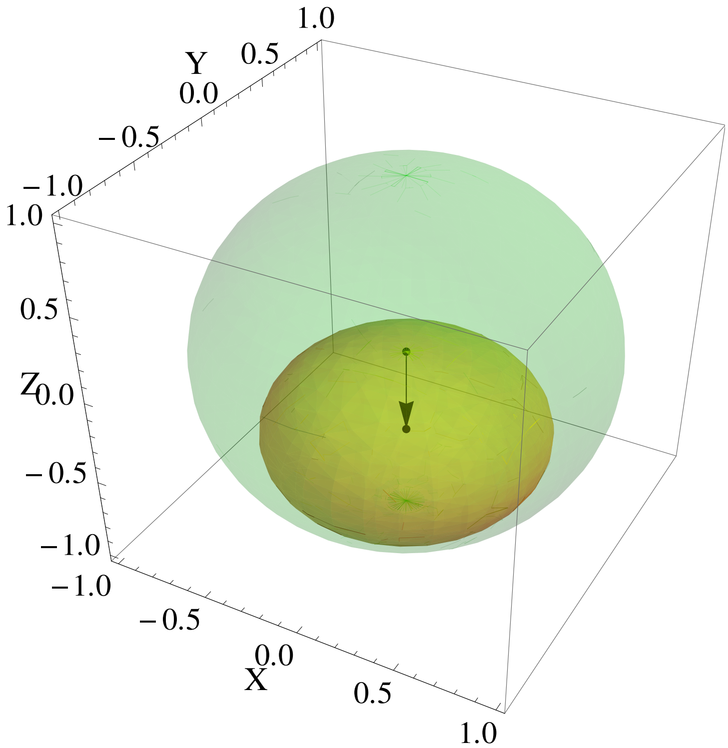

Figure 5.2: Under $\mathcal{E}_U$, the Bloch sphere (the outer sphere) is shrunk asymptotically to the inner solid oblate spheroid (eccentricity $e = \frac{1}{\sqrt{2}}$) by a volume factor $\frac{1}{4}$, centered at $(0, 0, -\frac{1}{2})$ (surface described by $\hat{n}^\infty(\theta, \phi)$.

where a and b are the semi-major and semi-minor axis, which are seen from the form of $R(\theta)$ to be $\frac{1}{\sqrt{2}}$ (corresponding to $\theta = \frac{\pi}{2}$) and $\frac{1}{2}$ (corresponding to $\theta = 0, \pi$), respectively.

5.5 Composability of AD and Unruh channels

The similarity of the Unruh channel $\mathcal{E}_U$ to the AD channel means that, for sufficiently small damping, from local observations on the degradation of quantum information, one cannot say whether one is in an accelerated frame and looking at Minkowski vacuum, or in an inertial frame, interacting dissipatively with a Minkowski vacuum. As one aspect of this equivalence, a genuine AD channel may be composed with an Unruh channel, resulting in an AD channel. Consider a state

$$\rho = \begin{pmatrix} \alpha & \beta \\ \beta^* & 1 - \alpha \end{pmatrix}, \tag{5.27}$$

with real α. The action of Unruh noise yields

$$\rho_U = \mathcal{E}_U(\rho) = \begin{pmatrix} \alpha \cos^2 r & \beta \cos r \\ \beta^* \cos r & 1 - \alpha \cos^2 r \end{pmatrix}, \tag{5.28}$$

while that of an AD channel on state ρ_U above is

$$\mathcal{E}_{UV}(\rho) = \left(\begin{array}{ll} \alpha \cos^2 r(1-\gamma) & \beta \cos r\sqrt{1-\gamma} \\ \beta^* \cos r\sqrt{1-\gamma} & 1+\alpha(-\cos^2 r + \gamma\cos^2 r) \end{array} \right). \tag{5.29}$$

Setting $\gamma'' = \gamma\cos^2 r + \sin^2 r$, the Kraus operators of noise $\mathcal{E}_{UV}$ are

$$\left(\begin{array}{cc} \sqrt{1-\gamma''} & 0 \\ 0 & 1 \end{array} \right), \quad \left(\begin{array}{cc} 0 & 0 \\ \sqrt{\gamma''} & 0 \end{array} \right), \tag{5.30}$$

which has the form of an AD channel. The same observation holds if the order of the AD and Unruh channels are inverted. This closure under composition is a manifestation of the semi-group property of AD channels.

5.6 Physical interpretation

The origin of the Unruh channel is, as noted, quite different from a conventional noise channel. We discuss below some aspects of this difference from a quantum information theoretic perspective.

In spite of the formal similarity of the Unruh effect to the AD channel, there are two differences from a conventional AD channel. One is that the damping parameter $\sin r$ in Eq. (5.18) can never go to 1. This means that the state remains logically reversible, and Rob can in principle reconstruct the initial state tomographically. For an AD channel, which is derived using the Born-Markov and rotating-wave approximations, one can invoke the fluctuation-dissipation theorem to require the convergence of the Bloch sphere at initial time to a single fixed point of the evolution asymptotically. The finite contraction factor derived in Section 5.4 thwarts this behavior.

Another point is that the rank of ρ_U in Eq. (5.5) is 2, and not 4. This is reflected in the fact that there are only 2 canonical Kraus operators, given in Eq. (5.18). This is somewhat surprising because we ascribe a finite Unruh temperature τ to the bath observed by the accelerated observer, which is expected to correspond to ρ_U of rank 4 [67]. This entails that the Unruh noise corresponds to only a single Lindblad channel corresponding to a de-excitation process. Now if the environment were a conventional finite-temperature bath, then we should have also the Lindblad excitation channel corresponding to the qubit absorbing a photon from this bath. The lack of the excitation channel here suggests that, from the physical perspective of an inertial detector, the Unruh background interacts as a vacuum, even though Rob views it as a thermal Rindler state in his own reference frame.

In the case of a conventional noise due to environmental decoherence, there is no simple way to reverse the noise. 'Turning off' the interaction with the environment leaves the qubit in a mixed state, while its further evolution is unitary. By contrast, the relativistic transformation corresponding to turning off the Unruh drive must be one that returns the thermal state to the maximally entangled form (5.3) with $a = 0$ (ignoring the effects of any local unitaries, or interaction with a genuine bath in the interim time). We expect that this inverse Unruh channel Φ_U^{-1} cannot be CP because, given the Unruh channel's formal similarity to the AD channel, the latter proceeds by setting up an entanglement between a system and its environment. Thus Φ^{-1} would formally be like a map on a state that is initially entangled, and thus in general lead to an NCP map.

Given the Unruh channel Φ_U, we require its inverse Φ_U^{-1} such that

$$\Phi_U^{-1}\left[\Phi_U(\rho)\right] = \rho. \tag{5.31}$$

Solving the above, we find that the required inverse channel is given by the *difference* of two CP (but non-trace preserving) maps, determined by the single Kraus operators, respectively:

$$K_1^\epsilon = \begin{pmatrix} 1/\cos r & 0 \\ 0 & 1 \end{pmatrix}, \quad K_2^\epsilon = \begin{pmatrix} 0 & 0 \\ \tan r & 0 \end{pmatrix}. \tag{5.32}$$

Thus, the inverse-Unruh channel

$$\Phi_U^{-1}(\rho) = K_1^\epsilon \rho K_1^\epsilon - K_2^\epsilon \rho K_2^\epsilon, \tag{5.33}$$

where the intervening *negative* sign is the tell-tale indication of Φ_U^{-1} being a non-CP map, a general Hermitian map that can formally apply even when the initial correlations between the system and the bath are non-classical [113, 114].

It may be verified that Φ_U^{-1} satisfies the completeness condition:

$$(K_1^\epsilon)^\dagger K_1^\epsilon - (K_2^\epsilon)^\dagger K_2^\epsilon = \mathbb{I}. \tag{5.34}$$

That the inverse-Unruh channel Φ_U^{-1} is not CP can be seen by its action on the maximally entangled state Ψ :

$$(\Phi_U^{-1} \otimes I)\Psi = \frac{1}{2}\begin{pmatrix} 1/\cos^2 r & 0 & 0 & 1/\cos r \\ 0 & -\tan^2 r & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/\cos r & 0 & 0 & 1 \end{pmatrix}. \tag{5.35}$$

The corresponding eigenvalues are $\{0, 0, (3+\cos 2r)/(4\cos^2 r), -\tan^2 r/2\}$. Since one of the eigenvalues $-\tan^2 r/2$ is negative, Φ_U^{-1} is an NCP map. Also note that $\mathrm{Tr}((\Phi_U^{-1} \otimes I)\Psi) = 1$, a consequence of Eq. (5.34). Thus the Φ_U^{-1} is trace preserving but not completely positive.

5.7 Discussion and conclusions

The Unruh channel for a fermionic qubit was shown to mimic the effect of an AD channel (of rank 2), even though the Unruh effect is associated with a non-zero temperature (suggesting a generalized amplitude damping channel, of rank 4). The damping parameter $\sin r$ attains the maximal value of $\frac{1}{\sqrt{2}}$, and not the maximum possible value of 1. Therefore, under infinite-time evolution, the Bloch sphere is contracted to a finite volume rather than to a point. Since the quantum channel capacity is known to vanish in both the fermionic and bosonic Unruh effects, as does nonlocality, it appears that channel capacity is better reflected in nonlocality than in entanglement. We construct the inverse Unruh channel, namely the map that returns Rob's noisy qubit to its initial pure form, and show that it is NCP.