CHAPTER II

EXTENSION OF NON-STANDARD ALMOST PERIODIC FUNCTIONS

2.1. In this chapter we shall prove an extension theorem of almost periodic functions in the context of Non-standard Analysis. We shall also give a necessary and sufficient condition that a set of Non-standard almost periodic functions (n.s.a.p.f.) be compact.

In order to prove the theorem, certain idea from the theory of Non-standard Analysis will be needed. We begin, therefore, with some abstract definitions and theorems and specialize them to apply to the problems at hand.

2.2. METRIC SPACES\(^{1)}\) : Let \( T \) be a metric space with distance function \( \delta(x,y) \). We assume that \( T \) and \( \mathbb{R} \) (\( \mathbb{R} \) are real numbers) are embedded, simultaneously, in a full structure \( *\mathbb{M} \) of \( \mathbb{M} \). We can develop the non-standard theory of the given metric space.

The topology of \( T \) is defined, by specifying as base the set of all open balls \( B \), where \( B = \{ q : \delta(p,q) < r \} \) for a point \( p \) in \( T \) and a positive real \( r \).

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\( 1) \) Robinson, A. \( (1) \)
For any point \( p \) in \( \ast T \) we define the monad of \( p \), \( \mathcal{M}(p) \) as the set of all points \( q \) such that \( \rho(p, q) \) is infinitesimal. Distinct monads are disjoint. We write \( p \bowtie q \) if \( \rho(p, q) \bowtie 0 \) i.e., if \( q \in \mathcal{M}(p) \).

A point \( p \) in \( \ast T \) will be called finite if there exists a standard point \( q \) such that \( \rho(p, q) \) is finite.

A metric space is bounded if there exists a real number \( m \), such that \( \rho(p, q) \leq m \) for all points \( p \) and \( q \) in the space.

**Theorem 2.2.1.** A metric space is bounded if and only if all points of \( \ast T \) are finite.

**Theorem 2.2.2.** (standard) A compact metric space is bounded.

2.3. **Sequences of functions. Compact mapping**: Let \( \{ f_n(x) \} \) be a sequence of functions which are defined on a set of points \( B \) in the metric space \( T \) and whose range is the metric space \( S \). According to the classical definition \( \{ f_n(x) \} \) converges on \( B \), to a function \( f(x) \), if for every positive \( \varepsilon \) there exists number \( \delta = \delta(\varepsilon) \) such that \( \rho(f(p), f_n(p)) < \varepsilon \) for all \( p \) in \( B \) and for all \( n > \delta \).
Theorem 2.3.1. The sequence of standard functions \( \{ f_n(x) \} \) converges to the standard function \( f(x) \) uniformly on \( BC \subseteq T \) (and hence on \( *B ) \) if and only if \( f(p) \preceq f_n(p) \) for all \( p \in *B \) and for all infinite \( n \).

Working with standard notions, let \( \{ f_n(x) \} \) be a sequence of functions which map a metric space \( T \) into a compact metric space \( S \). Suppose the sequence is uniformly equi continuous on \( T \). That is to say, for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( \rho(f_n(p), f_n(q)) < \epsilon \) provided \( \rho(p, q) < \delta \), \( n = 0, 1, 2, \ldots \).

Passing from \( M \) to \( *M \), we see that the condition of uniform equicontinuity affirms that for given standard \( \epsilon > 0 \) there exists a standard \( \delta > 0 \) such that \( \rho(p, q) < \delta \) implies \( \rho(f_n(p), f_n(q)) < \epsilon \) for any points \( p, q \) in \( *T \) and for any natural number \( n \), finite or infinite. This shows that \( f_n(x) \) is uniformly \( S \)-continuous on \( *T \).

Theorem 2.3.2. (Standard Ascoli): Let \( \{ f_n(x) \} \) be an equi continuous sequence of functions which map a compact metric space \( T \) into a compact metric space \( S \). Then there exists a sequence of natural numbers \( n_j \), \( n_0 < n_1 < n_2 < \ldots \) such that the sequence \( \{ f_{n_j}(x) \} \), \( j = 0, 1, 2, \ldots \) converges uniformly on \( T \) to a function \( F(x) \) which is uniformly continuous on \( T \).
REMARK: If \( f_n(x) \) are real valued, Ascoli's theorem is proved on the assumption that the functions are uniformly and collectively bounded. The appropriate space \( S \) is provided by some finite interval of real numbers.

**Theorem 2.3.3.** (standard): Let \( \{ f_n(x) \} \) be a sequence of compact mapping from a metric space \( T \) into a complete metric space \( S \). Suppose that \( \{ f_n(x) \} \) converges to a function \( f(x) \) uniformly on \( T \). Then \( f(x) \) is compact.

We now give an extension theorem of almost periodic functions in the context of Non-standard Analysis.

Let \( E_1 \) be an arbitrary metric space, \( E_2 \) be a complete metric space. We consider the set of all bounded functions \( f : E_1 \rightarrow E_2 \). Let us assume that \( E_1, E_2 \) and \( M \) are embedded simultaneously in a full structure \( N \). Let \( *N \) be the enlargement of \( N \). Then it is easy to see that \( *N \) contains enlargements \( *E_1, *E_2 \) and \( *M \) of \( E_1, E_2 \) and \( M \) respectively.

For \( f, g \in M \), we define its distance by

\[
\rho(f, g) = \sup_{x \in *E_1} d(f(x), g(x))
\]
It can be easily seen that \( \rho(f,g) \) satisfies the well known axioms of a metric. Then \( M \) becomes a complete metric space as \( E_2 \) is complete.

**DEFINITION 2.3.4.** A function \( f(x) \ (x \in \mathbb{E}_1) \) is said to be non-standard almost periodic function (n.s.a.p.f.), if it is continuous on \( E_1 \) and if for every standard \( \varepsilon > 0 \), there exists a standard \( \delta > 0 \) such that every interval of length \( \delta \) contains an element \( y = y(\varepsilon) \in \mathbb{E}_1 \) for which the relation

\[ d(f(x+y), f(x)) < \varepsilon \]

holds for all \( x \in \mathbb{E}_1 \).

Such an element \( y(\varepsilon) \) is called standard \( \varepsilon \)-periodic of the function \( f \).

It can easily be seen that every n.s.a.p.f. is bounded in metric space and therefore belong to the space \( M \).

**DEFINITION 2.3.5.** A set \( K \) of a metric space \( X \) is called standard \( \varepsilon \)-net for the set \( M \) of the space, if for every \( f \in M \), there exists an standard element \( f_\varepsilon \in K \) such that

\[ \rho(f, f_\varepsilon) < \varepsilon. \]

2) Sobolev, V.J. and Lusternik, L.A.
Now we shall state and prove an extension of Hausdorff’s theorem$^3$).

**Theorem 2.3.6.** The necessary condition for a set $M$ of a metric space $X$ is compact is that for every standard $\varepsilon > 0$, there exists a finite standard $\bar{\varepsilon}$-net for $M$. If, in addition, the space is complete, then it also satisfies the sufficient condition.

**Proof:** Necessary Condition: We assume that $M$ is compact. Let $f_1$ be an arbitrary element of $M$. If $\rho(f, f_1) < \varepsilon$ for each standard $\varepsilon$ and for all $f \in *M$, then we have found a finite standard $\varepsilon$-net. If, however, this is not the case, then there exists an element $f_2 \in M$ such that $\rho(f_1, f_2) \geq \varepsilon$. The statement 'for every element $f \in *M$ either $\rho(f_1, f) < \varepsilon$ or $\rho(f_2, f) < \varepsilon$' is true in $*M$ and hence, it is true in $M$. Then we have found a standard $\varepsilon$-net. If, however, this does not hold then there exists an element $f_3$ such that $\rho(f_1, f_3) \geq \varepsilon$, $\rho(f_2, f_3) \geq \varepsilon$

continuing in this way we determine an infinite sequence $\{f_n\}$ in $M$ such that $\rho(f_1, f_j) \geq \varepsilon$

$\textbf{3)}$ Sobolev, V.J. and Lusternik, L.A. (1)
for \( i \neq j \). There arises two possibilities. Either the procedure ceases after the \( k \)-th step, i.e., for every \( f \in \mathbb{M} \), one of the relations

\[
F(f, f_i) < \varepsilon, \ i = 1, 2, \ldots, k
\]

holds in \( \mathbb{M} \) and hence in \( \mathbb{N} \), or we can continue indefinitely the present procedure. In first case we would make a finite standard \( \varepsilon \)-net of \( \mathbb{M} \). The second possibility can not occur, since otherwise we would obtain an infinite sequence \( \{ f_n \} \) in \( \mathbb{M} \) such that

\[
F(f_i, f_j) \geq \varepsilon
\]

for \( i \neq j \). Evidently \( \{ f_n \} \) would have no limit point in \( \mathbb{M} \) and hence in \( \mathbb{N} \) which would contradict the assumption that \( \mathbb{M} \) is compact.

**Sufficient Condition**: We assume that \( X \) is complete and that to every standard \( \varepsilon > 0 \), there exists a finite standard \( \varepsilon \)-net in \( \mathbb{M} \). For every \( \varepsilon \) \( \in \mathbb{N} \) we construct a finite standard \( \varepsilon \)-net \( \left[ f_{1}^{(n)}, f_{2}^{(n)}, \ldots, f_{k}^{(n)} \right] \) for the set \( \mathbb{M} \). Then we choose arbitrary infinite subset \( S \) of \( \mathbb{M} \). Around every element \( f_{1}^{(1)}, f_{2}^{(1)}, \ldots, f_{k}^{(1)} \) of the standard \( \varepsilon_1 \)-net,
we place a closed sphere \( B_{\varepsilon_1} \) such that
\[
\rho(f, g) \leq 2 \varepsilon_1
\]
for every \( f, g \in B_{\varepsilon_1} \). Then every element of \( S \) is contained in one of these spheres. Since the number of the spheres is finite there is at least one sphere containing an infinite set of elements of \( S \). We denote this subset of \( S \) by \( S_1 \).

Around every element \( f_1(2) \), \( f_2(2) \), \( \ldots \), \( f_k(2) \) of the \( \varepsilon_2 \)-net we place a closed sphere \( B_{\varepsilon_2} \) such that
\[
\rho(f, g) \leq 2 \varepsilon_2
\]
for every \( f, g \in B_{\varepsilon_2} \). By the same reasoning as above, we obtain an infinite set \( S_2 \), situated in one of the constructed sphere \( B_{\varepsilon_2} \). Continuing this procedure, we obtain a sequence of subsets of \( S \): \( S_1 \supseteq S_2 \supseteq \ldots \supseteq S_n \) where the subset \( S_n \) is contained in a closed sphere \( B_{\varepsilon_n} \).

Now we choose an element \( f_1 \in S_1 \), an element \( f_2 \in S_2 \), different from \( f_1 \) an element \( f_3 \in S_3 \), different from \( f_1 \) and \( f_2 \) and so on and we obtain a sequence of
elements $S_\omega = \{ f_1, f_2, \ldots, f_n, \ldots \}$ which is a standard Cauchy net. $f_n \in S_n$ and $f_{n+p} \in S_{n+p}$ for every infinite natural number $p$ implies

$$\rho(f_{n+p}, f_n) \leq 2 \epsilon_n \to 0 \text{ as } n \to \infty.$$ 

By the hypothesis, the space $X$ is complete, so the sequence $S_\omega$ converges to an element $f \in X$. This proves the compactness of the set $M$.

**COROLLARY 2.3.7.** A set $M$ of a complete metric space $X$ is compact, if and only if, there exists a compact standard $\epsilon$-net of $M$ for every standard $\epsilon > 0$.

**PROOF:** Let $K$ be a compact standard $\frac{\epsilon}{2}$-net of the set $M$. Applying the theorem 2.3.5. to $K$, we find that there exists a finite standard $\frac{\epsilon}{2}$-net $K_0$ of $K$. Then $K_0$ is a finite standard $\epsilon$-net for $K$. For every $f \in \mathcal{M}$, there exists elements $f_1 \in K$ such that

$$\rho(f, f_1) < \frac{\epsilon}{2}.$$ 

Consequently, for every element $f \in \mathcal{M}$, there exists a standard $f_2$, such that

$$\rho(f, f_2) \leq \rho(f, f_1) + \rho(f_1, f_2) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$
i.e., \( k_0 \) is a finite standard \( \varepsilon \)-net for \( M \). Since the space \( X \) is complete, we conclude by theorem 2.3.6. that \( M \) is compact.

We shall now prove the following:

**Theorem 2.3.8.** A set \( P \) of n.s.a.p.i. is compact in the sense of metric of \( M \) if and only if:

(i) The functions of the set \( P \) are uniformly bounded and equicontinuous.

(ii) For every standard \( \eta > 0 \), there exists a number \( \eta \) in the interval \((a,a+\eta)\) which is an standard \( \eta \)-period for all functions of the set \( P \).

**Proof:** Necessary Condition: (i) The proof of (i) can easily be deduced by the corresponding assertion of Ascoli's theorem (Theorem 2.3.2.).

(ii): Since \( P \) is compact, for every standard \( \eta > 0 \), there exists a finite standard \( \eta/3 \)-net for \( P \). Let us denote these elements by \( f_1, f_2, \ldots, f_n \). Then for every element \( f \) in \( *P \), there exists an element \( f_i (1 \leq i \leq n) \) in \( P \) such that

\[
\hat{f}(f, f_i) < \frac{\eta}{3}.
\]
There exists a number $h > 0$ in every interval $(a, a+\ell)$ which is an standard $\frac{\eta}{3}$ -net, for which the following statement is true in \$P$ and hence in $P$. "For every $x \in \#E_1$ and for all $f_i$ ($i = 1, 2, \ldots, n$),

$$d\left(f_i(x + h), f_i(x)\right) \leq \frac{\eta}{3}.$$ 

Since $f_1$ is an standard $\frac{\eta}{3}$ -net for $P$, there exists for every element $f \in \#P$, an element $f_1$ such that

$$d(f(x + h), f(x)) \leq d(f(x + h), f_1(x + h)) + d(f_1(x + h), f_1(x)) + d(f_1(x), f(x))$$

$$\leq \frac{\eta}{3} + \frac{\eta}{3} + \frac{\eta}{3} = \eta.$$ 

for $x \in \#E_1$.

Therefore $h$ is an standard $\eta$ -period for all $f$ in \$P$. This completes the proof for the necessary part of the theorem.

**Sufficient Condition**: We assume that for a set $P$ of n.s.a.p. $f$. (i) and (ii) are satisfied and we choose a standard $\eta > 0$.

Let $\ell = l(\eta)$ be determined such that every interval of length $\ell$ contains an standard $\eta$ -period for every $f$ in \$P$. To every $f$ in \$P$, we associate a function $\widehat{f}$ defined by
\[ f(x) = \begin{cases} 
  f(x) & \text{if } -l < x \leq l \\
  f(x - r_n) & \text{if } n \leq x < (n+1)l, \quad n = 1, 2, \ldots \\
  n \leq x < (n+1)l, \quad n = -1, -2, \ldots 
\end{cases} \]

where \( r_n \) is an standard \( \eta \)-period for all \( f \) in \( P \), and its period lies in the interval \( (nl, (n+1)l) \).

We denote the set of all \( \bar{f} \) by \( P_\eta \). Then by theorem 2.3.3, \( P_\eta \) is compact in the sense of uniform convergence in the interval \([-l, l]\). But since \( x - r_n \in [-l, l] \), then by the definition of \( \bar{f} \), a sequence of these functions which converges uniformly in \([-l, l]\) also converges in metric space \( E_1 \).

For arbitrary \( f \in \ast P \) and the corresponding \( \bar{f} \in P_\eta \),

\[ d(f(x), \bar{f}(x)) = 0 \text{ if } -l < x < l \]

and

\[ d(f(x), \bar{f}(x)) = d(f(x), f(x - r_n)) \]

if \( \begin{cases} 
  -l < x < l & (n=1, 2, \ldots) \\
  n \leq x < (n+1)l & (n=-1, -2, \ldots) 
\end{cases} \)
Since \( r_n \) is an standard \( \eta \)-period for \( f \), therefore, for any arbitrary \( x \), we have

\[ d \left( f(x), \bar{f}(x) \right) < \eta. \]

Hence the compact set \( P_\eta \) forms an standard \( \eta \)-net for \( P \) in the space \( M \). By corollary 2.3.7., \( P \) is compact and therefore, we have shown that the condition (i) and (ii) are sufficient.