PRELIMINARY CONCEPTS OF FUZZY SET THEORY

Some of the basic concepts of the fuzzy sets and fuzzy Mathematical Programming [Bector and Chandra(2005)] used in solving fuzzy optimization problem are defined as.

Fuzzy set: Let $X$ be the universe whose generic element is denoted by $x$. A fuzzy set $A$ in $X$ is a function $A : X \rightarrow [0,1]$.

Membership Function: Fuzzy set $A$ is characterized by its membership function $\mu_A : X \rightarrow [0,1]$ which, associates with each $x$ in $X$, a real number $\mu_A(x)$ in $[0,1]$ representing the grade of membership of $x$ in $A$ and is interpreted as the degree to which $x$ belongs to $A$. An element mapping to the value 0 means that the member is not included in the given set, 1 describes a fully included member. Values strictly between 0 and 1 characterize the fuzzy members.

Support of a fuzzy set: The support of a fuzzy set $A$ in $X$, denoted by $S(A)$, is the crisp set given by $S(A) = \{x \in X : \mu_A(x) > 0\}$.

Normal Fuzzy Set: The height $h(A)$ of a fuzzy set $A$ is defined as

$$h(A) = \sup_{x \in X} \mu_A(x)$$

If $h(A)=1$, then the fuzzy set $A$ is called a normal fuzzy set, otherwise subnormal which can be normalized as $\frac{\mu_A(x)}{h(A)}, x \in X$.
**Standard Union:** The standard union of two fuzzy sets $A$ and $B$ is a fuzzy set $C$ whose membership function is given by $\mu_C(x) = \max(\mu_A(x), \mu_B(x))$ for all $x \in X$. This we express as $C = A \cup B$.

**Standard Intersection:** The standard intersection of two fuzzy sets $A$ and $B$ is a fuzzy set $D$ whose membership function is given by $\mu_D(x) = \min(\mu_A(x), \mu_B(x))$ for all $x \in X$. This we express as $C = A \cap B$.

**α-cut:** The $\alpha$-cut of the fuzzy set $A$ in $X$ is the crisp set $A_\alpha$ given by $A_\alpha = \{x \in X : \mu_A(x) \geq \alpha\}$ where $\alpha \in (0,1]$.

**Convex fuzzy set:** A fuzzy set $A$ in $\mathbb{R}^n$ is said to be a convex fuzzy set if its $\alpha$-cuts $A_\alpha$ are (crisp) convex sets for all $\alpha \in (0,1]$.

**Theorem 1:** A fuzzy set $A$ in $\mathbb{R}^n$ is a convex fuzzy set iff for all $x_1, x_2 \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$

$$\mu_A(\lambda x_1 + (1-\lambda) x_2) \geq \min\left(\mu_A(x_1), \mu_A(x_2)\right)$$

**Zadeh’s Extension Principle:** Let $f : X \rightarrow Y$ be a crisp function and $F(X)$ (respectively $F(Y)$) be the set of all fuzzy sets of $X$ (respectively $Y$). The function $f : X \rightarrow Y$ induces two functions $f : F(X) \rightarrow F(Y)$ and $f^{-1} : F(Y) \rightarrow F(X)$. The extension principle gives formulas to compute the membership function of fuzzy sets $f(A)$ in $Y$ (respectively $f^{-1}(B)$ in $X$) in terms of membership function of fuzzy set $A$ in $X$ (respectively $B$ in $Y$). The principle states that

1. $\mu_{f(A)}(y) = \sup_{x \in X, f(x) = y} (\mu_A(x)), \forall A \in F(X)$

2. $\mu_{f^{-1}(B)}(x) = \mu_B(x), \forall B \in F(Y)$

If the function $f$ maps a $n$-tuple in $X$ to a point in $Y$ i.e. $X = X_1 \times X_2 \times \ldots \times X_n$ and $f : X \rightarrow Y$ given by $y = f(x_1, x_2, \ldots, x_n)$. Let $A_1, A_2, \ldots, A_n$ be $n$ fuzzy sets in $X_1, X_2, \ldots$, 

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$X_n$ respectively. The extension principle of Zadeh allows to extend the crisp function $y = f(x_1, x_2, ..., x_n)$ to act on $n$ fuzzy subsets of $X$, namely $A_1, A_2, ..., A_n$ such that $B = f(A_1, A_2, ..., A_n)$

The fuzzy set $B$ is defined as

$$B = \left\{ (y, \mu_B(y)) : y = f(x_1, x_2, ..., x_n), (x_1, x_2, ..., x_n) \in X_1 \times ... \times X_n \right\}$$

and $\mu_B(y) = \sup_{x \in X, y = f(x)} \min\left(\mu_{A_1}(x_1), ..., \mu_{A_n}(x_n)\right)$

**Fuzzy number:** A fuzzy set $A$ in $\mathbb{R}$ is called a fuzzy number if it satisfies the following conditions:-

(i) $A$ is normal.

(ii) $A$ is convex.

(iii) $\mu_A$ is upper semi continuous.

(iv) Support of $A$ is bounded.

**Theorem 2** Let $A$ be a fuzzy set in $\mathbb{R}$. Then $A$ is a fuzzy number if and only if there exists a closed interval (which may be singleton) $[a, b] \neq \emptyset$ such that where

$$\mu_A(x) = \begin{cases} 1, & x \in [a, b] \\ l(x), & x \in (-\infty, a) \\ r(x), & x \in (b, \infty) \end{cases}$$

(i) $l : (-\infty, a) \to [0, 1]$ is non-decreasing, continuous from the right and $l(x) = 0$ for $x \in (-\infty, w_1), w_1 < a$

(ii) $r : (b, \infty) \to [0, 1]$ is non-increasing, continuous from the left and $r(x) = 0$ for $x \in (w_2, \infty), w_2 > b$ and $\mu_A(x)$ is called ‘Membership Function’ of fuzzy set $A$ on $\mathbb{R}$.
Special Types of fuzzy numbers

Triangular fuzzy number (TFN): A fuzzy number \( A \) denoted by the triplet \( A = (a_l, a, a_u) \) having the shape of a triangle is called a TFN if its membership function \( \mu_A(x) \) is given by

\[
\mu_A(x) = \begin{cases}
0 & \text{for } x < a_l \\
\frac{x-a_l}{a-a_l} & \text{for } a_l \leq x \leq a \\
\frac{a_u-x}{a_u-a} & \text{for } a < x \leq a_u \\
0 & \text{for } a_u \leq x
\end{cases}
\]

The \( \alpha \)-cut of a TFN is the closed interval

\[
A_\alpha = [a^L_\alpha, a^R_\alpha] = [(a - a_l)\alpha + a_l, (a - a_u)\alpha + a_u], \alpha \in (0,1].
\]

Trapezoidal membership function (TrFN): A fuzzy number \( A \) denoted by the quadruplet \( A = (a_l, a, a, a_u) \) having the shape of a trapezoid is called a TrFN if its membership function \( \mu_A(x) \) is given by

\[
\mu_A(x) = \begin{cases}
0 & \text{for } x < a_l \\
\frac{x-a_l}{a-a_l} & \text{for } a_l \leq x < a \\
1 & \text{for } a \leq x \leq \bar{a} \\
\frac{a_u-x}{a_u-a} & \text{for } \bar{a} < x \leq a_u \\
0 & \text{for } a_u < x
\end{cases}
\]

The \( \alpha \)-cut of a TrFN is the closed interval

\[
A_\alpha = [a^L_\alpha, a^R_\alpha] = [(a - a_l)\alpha + a_l, (\bar{a} - a_u)\alpha + a_u], \alpha \in (0,1].
\]
Ranking of fuzzy numbers

Ranking of fuzzy number is an important issue in the study of fuzzy set theory and is useful in various applications. Fuzzy mathematical programming is one of the applications. There are numerous methods proposed in literature for ranking the fuzzy numbers such as ranking function (index) approach, k-preference index approach and possibility theory approach, useful in particular context but not in general. We use the Ranking function (index) approach for ranking the fuzzy numbers for our problem and the same is illustrated below.

Ranking function (index) approach

Let \( \mathbb{N}(\mathbb{R}) \) be the set of all fuzzy numbers in \( \mathbb{R} \) and \( A, B \in \mathbb{N}(\mathbb{R}) \). Define a function \( F: \mathbb{N}(\mathbb{R}) \rightarrow \mathbb{R} \), called a ranking function or ranking index, where \( F(A) \leq F(B) \) is equivalent to \( A(\leq)B \). Following indices are proposed by Yager (1981).

(i) \( F_1(A) = \left( \int_{a_l}^{a_u} \int_{a_l}^{a_u} \frac{\mu_{A(x)}}{x} dx \right) \), where \( a_l \) and \( a_u \) are the lower and upper limits of the support of \( A \). The value \( F_1(A) \) represents the centroid of the fuzzy number \( A \in \mathbb{N}(\mathbb{R}) \). For example, if \( A = (a_l, a, a_u) \) is a triangular fuzzy number (TFN) where \( a_l \) and \( a_u \) are the lower and upper limits of the support of \( A \) and \( a \) is the modal value then \( F_1(A) = \left( a_l + a + a_u \right)/3 \).

(ii) \( F_2(A) = \left( \int_0^{a_{\text{max}}} m[a^L_{\alpha}, a^R_{\alpha}] d\alpha \right) \), where \( a_{\text{max}} \) is the height of \( A \), \( a_{\alpha} = [a^L_{\alpha}, a^R_{\alpha}] \) is an \( \alpha \)-cut, \( \alpha \in (0,1] \), and \( m[a^L_{\alpha}, a^R_{\alpha}] \) is the mean value of elements of the \( \alpha \)-cut. For example, if \( A = (a_l, a, a_u) \) is a TFN, \( a_{\text{max}} = 1 \) and

\[
A_{\alpha} = \left[ a^L_{\alpha}, a^R_{\alpha} \right] = (a - a_l)\alpha + a_l, (a - a_u)\alpha + a_u
\]

then \( m[a^L_{\alpha}, a^R_{\alpha}] = \left( 2a - a_l - a_u \right)\alpha + (a_l + a_u) \)/2

and \( F_2(A) = \left( a_l + 2a + a_u \right)/4 \).
Flow chart procedure of fuzzy optimization problems

Fuzzy Optimization Algorithm

1. Compute the crisp equivalent of the fuzzy parameters using a defuzzification function (ranking of fuzzy numbers) \( F_2(A) = \left( a_1 + 2a + a_u \right)/4 \). Same defuzzification function is to be used for each of the parameters.

2. Incorporate the objective function of the fuzzifier min (max) as a fuzzy constraint with a restriction (aspiration) level. The inequalities are defined softly if the requirement (resource) constants are defined imprecisely.
3. Define appropriate membership functions for fuzzy inequalities. The membership function for the fuzzy less than or equal to and greater than or equal to type are given as

\[
\mu_{C}(X) = \begin{cases} 
1 & : C(X) \leq C_0 
\frac{C_0^* - C(X)}{C_0 - C_0^*} & : C_0 \leq C(X) < C_0^* 
0 & : C(X) > C_0^*
\end{cases}
\]

respectively, where \( C_0 \) is the restriction and \( C_0^* \) is the tolerance levels to the fuzzy total cost. The membership functions can be a linear or piecewise linear function that is concave or quasiconcave.

4. Employ the extension principle[Bector and Chandra(2005)] to identify the fuzzy decision, which results in a crisp mathematical programming problem given by

Maximize \( \alpha \)

subject to \( \mu_i(T) \geq \alpha \ ; i = 1,2,...,n \)
\( \alpha \geq 0 \ ; T \geq 0 \) \( (*) \)

5. The problem can be looked as a fuzzy multiple objective mathematical programming problem. Further each objective can have different level of importance and can be assigned weights to measure the relative importance. The resulting problem can be solved by the weighted min max approach. The crisp formulation of the weighted problem is given as

Maximize \( \alpha \)

subject to \( \mu_i(T) \geq w_i \alpha \ ; i = 1,2,...,n \)
\( \alpha \geq 0 \ ; T \geq 0 \ ; \sum_{i=1}^{n} W_i \) \( (**) \)

\( (*) \) Where \( \alpha \) represents the degree up to, which the aspiration of the decision maker is met
If a feasible solution is not obtainable for the problem (*) or (*) then we can use fuzzy goal programming approach to obtain a compromised solution [Zimmermann, 1996].

The above crisp optimization problem can be solved by the standard crisp mathematical programming algorithms. The optimal solution of this problem is optimal for the initial problem.