Chapter-4

Optimal Advertising and Pricing Policies of Successive Generation Products in Segmented Market

One constant in the technological management is the change – it is virtually accepted that every generation of technology is replaced by a newer generation. The newer technology products provide an opportunity to switch from earlier generations to the later ones. These substitution effects will diminish the potential, if not the actual sale of earlier generation products; customers who would otherwise have adopted the earlier generations will instead, adopt the later ones, and who have adopted the earlier generations may switch to the later ones. The earlier technology products will continue to diffuse in potential buyers even though the substitution process is under way.

The spread of innovation in market is termed as diffusion. Innovation diffusion is the market penetration process of the new products and services, which is driven by social influences. Such influences include all the inter-dependences among consumers that affect various market players with or without their explicit knowledge. Diffusion theory is extensively used in marketing to capture the dynamics involved in life cycle of a product. Many researchers have lucratively applied innovation diffusion models to study the adoption behavior of the product over its life cycle and to make imperative decisions related to product modification, price differentiation, resource optimization etc. The most commonly and widely used model in this field is Bass advertising model (Bass 1969) as it captures the interaction between advertising and sales in an instinctively satisfying manner. This model does not show the impact of marketing variables. Many authors used the Bass model to study the impact of marketing variables on new product diffusion.


Multiple generations of many durable products serve the market simultaneously. For instance with the advent of generations after generation of mobile networks, it has become important to understand the concept behind each. This first generation (1G) analog system for mobile communications was introduced during the 1970s, which was followed by second generation (2G) digital cellular systems which came into being at the end of the 1980s. Third generation (3G) systems came into existence in 2001 and after sometime fourth generation products came in the market. Fifth generation (5G) systems are the latest system in the block. The major characteristics of such markets are rapid pace of technological change and overlapping product life cycles. The newer technology widens the market by its greater application potential (referred as the market expansion effect) and also provides an opportunity to substitute the most recent
technology for earlier ones (referred as the technology substitution effect). Technology substitution and market expansion effects have significant impact on the strategy adopted by the firm with regard to these products. The Norton-Bass model and some variation of the model (Norton and Bass Model (1987, 1992)) have been used for forecasting innovation diffusion of multiple generation of a number of products and have successfully predicted the diffusion process of multiple generations. Mahajan and Muller (1996) model of successive technology substitution describe the system of IBM mainframe computers in use which are based on an extension of the basic Bass model and have developed normative guidelines for the introduction timing of a new generation.

Present globalized markets comprise multi-cultural customers with diversified tastes and needs. As a result firms need to constantly innovate and promote their products for sustainable growth. However, it is seldom possible to satisfy every customer demand by treating them alike. This necessitates the need for segregating markets into various segments comprising of customers with similar demand characteristics. While market segmentation has long been a popular topic for acadmicians an marketing research, only a few papers on dynamic advertising model deal with market segmentation (Little and Lodish (1969), Seidmann et al. (1987), Buratto et al. (2005, 2006)). Little and Lodish (1969) analyzed a discrete time stochastic model of multiple media selection in a segmented. Seidmann et al. (1987) proposed a general sales-advertising model in which the state of the system represents a population distribution over a parameter space and they show that such models are well poised and that there exists an optimal conrol. Buratto et al. (2005, 2006) discussed the optimal advertising policy for a new product introduction in a segmented market. Jha et al. (2009) used the concept of market segmentation in diffusion model for a new product and studied the optimal advertising effectiveness rate.

We develop an advertising and pricing control model with the objective to maximize the total profit. In this advertising and pricing control model the current sales rate of the product depends on advertising effort, price and cumulative sales for each segment. This system evolves over time and so we
call it dynamic system. This dynamic system can be solved by many optimization techniques. We will use optimal control theory to solve and study this dynamic system. And we will discuss optimal advertising and pricing policies for this dynamic system.

This chapter is divided into three sections: section 4.1, section 4.2 and section 4.3. Section 4.1 will discuss the optimal advertising and pricing policies for the two generation durable product in the single market. Section 4.2 will give the optimal advertising and pricing policies for the two generation durable product in the segmented market. In this section; first we will discuss these policies for the model where current sales rate is the general function of advertisement, price and cumulative sale. Section 4.3 gives optimal advertising and pricing policies for the sales model where current sales rate is the particular function of advertisement, price and cumulative sale. The differential evolution approach has bee used to solve numerical illustration for this particular case.

4.1 OPTIMAL ADVERTISING AND PRICING POLICIES OF SUCCESSIVE GENERATION PRODUCT

This section presents a dynamic optimal control model for consumer sales of a two generation consumer durable product. The analysis is restricted to single firm which only manipulates price and advertising expenditures in segmented market over a finite planning horizon \([0, T]\). We begin our analysis by stating the following assumptions that the firm markets only two products over a planning horizon and first generation product has been launched at the beginning of the planning horizon. The point of departure of our research from the existing literature is our consideration of the firm’s decision to introduce an advanced version of the product at time \(t=\tau\) (where \(\tau \in (0, T)\)). From time \(t=\tau\) onwards, firm sells both the products.

Let \(\dot{x}(t), x(t), p(t)\) and \(u(t)\) represents the current sales rate, cumulative sales, price and advertising expenditure of product at time \(t\) respectively. Current
sales rate is a function of price, advertising expenditure and cumulative sales. So a general diffusion model in segmented market is given by

\[ \frac{dx(t)}{dt} = f(x(t), u(t), p(t)) \]

In the above differential equation, we assume that the demand function \( f \) is twice differentiable, which increases with advertising and decreases with price.

**Notations:**

- \( x_1(t) \) : Cumulative sales of the 1\(^{st}\) generation product by time \( t \),
- \( x_2(t) \) : Cumulative sales of the 2\(^{nd}\) generation product by time \( t \),
- \( x_1(t) \) : Current sales of the 1\(^{st}\) generation product at time \( t \),
- \( x_2(t) \) : Current sales of the 2\(^{nd}\) generation product at time \( t \),
- \( p_1(t) \) : Price of the 1\(^{st}\) generation product over the planning period,
- \( p_2(t) \) : Price of the 2\(^{nd}\) generation product over the planning period,
- \( u_1(t) \) : The firm’s rate of advertising expenditure on the 1\(^{st}\) generation product at time \( t \),
- \( u_2(t) \) : The firm’s rate of advertising expenditure on the 2\(^{nd}\) generation product at time \( t \),
- \( c_1(t) \) : Marginal cost of production of the 1\(^{st}\) generation product by time \( t \),
- \( c_2(t) \) : Marginal cost of production of the 2\(^{nd}\) generation product by time \( t \),
- \( r \) : The discount rate
- \( \lambda = \lambda(t) \) : Co-state variable of the 1\(^{st}\) generation product,
- \( \mu = \mu(t) \) : Co-state variable of the 2\(^{nd}\) generation product,
- \( t = 0 \) : Introduction time of 1\(^{st}\) generation product,
- \( t = \tau \) : Introduction time of 2\(^{nd}\) generation product,
ψ : Proportion of the potential adopters of the 1st generation product who did not buy the 1st generation product but will buy the 2nd generation product

Based on the above assumptions, the sales rate of the first generation product for the interval \(0 < t < \tau\), before the introduction of the second generation product is described by the following differential equation

\[
\dot{x}_1(t) = f(x_1(t), u_1(t), p_1(t)) \quad \forall \ t < \tau
\]  

(4.1.1)

Second generation of product is introduced in the market at time \(\tau\). As soon as second generation of the product is introduced a change in the sales rate of the first generation product is observed. These changes in the sales rate can be accounted due to the fact that second generation product have a substitution effect on the first generation. The resulting demand rate equations in time interval \(\tau < t \leq T\) are

\[
\begin{align*}
\dot{x}_1(t) &= (1-\psi)f(x_1(t), u_1(t), p_1(t)) \quad \forall \ t < T \\
\dot{x}_2(t) &= g(x_1(t), x_2(t), u_1(t), u_2(t), p_1(t), p_2(t)) \quad \forall \ t < T
\end{align*}
\]  

(4.1.2)  
(4.1.3)

Let us assume that the demand functions \(f\) and \(g\) for both the generations are twice differentiable and increase with advertising and decrease with price. Therefore we have

\[
f_{p_1} < 0, \ g_{p_2} < 0, \ f_{u_1} > 0 \ and \ g_{u_2} > 0
\]

(4.1.4)

The optimal control problem is given by

\[
\max_{p_1, p_2, u_1, u_2} \int_0^T e^{-\alpha t} \left[ p_1(t) - c_1(t) \right] \dot{x}_1(t) + \left[ p_2(t) - c_2(t) \right] \dot{x}_2(t) - u_1(t) - u_2(t) \ dt
\]  

Subject to

\[
\begin{align*}
\dot{x}_1(t) &= f(x_1(t), u_1(t), p_1(t)) \quad \forall t, \ t < \tau \\
\dot{x}_1(t) &= (1-\psi)f(x_1(t), u_1(t), p_1(t)) \quad \forall \ t < T \\
\dot{x}_2(t) &= g(x_1(t), x_2(t), u_1(t), u_2(t), p_1(t), p_2(t)) \quad \forall \ t < T
\end{align*}
\]
Where

\[ \psi = \begin{cases} 
0 & \text{if } t < \tau \\
\psi & \text{otherwise}
\end{cases} \]

and \( x_1(0) = x_{10} \), \( x_2(0) = 0 \) \( \forall t \leq \tau \)

This is a dynamic optimization problem, which is the generalization of many previously published papers (Padmanabhan and Bass, 1993; Teng and Thompson, 1996). Here we have four control variables \( u_1(t), u_2(t), p_1(t), p_2(t) \) and two state variables \( x_1(t), x_2(t) \).

In the above optimal control problem, objective function is discontinuous at \( t = \tau \). Therefore we convert the above problem in two stage optimal control problem. Stage two of the problem focuses on the profit maximization problem subsequent to second generation product and is defined as

\[
\max_{p_1, p_2, u_1, u_2} J_2 = \left[ \int_{\tau}^{T} e^{-\alpha} \left\{ \left[ p_1(t) - c_1(t) \right]\ddot{x}_1(t) + \left[ p_2(t) - c_2(t) \right]\ddot{x}_2(t) - u_1(t) - u_2(t) \right\} dt \right]
\]  

(4.1.5)

Subject to

\[
\ddot{x}_1(t) = (1 - \psi) f(x_1(t), u_1(t), p_1(t))
\]

\[
\ddot{x}_2(t) = g(x_1(t), x_2(t), u_1(t), u_2(t), p_1(t), p_2(t))
\]

Where \( x_1(\tau) = x_{1\tau} \), \( x_2(\tau) = 0 \)

The solution of above problem yields the optimal advertising \( (u_1^*, u_2^*) \) and price policies \( (p_1^*, p_2^*) \) in the time horizon \( (\tau, T] \).

**The stage one of the optimal control problem is**

\[
\max_{p_1, u_1} J_1 = \left[ \int_{0}^{\tau} e^{-\alpha} \left\{ \left[ p_1(t) - c_1(t) \right]\ddot{x}_1(t) - u_1(t) \right\} dt + J_2 \right]
\]  

(4.1.6)

Subject to

\[
\ddot{x}_1(t) = f(x_1(t), u_1(t), p_1(t))
\]

Where \( x_1(0) = 0 \)
Maximum Principle is applied to study and solve both stages of the problem.

**Analytical Results: Stage Two Optimal Control Problem When** \( t \in (\tau, T] \)

Using maximum principle Hamiltonian is defined as

\[
H = (p_1(t) - c_1(t) + \lambda)\dot{x}_1(t) + (p_2(t) - c_2(t) + \mu)\dot{x}_2(t) - u_1 - u_2 \tag{4.1.7}
\]

The following necessary conditions hold for optimal values of \( p_1, p_2, u_1 \) and \( u_2 \) respectively

\[
H_{p_1} = 0 \Rightarrow (p_1 - c_1 + \lambda)(1-\psi)f_{x_1} + (1-\psi)f + (p_2 - c_2 + \mu)g_{p_1} = 0 \tag{4.1.8}
\]

\[
H_{p_2} = 0 \Rightarrow (p_2 - c_2 + \mu)g_{p_2} + g = 0 \tag{4.1.9}
\]

\[
H_{u_1} = 0 \Rightarrow (p_1 - c_1 + \lambda)(1-\psi)f_{u_1} + (p_2 - c_2 + \mu)g_{u_1} - 1 = 0 \tag{4.1.10}
\]

\[
H_{u_2} = 0 \Rightarrow (p_2 - c_2 + \mu)g_{u_2} - 1 = 0 \tag{4.1.11}
\]

These optimality conditions yield the optimal prices and advertising expenditure as:

\[
p_1^* = \left( \frac{\eta_1}{\eta_1 - 1} \right) \left( c_1 - \lambda - \frac{g_{p_1}(p_2 \eta_2)}{f_{p_1} \eta_2 (1-\psi)} \right), \quad p_2^* = \frac{\eta_2}{\eta_2 - 1} (c_2 - \mu) \tag{4.1.12}
\]

\[
u_1^* = \beta_1 \left( \frac{p_1 (1-\psi) - \frac{g_{p_1}}{f_{p_1}} \frac{p_2}{\eta_2}}{\eta_1} \right), \quad u_2^* = g \frac{\beta_1 p_2}{\eta_2} \tag{4.1.13}
\]

Where \( \eta_1, \eta_2, \beta_1, \beta_2 \) are elasticity of price and advertising with respect to demand respectively. It is observed from above equations that price and advertising elasticities directly influence the optimal price and advertising expenditure polices.

From the necessary optimality conditions for adjoint variables

\[
\frac{d\lambda(t)}{dt} = -\frac{\partial H^*}{\partial x_1(t)}, \quad \frac{d\mu(t)}{dt} = -\frac{\partial H^*}{\partial x_2(t)}
\]

we have adjoint variables \( \lambda(t) \) and \( \mu(t) \) that satisfies the following differential equations

\[
\lambda(t) = r \lambda - \frac{\partial H}{\partial \lambda}, \quad \mu(t) = \frac{\partial H}{\partial \mu} - \left\{ -c_{1x_1}(1-\theta)f + (p_1 - c_1 + \lambda)(1-\theta)f_{x_1} + (p_2 - c_2 + \mu)g_{x_1} \right\} \tag{4.1.14}
\]
\[
\dot{\mu}(t) = r\mu - \frac{\partial H}{\partial x_2} = r\mu - \left\{ (p_2 - c_2 + \mu)g_{x_1} - c_{x_1}g \right\} 
\]
(4.1.15)

With transversality conditions \( \lambda(T) = 0 \) and \( \mu(T) = 0 \)

Solving the above differential equations, we get

\[
\lambda(t) = e^{\alpha t} \int_{t}^{T} e^{-\alpha s} \left( (p_1 - c_1 + \lambda)(1-\theta)f_{x_1} + (p_2 - c_2 + \mu)g_{x_1} - c_{x_1}g \right) ds 
\]
(4.1.16)

\[
\mu(t) = e^{\alpha t} \int_{t}^{T} e^{-\alpha s} \left( (p_2 - c_2 + \mu)g_{x_1} - c_{x_1}g \right) ds 
\]
(4.1.17)

The theoretical results of advertising expenditure and price polices are almost identical to those of Padmanabhan and Bass (1993)

**Proposition 1:** With experience curve effects alone, the price of second generation product monotonically decreases over time.

**Proposition 2:** If \( g_{p_1} = 0 \) i.e. if sales of the 2\(^{nd}\) generation product are independent of the price of the 1\(^{st}\) generation product, then technological substitution has no effect on the price of the 1\(^{st}\) generation product. In case of advertising, if \( g_{p_1} = 0 \) and \( g_{u_1} = 0 \) then advertising expenditure for the 1\(^{st}\) generation product depends upon technological substitution and as technological substitution increases, the advertising expenditure for the 1\(^{st}\) generation product decreases over time.

**Proposition 3:** If \( g_{p_1} > 0 \) and \( g_{u_1} = 0 \) then the effect of technological substitution in the price problem is to cause the 1\(^{st}\) generation product to be priced higher and advertising policy recommends that advertising expenditure of 1\(^{st}\) generation product increases after introduction of the more advanced version.

**Proposition 4:** If \( g_{p_1} < 0 \) and \( g_{u_1} = 0 \) then the effect of technological substitution reduces the price and advertising expenditure of the 1\(^{st}\) generation product.

**Stage One Optimal Control Problem When** \( t \in [0, \tau] \)

The optimal control problem is defined as

\[
\max_{\tilde{p}, \tilde{u}} J = \int_{0}^{\tau} e^{-\alpha t} \left\{ \left[ p_1(t) - c_1(t) \right] \dot{x}_1(t) - u_1(t) \right\} dt + J_2 
\]
(4.1.18)

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Subject to
\[ x_i(t) = f(x_i(t), u_i(t), p_i(t)) \quad \text{and} \quad x_i(0) = 0 \]

Here \( J_2 \) is used as salvage value. Using maximum principle the Hamiltonian is defined as
\[ H_2 = \left( p_i(t) - c_i(t) + \alpha(t) \right) \dot{x}_i(t) - u_i \]  

(4.1.19)

Confining our interest to interior solutions, we have the first order necessary conditions \( H_{2n} = 0 \) and \( H_{2n} = 0 \). The optimality conditions yield the optimal prices \( p_i^* \) and advertising \( u_i^* \) as
\[ p_i^* = \left( \frac{\eta_i}{\eta_i - 1} \right) (c_i - \alpha) \]  

(4.1.20)
\[ u_i^* = \beta_i (p_i - c_i + \alpha) f \]  

(4.1.21)

where \( \alpha(t) \) is the current value adjoint variable which satisfies the differential equation
\[ \dot{\alpha}(t) = r\alpha - \frac{\partial H_2}{\partial x_1} = r\alpha - (p_i - c_i + \alpha) f_{x_i} + c_{x_1} f \]  

(4.1.22)

with the transversality condition at \( t = \tau \), \( \alpha(\tau) = \frac{\partial J_2}{\partial x_1} \), the solution of the adjoint differential equation is
\[ \alpha(t) = e^{-\tau(t-i)} \alpha(t) + e^\tau \int_t^\tau e^{\tau-s} \left( \frac{f_{x_i}}{u_i} - f c_{x_1} \right) ds \]  

(4.1.23)

4.2 OPTIMAL ADVERTISING AND PRICING POLICIES OF SUCCESSIVE GENERATION PRODUCT IN SEGMENTED MARKET

This section presents a dynamic optimal control model for consumer sales of a two generation consumer durable product in segmented market. The analysis is restricted to single firm which only manipulates price and advertising expenditures in segmented market over a finite planning horizon \([0, T]\). We begin our analysis by stating the following assumptions that the firm markets only two products over a planning horizon and first generation product has been launched at the beginning of the planning horizon. The point
of departure of our research from the existing literature is our consideration of the firm’s decision to introduce an advanced version of the product at time \( t = r \) (where \( r \in (0, T) \)). From time \( t = r \) onwards, firm sells both the products.

**Notations**

- \( x_{1i}(t) \) : Cumulative sales of the 1\(^{st}\) generation product in \( i^{th} \) segment by time \( t \),
- \( x_{2i}(t) \) : Cumulative sales of the 2\(^{nd}\) generation product in \( i^{th} \) segment by time \( t \),
- \( x_{1i}(t) \) : Current sales of the 1\(^{st}\) generation product in \( i^{th} \) segment at time \( t \),
- \( x_{2i}(t) \) : Current sales of the 2\(^{nd}\) generation product in \( i^{th} \) segment at time \( t \),
- \( p_{1i}(t) \) : Price of the 1\(^{st}\) generation product in \( i^{th} \) segment at time \( t \),
- \( p_{2i}(t) \) : Price of the 2\(^{nd}\) generation product in \( i^{th} \) segment at time \( t \),
- \( u_{1i}(t) \) : The firm’s rate of advertising expenditure on the 1\(^{st}\) generation product in \( i^{th} \) segment at time \( t \),
- \( u_{2i}(t) \) : The firm’s rate of advertising expenditure on the 2\(^{nd}\) generation product in \( i^{th} \) segment at time \( t \),
- \( c_{1i}(t) \) : Marginal cost of production of the 1\(^{st}\) generation product in \( i^{th} \) segment at time \( t \),
- \( c_{2i}(t) \) : Marginal cost of production of the 2\(^{nd}\) generation product in \( i^{th} \) segment at time \( t \),
- \( r \) : The discount rate
- \( \lambda_{1i} = \dot{\lambda}_{1i}(t) \) : Co-state variable of the 1\(^{st}\) generation product in \( i^{th} \) segment at time \( t \),
- \( \lambda_{2i} = \dot{\lambda}_{2i}(t) \) : Co-state variable of the 2\(^{nd}\) generation product in \( i^{th} \) segment at time \( t \),
\( t = 0 \) : Introduction time of \( 1^{st} \) generation product,

\( t = \tau \) : Introduction time of \( 2^{nd} \) generation product,

\( \psi_{i} \) : Proportion of the potential adopters of the \( 1^{st} \) generation product who did not buy the \( 1^{st} \) generation product but will buy the \( 2^{nd} \) generation product in \( i^{th} \) segment

Based on the above assumptions, the sales rate of the first generation product for the interval \( 0 < t < \tau \), before the introduction of the second generation product, is described by the following differential equation for each segment

\[
\dot{x}_{i}(t) = \frac{dx_{i}(t)}{dt} = f_{i}(x_{i}(t), u_{i}(t), p_{i}(t)) \quad \forall i, \ t < \tau
\]  

(4.2.1)

Second generation of product is introduced in the market at time \( \tau \), as soon as second generation of the product is introduced a change in the sales rate of the first generation product is observed. These changes in the sales rate can be accounted due to the fact that second generation product have a substitution effect on the first generation. The resulting demand rate equations for each segment in time interval \( \tau < t \leq T \) are

\[
\dot{x}_{i}(t) = (1-\psi_{i})f_{i}(x_{i}(t), u_{i}(t), p_{i}(t)) \quad \forall i
\]  

(4.2.2)

\[
\dot{x}_{2i}(t) = g_{i}(x_{i}(t), x_{2i}(t), u_{i}(t), u_{2i}(t), p_{i}(t), p_{2i}(t)) \quad \forall i
\]  

(4.2.3)

Where, \( x_{i}(t), x_{2i}(t) \); \( p_{i}(t), p_{2i}(t) \); \( u_{i}(t), u_{2i}(t) \) are cumulative sales, price, rate of advertising expenditure of the \( i^{th} \) and \( 2^{nd} \) generation product in \( i^{th} \) segment by time \( t \).

Let us assume that the demand functions \( x_{i}(t), x_{2i}(t) \) for the \( 1^{st} \) and \( 2^{nd} \) generations are twice differentiable and increase with advertising and decrease with price in each segment. Therefore we have

\[
\frac{\partial f_{i}}{\partial p_{i}} < 0, \quad \frac{\partial g_{i}}{\partial p_{2i}} < 0, \quad \frac{\partial f_{i}}{\partial u_{i}} > 0, \quad \frac{\partial g_{i}}{\partial u_{2i}} > 0
\]
The optimal control problem for \([0, T]\) is given by

\[
\max_{p_1, p_2, u_1, u_2} \sum_{i=1}^{m} \int_{0}^{T} e^{-\alpha t} \left\{ \left[ p_1(t) - c_1(t) \right] \dot{x}_1(t) + \left[ p_2(t) - c_2(t) \right] \dot{x}_2(t) - u_1(t) - u_2(t) \right\} \, dt
\]  

(4.2.4)

Subject to

\[
\dot{x}_1(t) = f_i(x_i(t), u_i(t), p_i(t)) \quad \forall i, \ t < \tau
\]

\[
\dot{x}_1(t) = (1 - \psi_i) f_i(x_i(t), u_i(t), p_i(t)) \quad \forall i \ t \geq \tau
\]

\[
\dot{x}_2(t) = g_i(x_i(t), x_i(t), u_i(t), u_2(t), p_i(t), p_2(t)) \quad \forall i \ t \geq \tau
\]

Where

\[
\psi_i = \begin{cases} 
0 & \text{if } t < \tau \\
\psi_i & \text{otherwise}
\end{cases}
\]

and \(x_i(0) = x_i^0, x_2(t) = 0 \quad \forall t \leq \tau\)

And \(c_1(x_i(t), t), c_2(x_2(t), t)\) is marginal cost of production of the \(1^{st}\) and \(2^{nd}\) generation product in \(i^{th}\) segment by time \(t\) respectively and \((p_i(t) - c_i(t)) > 0, (p_i(t) - c_i(t)) > 0 \quad \forall i\). Here we have an optimal control problem with multiple control variables \(p_1, p_2, u_1, u_2\) and state variables \(x_1, x_2\) in segmented market.

In this optimal control problem, objective function is discontinuous at \(t = \tau\). Therefore we convert the above problem in two stage optimal control problem. Stage two of the problem focuses on the profit maximization problem subsequent to second generation product (i.e. \(t \geq \tau\)), which is as follows

\[
\max_{p_1, p_2, u_1, u_2} J_2 = \sum_{i=1}^{m} \int_{\tau}^{T} e^{-\alpha (t-\tau)} \left\{ \left[ p_1(t) - c_1(t) \right] \dot{x}_1(t) + \left[ p_2(t) - c_2(t) \right] \dot{x}_2(t) - u_1(t) - u_2(t) \right\} \, dt
\]  

(4.2.5)

subject to

\[
\dot{x}_1(t) = (1 - \psi_i) f_i(x_i(t), u_i(t), p_i(t)) \quad \forall i \ t \geq \tau
\]

\[
\dot{x}_2(t) = g_i(x_i(t), x_i(t), u_i(t), u_2(t), p_i(t), p_2(t)) \quad \forall i \ t \geq \tau
\]

Where \(x_1(t) = x_i^\tau, x_2(t) = 0\).

We can write above equation (4.2.5) as
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\[
\max_{p_i, q_i, a_i, b_i} J_2 = \sum_{i=1}^{m} J_i \tag{4.2.6}
\]

Where \( J_i = \int_{\tau}^{T} e^{-\tau} \{ [p_i(t) - c_i(t)] \dot{x}_i(t) + [p_{2i}(t) - c_{2i}(t)] \dot{x}_{2i}(t) - u_{ii}(t) - u_{2i}(t) \} dt \)

Is a profit function of \( i^{th} \) segment. Therefore, the segment dependent product introduction problem is equivalent to determining it for all segments and advertising expenditure rate \( u_{ii}, u_{2i} \) with associated sales rate \( x_{ii}(t), x_{2i}(t) \) satisfying equation (4.2.2) and (4.2.3).

The solution of above problem yields the optimal advertising \( (u_{ii}^*, u_{2i}^*) \) and price policies \( (p_{ii}^*, p_{2i}^*) \) for each segment in the time horizon \((\tau, T] \).

The stage one of the optimal control problem is

\[
\max_{p_i^0, \lambda_i, \mu_i} \int_{\tau}^{T} e^{-\tau} \{ [p_i(t) - c_i(t)] \dot{x}_i(t) - u_{ii}(t) \} dt + J_2 \tag{4.2.7}
\]

s.t.

\[ \dot{x}_i(t) = f_i(x_i, u_i, p_i) \quad \forall i, \quad t < \tau \]

Where \( x_i(0) = 0 \quad \forall i \).

Maximum Principle is applied to study and solve both stages of the problem.

**Analytical Results: Stage Two Optimal Control Problem when \( t \in [\tau, T] \)**

In this section, optimal prices \( (p_{ii}^*, p_{2i}^*) \) and optimal advertising expenditures \( (u_{ii}^*, u_{2i}^*) \) from the model formulated in (4.2.6) are discussed. To solve the proposed problem, Pontryagain Maximum Principle is applied; we define the Hamiltonian for each segment as

\[
H_i = (p_i(t) - c_i(t) + \lambda_i) \dot{x}_i(t) + (p_{2i}(t) - c_{2i}(t) + \mu_i) \dot{x}_{2i}(t) - u_{ii} - u_{2i} \tag{4.2.8}
\]
The following necessary conditions hold for optimal values of $p_{1i}, p_{2i}, u_{1i}, u_{2i}$ respectively

\[ \frac{\partial H}{\partial p_{1i}} = 0 \Rightarrow (p_{1i} - c_{1i} + \lambda_i)\frac{\partial g_i}{\partial p_{1i}} + (1 - \psi_i)\frac{\partial f_i}{\partial p_{1i}} + (p_{2i} - c_{2i} + \mu_i)\frac{\partial g_i}{\partial p_{1i}} = 0 \]  

\[ (4.2.9) \]

\[ \frac{\partial H}{\partial p_{2i}} = 0 \Rightarrow (p_{2i} - c_{2i} + \mu_i)\frac{\partial g_i}{\partial p_{2i}} + g_i = 0 \]  

\[ (4.2.10) \]

\[ \frac{\partial H}{\partial u_{1i}} = 0 \Rightarrow (p_{1i} - c_{1i} + \lambda_i)(1 - \psi_i)\frac{\partial f_i}{\partial u_{1i}} + (p_{2i} - c_{2i} + \mu_i)\frac{\partial g_i}{\partial u_{1i}} - 1 = 0 \]  

\[ (4.2.11) \]

\[ \frac{\partial H}{\partial u_{2i}} = 0 \Rightarrow (p_{2i} - c_{2i} + \mu_i)\frac{\partial g_i}{\partial u_{2i}} - 1 = 0 \]  

\[ (4.2.12) \]

These optimality conditions yield the optimal prices and advertising expenditure as follows

\[ p_{1i}^* = \left( \frac{\eta_{1i}}{\eta_{1i} - 1} \right) c_{1i} - \lambda_i - \left( \frac{\partial g_i}{\partial p_{1i}} \right) \left( \frac{p_{2i}}{\eta_{2i}(1 - \psi_i)} \right) \]  

\[ p_{2i}^* = \frac{\eta_{2i}}{\eta_{2i} - 1} (c_{2i} - \mu_i) \]  

\[ (4.2.13) \]

\[ u_{1i}^* = \frac{\beta_{1i}}{\eta_{1i}} \left( p_{1i} (1 - \psi_i) \right) \left( \frac{\partial g_i}{\partial p_{1i}} \right) \left( \frac{p_{2i}}{\eta_{2i}} \right) \]  

\[ u_{2i}^* = g_i \frac{\beta_{2i}}{\eta_{2i}} \frac{p_{2i}}{\eta_{2i}} \]  

\[ (4.2.14) \]

where $\eta_{1i}, \eta_{2i}, \beta_{1i}, \beta_{2i}$ are elasticity of price and advertising with respect to demand respectively for each segment. It is observed from equations (4.2.13) and (4.2.14) that price and advertising elasticities directly influence the optimal price and advertising expenditure policies for each segment.

We state some immediate consequences of the optimality conditions (4.2.13) and (4.2.14). The impact of technological substitution on the price of the first generation product depends on $\psi_i$ (substitutability between the two products) and the ratio of sensitivity of sales of the two products to price of the first generation product; the effect of technological substitution alone on the price of the first generation product directly; when $\psi_i$ approaches one, then the optimal price gets down to marginal cost. In other way, as substitution increases the firm starts following a policy that closely resembles the myopic policy with respect to the first generation product. From the necessary optimality conditions for adjoint variables $\frac{d\lambda_i(t)}{dt} = -\frac{\partial H^*}{\partial \lambda_i(t)}$. 
\[ d\mu_i(t)/dt = -\partial H_i/\partial x_{2i}(t), \]
we have adjoint variables \( \lambda_i(t) \) and \( \mu_i(t) \) that satisfies the following differential equations respectively:

\[ \dot{\lambda}_i(t) = r \lambda_i(t) - \left\{ (\partial c_i / \partial x_i)(1-\psi_i) + \partial g_i / \partial x_i \right\} + (p_{2i} - c_{2i} + \mu_i)(\partial g_i / \partial x_i) \]  \( (4.2.15) \)

\[ \dot{\mu}_i(t) = r \mu_i(t) - \partial H_i/\partial x_{2i} = r \mu_i(t) - \left\{ (p_{2i} - c_{2i} + \mu_i)(\partial g_i / \partial x_i) - (\partial c_i / \partial x_i) \right\} \]  \( (4.2.16) \)

With transversality conditions \( \lambda(T) = 0 \) and \( \mu(T) = 0 \). Solving the equations \((4.2.15)\) and \((4.2.16)\), we get:

\[ \dot{\lambda}_i(t) = e^{t-T} \int_t^T e^{-s} \left\{ (p_{2i} - c_{2i} + \mu_i)(\partial g_i / \partial x_i) - (\partial c_i / \partial x_i) \right\} ds \]  \( (4.2.17) \)

\[ \dot{\mu}_i(t) = e^{t-T} \int_t^T e^{-s} \left\{ (p_{2i} - c_{2i} + \mu_i)(\partial g_i / \partial x_i) - (\partial c_i / \partial x_i) \right\} g_i ds \]  \( (4.2.18) \)

The theoretical results of advertising expenditure and price policies are almost identical to those of Padmanabhan and Bass (1993).

**Proposition 1**: With experience curve effects alone, the price of second generation product monotonically decreases over time in each segment.

**Proposition 2**: If \( \partial g_i / \partial p_{2i} = 0 \) i.e. if sales of the 2nd generation product are independent of the price of the 1st generation product, then technological substitution has no effect on the price of the 1st generation product. In case of advertising, if \( \partial g_i / \partial p_{2i} = 0 \) and \( \partial g_i / \partial u_{2i} = 0 \), then advertising expenditure for the 1st generation product depends upon technological substitution and as technological substitution increases, the advertising expenditure for the 1st generation product decreases over time.

**Proposition 3**: If \( \partial g_i / \partial p_{2i} > 0 \) and \( \partial g_i / \partial u_{2i} = 0 \), then the effect of technological substitution in the price problem is to cause the 1st generation product to be priced higher and advertising policy recommends that advertising expenditure of 1st generation product increases after introduction of the more advanced version.

**Proposition 4**: If \( \partial g_i / \partial p_{2i} < 0 \) and \( \partial g_i / \partial u_{2i} = 0 \), then the effect of technological substitution reduces the price and advertising expenditure of the 1st generation product.
We have also seen that the technological substitution does not have a direct effect on the price and advertising expenditure of the second generation product but has an indirect effect through its impact on the adjoint variables.

**Stage One Optimal Control Problem When** \( t \in [0, \tau] \)

In this section, optimal prices \( (p_i^*) \) and optimal advertising expenditures \( (u_i^*) \) from the stage one formulated in (4.2.7) are discussed. Here we use \( J_2 \) as a salvage value. The stage one optimal control problem is

\[
\max_{p_i, u_i} J = \sum_{i=1}^{n} J_i + J_2
\]

s.t.

\[
\dot{x}_i(t) = f_i(x_i, u_i, p_i), \quad x_i(0) = 0 \quad \forall i, \quad t < \tau
\]

Using Maximum Principle the Hamiltonian is defined as

\[
H_i = \left( p_i(t) - c_i(t) + \alpha_i(t) \right) x_i(t) - u_i
\]

Confining our interest to interior solutions, we have the first order necessary conditions \( \partial H_i / \partial p_i = 0 \) and \( \partial H_i / \partial u_i = 0 \). Therefore optimality conditions yield the optimal prices \( p_i^* \) and advertising \( u_i^* \).

\[
p_i^* = \left( \frac{\eta_i}{\eta_i - 1} \right) (c_i - \alpha_i)
\]

\[
u_i^* = \beta_i \left( p_i - c_i + \alpha_i \right) f_i
\]

where \( \alpha_i(t) \) is the current value adjoint variable which satisfies the differential equation

\[
\dot{\alpha}_i(t) = r \alpha_i - \frac{\partial H_2}{\partial x_i} + r \alpha_i - \left( p_i - c_i + \alpha_i \right) \left( \frac{\partial f_i}{\partial x_i} \right) + \left( \frac{\partial c_i}{\partial x_i} \right) f_i
\]

with the transversality condition \( \alpha_i(\tau) = \tilde{c}J_i / \tilde{c}x_i \), the solution of the adjoint differential equation is

\[
\alpha_i(t) = e^{-(r-\tau)t} \alpha_i(\tau) + e^{\tau} \int_{t}^{\tau} e^{-s} \left( \frac{\partial f_i}{\partial x_i} \right) - \left( \frac{\partial c_i}{\partial x_i} \right) f_i ds
\]

The basic theoretical results of advertising expenditure and price policies are almost identical to those of Padmanabhan and Bass (1993).
4.3 OPTIMAL ADVERTISING AND PRICING POLICIES OF SUCCESSIVE GENERATION PRODUCT IN SEGMENTED MARKET FOR PARTICULAR CASE

In the previous section, we discussed general case of advertising and pricing policies for two generation durable product in segmented market. Here, we extend the Padmanabhan and Bass (1993) model by adding advertising expenditure as a control variable. This will be in two forms: first tell about segment specific advertisement and second about single advertising channel mode. And in the end we will solve this numerically by differential evolution.

Notations

\( a_{1i} \): Coefficient of innovation of 1\(^{st}\) product in \( i \)\(^{th}\) segment,

\( a_{2i} \): Coefficient of innovation of 2\(^{nd}\) product in \( i \)\(^{th}\) segment,

\( b_{1i} \): Coefficient of imitation of 1\(^{st}\) product in \( i \)\(^{th}\) segment,

\( b_{2i} \): Coefficient of imitation of 1\(^{st}\) product in \( i \)\(^{th}\) segment,

\( \bar{X}_{1i} \): Total potential market for 1\(^{st}\) product in \( i \)\(^{th}\) segment,

\( \bar{X}_{2i} \): Total potential market for 2\(^{nd}\) product in \( i \)\(^{th}\) segment,

4.3.1 Segment Specific Advertising:

Current sales rate is taken as particular case

\[
\frac{dx(t)}{dt} = \left( a + b \frac{x(t)}{\bar{X}} \right) u(t) (\bar{X} - x(t)) e^{-x_p} = f \text{ (say)}
\]

The sales rate of the first generation product for the \( i \)\(^{th}\) segment in time interval \( 0 < t < \tau \) is described by the following differential equation

\[
\dot{x}_{1i}(t) = \left( a_i + b_i \frac{x_{1i}(t)}{\bar{X}_{1i}} \right) u_i(t) (\bar{X}_{1i} - x_{1i}(t)) e^{-x_{1i}} \quad \forall i, \ t < \tau \tag{4.3.1}
\]

As soon as second generation of the product is introduced, a change in the sales rate of the first generation product is observed. These changes in the sales rate can be accounted due to the fact that second generation product have a substitution effect on the first generation. The resulting demand rate equations for each segment in time interval \( \tau < t \leq T \) are
\[ \dot{x}_i(t) = (1 - \psi_i) \left( a_i + b_i \frac{x_i(t)}{X_i} \right) u_i(t) (X_i - x_i(t)) e^{-x_i(t)} \quad \forall i \tag{4.3.2} \]

\[ \dot{x}_2(t) = \left( a_2 + b_2 \frac{x_2(t)}{X_2} \right) u_2(t) (X_2 - x_2(t)) e^{-x_2(t)} + \psi_i \left( a_i + b_i \frac{x_i(t)}{X_i} \right) u_i(t) (X_i - x_i(t)) e^{-x_i(t)} \quad \forall i \tag{4.3.3} \]

Where, \( x_1(t), x_2(t), p_1(t), p_2(t), u_1(t), u_2(t) \) are cumulative sales, price, rate of advertising effort of the \( I^{th} \) and \( 2^{nd} \) generation product in \( I^{th} \) segment by time \( t \) respectively. We use a parameter, \( \psi_i \) to capture the cannibalization effect (Padmanabhan and Bass 1993).

Let us assume that the demand functions \( x_1(t), x_2(t) \) for both the generations are twice differentiable and increase with advertising and decrease with price in each segment. Therefore we have

\[ \frac{\partial \dot{x}_1}{\partial p_1} < 0, \quad \frac{\partial \dot{x}_2}{\partial p_2} < 0, \quad \frac{\partial \dot{x}_1}{\partial u_1} > 0, \quad \frac{\partial \dot{x}_2}{\partial u_2} > 0 \]

The optimal control problem for this case given by

\[ \max_{P, \psi_1, \psi_2, \psi_0} J = \int e^{-\tau} \left[ \sum_{i=1}^{n} \left( \left( p_i(t) - c_i(t) \right) \dot{x}_i(t) + \left( p_2(t) - c_2(t) \right) \dot{x}_2(t) \right) - w_i(u_i(t)) - w_2(u_2(t)) \right] dt \tag{4.3.4} \]

subject to

\[ \dot{x}_i(t) = \left( a_i + b_i \frac{x_i(t)}{X_i} \right) u_i(t) (X_i - x_i(t)) e^{-x_i(t)} \quad \forall i, \quad t < \tau \]

\[ \dot{x}_i(t) = (1 - \psi_i) \left( a_i + b_i \frac{x_i(t)}{X_i} \right) u_i(t) (X_i - x_i(t)) e^{-x_i(t)} \quad \forall i \quad \forall i \]

\[ \dot{x}_2(t) = \left( a_2 + b_2 \frac{x_2(t)}{X_2} \right) u_2(t) (X_2 - x_2(t)) e^{-x_2(t)} + \psi_i \left( a_i + b_i \frac{x_i(t)}{X_i} \right) u_i(t) (X_i - x_i(t)) e^{-x_i(t)} \quad \forall i \quad \forall t \geq \tau \]

Where

\[ \psi_i = \begin{cases} 0 & \text{if } t < \tau \\ \psi_i & \text{otherwise} \end{cases} \]

\[ x_i(0) = x^0_i, \quad x_2(t) = 0 \quad \forall t \leq \tau \]
And $c_{i_1}(x_{i_1}(t), t), c_{i_2}(x_{i_2}(t), t)$ are marginal cost of production of the $i^{th}$ and $2nd$ generation product in $i^{th}$ segment respectively, $(p_{i_1}(t) - c_{i_1}(t)) > 0,$ \((p_{i_2}(t) - c_{i_2}(t)) > 0 \forall i.$ The advertising cost rate is $w_{i_1}(u_{i_1}(t)) = \alpha_{i_1} u_{i_1} + \beta_{i_1},$ $w_{i_2}(u_{i_2}(t)) = \alpha_{i_2} u_{i_2} + \beta_{i_2}$ where $\alpha_{i_1}, \alpha_{i_2}$ and $\beta_{i_1}, \beta_{i_2}$ are nonnegative constants. The numbers $\beta_{i_1}, \beta_{i_2}$ represents the fixed costs of just listing the product in the catalog for $i^{th}$ and $2^{nd}$ generation product in segmented market (In numerical illustration, we assume that $\alpha_{i_1} = \alpha_{i_2} = 1; \beta_{i_1} = \beta_{i_2} = 0$).

This dynamic optimization problem is an optimal control problem with multiple control variables $p_{i_1}, p_{i_2}, u_{i_1}, u_{i_2}$ and state variables $x_{i_1}, x_{i_2}$ in segmented market.

In this optimal control problem, objective function is discontinuous at $t=\tau$. Therefore we convert the above problem in two stage optimal control problem.

Stage two of the problem focuses on the profit maximization problem subsequent to second generation product, which is as follows

$$\max_{p_{i_1}, p_{i_2}, u_{i_1}, u_{i_2}} J_2 = \int_{\tau}^{T} e^{-\eta t} \left( \sum_{i=1}^{m} \left( (p_{i_1}(t) - c_{i_1}(t)) \dot{x}_{i_1}(t) + (p_{i_2}(t) - c_{i_2}(t)) \dot{x}_{i_2}(t) \right) \right) dt \quad (4.3.5)$$

subject to

$$\dot{x}_{i_1}(t) = (1 - \psi_{i_1}) \left( a_{i_1} + b_{i_1} \frac{x_{i_1}(t)}{X_{i_1}} \right) u_{i_1}(t) \left( X_{i_1} - x_{i_1}(t) \right) e^{-\eta_{i_1} t} = (1 - \psi_{i_1}) f_i \quad \text{(say)}$$

$$\dot{x}_{i_2}(t) = \left( a_{i_2} + b_{i_2} \frac{x_{i_2}(t)}{X_{i_2}} \right) u_{i_2}(t) \left( X_{i_2} - x_{i_2}(t) \right) e^{-\eta_{i_2} t} + \psi_{i_2} \left( a_{i_2} + b_{i_2} \frac{x_{i_2}(t)}{X_{i_2}} \right) u_{i_2}(t) \left( X_{i_2} - x_{i_2}(t) \right) e^{-\eta_{i_2} t} = g_i \quad \text{(say)}$$

Where, $x_{i_1}(\tau) = x_{i_1}^*, x_{i_2}(\tau) = 0$

The solution of above problem yields the optimal advertising $(u_{i_1}^*, u_{i_2}^*)$ and price policies $(p_{i_1}^*, p_{i_2}^*)$ for each segment in the time horizon $[\tau, T]$.

The stage one of the optimal control problem is

$$\max_{p_{i_1}, p_{i_2}, u_{i_1}, u_{i_2}} J_1 = \int_{\tau}^{T} e^{-\eta t} \left( \sum_{i=1}^{m} \left[ (p_{i_1}(t) - c_{i_1}(t)) \dot{x}_{i_1}(t) - u_{i_1}(t) \right) \right) dt + J_2 \quad (4.3.6)$$
Optimal Production and Inventory Policies for Deterioration Items in Segmented Market

Subject to
\[
\dot{x}_i(t) = \left( a_i + b_i \frac{x_i(t)}{X_i} \right) u_{i_0}(t) \left( X_i - x_i(t) \right) e^{-(x_i)/p_i} \quad \forall i, \quad t < \tau
\]

Where \( x_i(0) = 0 \quad \forall i \).

Maximum Principle is applied to study and solve both stages of the problem.

Analytical Solution: Stage Two Optimal Control Problem when \( t \in [\tau, T] \)

We can write equation (4.3.5) as
\[
\max_{p_{i_0}, p_{i_1}, u_{i_0, i_1}} \sum_{i=1}^{m} J_{2,i} = 0
\]

Where
\[
J_{2,i} = \int_{\tau}^{T} e^{-\tau} \left[ p_{i_0}(t) - c_{i_0}(t) \right] \dot{x}_{i_0}(t) + \left[ p_{i_1}(t) - c_{i_1}(t) \right] \dot{x}_{i_1}(t) - (\alpha_{i_0} u_{i_0}(t) + \beta_{i_0}) - (\alpha_{i_1} u_{i_1}(t) + \beta_{i_1}) dt
\]

This is a profit function of \( i^{th} \) segment. Therefore, the segment dependent product introduction problem is equivalent to determining it for all segments; and advertising expenditure rate \( u_{i_0}, u_{i_1} \) with associated sales rate \( x_{i_0}(t), x_{i_1}(t) \) satisfy equation (4.3.2) and (4.3.3).

The solution of above problem yields the optimal advertising policy \((u_{i_0}^*, u_{i_1}^*)\) and price policy \((p_{i_0}^*, p_{i_1}^*)\) for each segment in the time horizon \([\tau, T]\). To solve the problem, Pontryagin Maximum Principle has been applied. We define the Hamiltonian as
\[
H_{2,i} = \left\{ (p_{i_0}(t) - c_{i_0}(t) + \lambda_{i_1}) \dot{x}_{i_0}(t) + (p_{i_1}(t) - c_{i_1}(t) + \lambda_{i_2}) \dot{x}_{i_1}(t) \right\} - (\alpha_{i_0} u_{i_0} + \beta_{i_0}) - (\alpha_{i_1} u_{i_1} + \beta_{i_1}) \quad (4.3.7)
\]

The following necessary conditions hold for optimal values of \( p_{i_0}, p_{i_1}, u_{i_0, i_1} \) respectively
\[
(H_{2,i})_{p_{i_0}} = 0 \quad \Rightarrow \quad (p_{i_0} - c_{i_0} + \lambda_{i_1})(1 - \psi_{i_1})(f_{i_1})_{p_{i_0}} + (1 - \psi_{i_1})f_{i_1} + (p_{i_1} - c_{i_1} + \lambda_{i_2})(g_{i_1})_{p_{i_0}} = 0 \quad (4.3.8)
\]
\[
(H_{2,i})_{p_{i_1}} = 0 \quad \Rightarrow \quad (p_{i_1} - c_{i_1} + \lambda_{i_2})(g_{i_1})_{p_{i_1}} + g_{i_1} = 0 \quad (4.3.9)
\]
These optimality conditions yield the optimal prices and advertising expenditure as follows

\[ p^*_i = \left( \frac{\eta_i}{\eta_i - 1} \right) \left( c_{i1} - \lambda_{i1} \right) + \left( g_i \right)_{x_{i1}} \left( p_{2i} \right)_{x_{i1}} \eta_{2i}, \quad p^*_2 = \frac{\eta_{2i}}{\eta_{2i} - 1} \left( c_{2i} - \lambda_{2i} \right) \]

\[ u^*_i = \left( \frac{\beta_i}{\beta_i - 1} \right) \left( p_{1i} \left( 1 - \psi_i \right) \right) + \left( g_i \right)_{x_{i1}} \left( p_{2i} \right)_{x_{i1}} \eta_{2i}, \quad u^*_2 = g_i \beta_{2i} p_{2i} \eta_{2i} \]

Where \( \eta_i, \eta_{2i}; \beta_i, \beta_{2i} \) are elasticity of price and advertising with respect to demand respectively for 1\(^{st}\) and 2\(^{nd}\) generation in segmented market. It is observed from equations (4.3.12) and (4.3.13) that price and advertising elasticity directly influence the optimal price and advertising expenditure polices.

From the necessary optimality conditions for adjoint variables, we have adjoint variables \( \lambda_{i1}(t) \) and \( \lambda_{2i}(t) \) that satisfy the following differential equations respectively

\[ \dot{\lambda}_{i1}(t) = r \lambda_{i1} + \left( \frac{\partial H}{\partial x_{i1}} \right) - \left( \frac{\partial}{\partial x_{i1}} \left( p_{1i} \left( 1 - \psi_i \right) \right) \right) + \left( p_{2i} - c_{2i} + \lambda_{2i} \right) \left( g_i \right)_{x_{i1}} \]

\[ \dot{\lambda}_{2i}(t) = r \lambda_{2i} + \left( \frac{\partial H}{\partial x_{2i}} \right) - \left( p_{2i} - c_{2i} + \lambda_{2i} \right) \left( g_i \right)_{x_{2i}} - \left( c_{2i} \right)_{x_{2i}} \]

With transversality conditions, \( \lambda_{i1}(T) = 0, \lambda_{2i}(T) = 0 \). On solving the equations (4.3.14) and (4.3.15), we get

\[ \lambda_{i1}(t) = e^{rt} \int_{t}^{T} e^{-rs} \left( p_{1i} \left( 1 - \psi_i \right) \right) ds \]

\[ \lambda_{2i}(t) = e^{rt} \int_{t}^{T} e^{-rs} \left( p_{2i} - c_{2i} + \lambda_{2i} \right) ds \]
Stage One Optimal Control Problem when $t \in [0, \tau]$

In this section, optimal prices $p_i^*$ and optimal advertising expenditures $u_i^*$ from the stage one formulated in (4.3.6) are discussed. Here we use $J_{2i}$ as a salvage value. The stage one optimal control problem is

$$\max_{p_i, u_i} J_u = \int_0^\tau e^{-\alpha t} \left\{ [p_i(t) - c_i(t)] \dot{x}_i(t) - \left( \alpha_i u_i(t) + \beta_i \right) \right\} dt + J_{2i}$$

(4.3.18)

s.t.

$$\dot{x}_i(t) = f(x_i(t), u_i(t), p_i(t)), \quad x_i(0) = 0$$

(4.3.19)

To solve the above problem, Pontryagin Maximum Principle has been applied. Therefore we define the Hamiltonian as

$$H_i = \left( p_i(t) - c_i(t) + \mu_i(t) \right) \dot{x}_i(t) - \left( \alpha_i u_i + \beta_i \right)$$

(4.3.20)

Confining our interest to interior solutions, we have the first order necessary conditions, $(H_i)_{p_i} = 0$ and $(H_i)_{u_i} = 0$. Therefore optimality conditions yield the optimal prices $p_i^*$ and advertising $u_i^*$ as:

$$p_i^* \left( \begin{array}{c} \eta_i \\ \eta_i - 1 \end{array} \right) (c_i - \mu_i)$$

(4.3.21)

$$u_i^* = \beta_i \left( p_i - c_i + \mu_i \right) f_i$$

(4.3.22)

where $\mu_i$ is the current value adjoint variable which satisfies the differential equation

$$\dot{\mu}_i(t) = r \mu_i - \frac{\partial (H_{2i})}{\partial x_i} = r \mu_i - \left( p_i - c_i + \mu_i \right) (f_i)_{x_i} + \left( c_i' \right)_{x_i} f_i$$

(4.3.23)

with the transversality condition $\mu_i(\tau) = \frac{\partial J_{2i}}{\partial x_i}$, the solution of the adjoint differential equation is

$$\mu_i(t) = e^{-r(t-\tau)} \mu_i(\tau) + e^{-r} \int_t^\tau e^{r(s-t)} \left( f_i \right)_{x_i} - f_i \left( c_i' \right)_{x_i} ds$$

(4.3.24)

The theoretical results of advertising expenditure and price polices are almost identical to those of Padmananabhan and Bass (1993)
4.3.2 Single Advertising Channel Model

In this section, we assume that there is a single advertising media with a fixed segment spectrum, which reaches several segments with different effectiveness rate. The sales rate of the first generation product for the \( i \)th segment in time interval \( 0 < t < \tau \), before the introduction of the second generation product, is described by the following differential equation:

\[
\dot{x}_i(t) = \left( a_i + b_i \frac{x_i(t)}{\bar{X}_i} \right) \gamma_i u_i(t)(\bar{X}_u - x_i(t)) e^{-\alpha_i t / \rho_i}, \quad x_i(0) = x_i(0) \quad \forall i, \quad t < \tau \quad (4.3.25)
\]

Where, \( \gamma_i > 0 \quad \forall i \) is the channel medium spectrum, and its elements \( \gamma_{ii} \), provide different effectiveness rate of the advertising media on the market segments and \( \sum_{i=1}^{n} \gamma_{ii} = 1 \).

As soon as second generation of the product is introduced a change in the sales rate of the first generation product is observed. These changes in the sales rate can be accounted due to the fact that second generation product have a substitution effect on the first generation. The resulting demand rate equations for each segment in time interval \( r < t \leq T \) are

\[
\dot{x}_i(t) = (1 - \psi_i) \left( a_i + b_i \frac{x_i(t)}{\bar{X}_i} \right) \gamma_i u_i(t)(\bar{X}_u - x_i(t)) e^{-\alpha_i t / \rho_i} \quad \forall i \quad (4.3.26)
\]

\[
\dot{x}_2(t) = \left( a_2 + b_2 \frac{x_2(t)}{\bar{X}_2} \right) \gamma_2 u_2(t)(\bar{X}_u - x_2(t)) e^{-\alpha_2 t / \rho_2} + \psi_i \left( a_i + b_i \frac{x_i(t)}{\bar{X}_i} \right) \gamma_i u_i(t)(\bar{X}_u - x_i(t)) e^{-\alpha_i t / \rho_i} \quad \forall i \quad (4.3.27)
\]

Where, \( u_1(t), u_2(t) \) is rate of advertising effort of the 1st and 2nd generation product.

The total present value of profit over the planning horizon in segmented market can be written as:

\[
\max_{\rho_i, \beta_i, \rho_1, \beta_1, \rho_2, \beta_2} J = \int_0^{\tau} \left[ \sum_{i=1}^{n} \left( (p_i(t) - c_i(t)) \dot{x}_i(t) + (p_2(t) - c_2(t)) \dot{x}_2(t) \right) - (\alpha_1 u_1(t) + \beta_1) - (\alpha_2 u_2(t) + \beta_2) \right] dt
\]
subject to (4.3.25), (4.3.26) and (4.3.27).

Here we have an optimal control problem with multiple control variables $p_{1i}, p_{2i}, u_1, u_2$ and state variables $x_{1i}, x_{2i}$ in segmented market.

The objective function in above optimal control problem is discontinuous at $t=\tau$. Therefore we rewrite the above problem in two stage optimal control problem. Stage two of the problem focuses on the profit maximization problem subsequent to second generation product, which is as follows

$$\max_{\substack{p_1, p_2, u_1, u_2 \leq 0 \text{ for each segment} \in [t, T]}} J_2 = \int_{t}^{T} e^{-\alpha t} \left( \sum_{i=1}^{m} \left[ \left( p_{1i}(t) - c_{1i}(t) \right) x_{1i}(t) + \left( p_{2i}(t) - c_{2i}(t) \right) x_{2i}(t) \right] \right) \, dt$$  \hspace{1cm} (4.3.28)

subject to

$$\dot{x}_{1i}(t) = (1-\psi_i) \left( a_i + b_i \frac{x_{1i}(t)}{X_{1i}} \right) y_{1i}(t) \left( X_{1i} - x_{1i}(t) \right) e^{-\alpha x_{1i}} \quad \forall i$$

$$\dot{x}_{2i}(t) = \left( a_{2i} + b_{2i} \frac{x_{2i}(t)}{X_{2i}} \right) y_{2i}(t) \left( X_{2i} - x_{2i}(t) \right) e^{-\alpha x_{2i}} + \psi_i \left( a_i + b_i \frac{x_{1i}(t)}{X_{1i}} \right) y_{1i}(t) \left( X_{1i} - x_{1i}(t) \right) e^{-\alpha x_{1i}} \quad \forall i$$

Where $x_{1i}(\tau) = x_{1i}^\ast$, $x_{2i}(\tau) = 0$.

The solution of above problem yields the optimal advertising $(u_1^\ast, u_2^\ast)$ and price policies $(p_{1i}^\ast, p_{2i}^\ast)$ for each segment in the time horizon $[t, T]$.

The stage one of the optimal control problem is

$$\max_{\substack{p_1 \geq 0 \text{ for each segment} \in [0, t]}} J_1 = \int_{0}^{t} e^{-\alpha t} \left( \left( \sum_{i=1}^{m} \left[ p_{1i}(t) - c_{1i}(t) \right] x_{1i}(t) \right) - \left( \alpha_1 u_1(t) + \beta_1 \right) \right) \, dt + J_2$$  \hspace{1cm} (4.3.29)

Subject to

$$\dot{x}_{1i}(t) = \left( a_i + b_i \frac{x_{1i}(t)}{X_{1i}} \right) u_1(t) \left( X_{1i} - x_{1i}(t) \right) e^{-\alpha x_{1i}} \quad \forall i, \quad t < \tau$$

Where $x_{1i}(0) = 0 \forall i$.

Both equations (4.3.28) and (4.3.29) are optimal control problems and Maximum Principle is applied to study and solve both stages of the problem.
Analytical Solution: Stage Two Optimal Control Problem When \( t \in [\tau, T] \)

Applying the Maximum Principle as before to characterize the optimal policy. We formulate the Hamiltonian as follows

\[
H = \sum_{i=1}^{n} \left[ \left( p_i(t) - c_i(t) + \lambda_u \right) \left( 1 - \psi_i \right) \left( a_i + b_i \frac{x_i(t)}{X_i} \right) g_i(t) u_i(t) (X_u - x_i(t)) e^{-x_i} \right] + \left[ \left( p_i(t) - c_i(t) + \lambda_x \right) \left( 1 - \psi_i \right) \left( a_i + b_i \frac{x_i(t)}{X_i} \right) g_i(t) u_i(t) (X_u - x_i(t)) e^{-x_i} \right] \\
- \left( \alpha u_i + \beta_i \right) - \left( \alpha u_i + \beta_i \right)
\]

The following necessary conditions hold for optimal values of \( p_i, p_i, u_i, u_2 \) respectively

\[
H_{p_i} = 0 \Rightarrow \left( p_i - c_i + \lambda_u \right) (1 - \psi_i) (f_i)_{p_i} + (1 - \psi_i) f_i + (p_i - c_i + \lambda_x) (g_i)_{p_i} = 0 \quad (4.3.31)
\]

\[
H_{p_i} = 0 \Rightarrow \left( p_i - c_i + \lambda_x \right) (g_i)_{p_i} + g_i = 0 \quad (4.3.32)
\]

\[
H_n = 0 \Rightarrow \sum_{i=1}^{n} \left[ \left( p_i - c_i + \lambda_u \right) (1 - \psi_i) (f_i)_{n} + \left( p_i - c_i + \lambda_x \right) (g_i)_{n} \right] - 1 = 0 \quad (4.3.33)
\]

\[
H_n = 0 \Rightarrow \sum_{i=1}^{n} \left( p_i - c_i + \lambda_x \right) (g_i)_{n} - 1 = 0 \quad (4.3.34)
\]

Where, \( f_i = \left( a_i + b_i \frac{x_i(t)}{X_i} \right) \left( 1 - \psi_i \right) \left( a_i + b_i \frac{x_i(t)}{X_i} \right) g_i(t) u_i(t) (X_u - x_i(t)) e^{-x_i} \),

\[
g_i = \dot{x}_i(t) = \left( a_i + b_i \frac{x_i(t)}{X_i} \right) \left( 1 - \psi_i \right) \left( a_i + b_i \frac{x_i(t)}{X_i} \right) g_i(t) u_i(t) (X_u - x_i(t)) e^{-x_i}
\]

These optimality conditions yield the optimal prices and advertising expenditure as follows

\[
p_i^* = \left( \frac{\eta_i}{\eta_i - 1} \right) \left( c_i - \lambda_i \right) - \left( \frac{p_i, (g_i)_{p_i}}{\eta_i \left( 1 - \psi_i \right)} \right), \quad p_i^* = \left( \frac{\eta_i}{\eta_i - 1} \right) \left( c_i - \lambda_i \right)
\]

\[
u_i^* = \sum_{i=1}^{n} \left[ \left( 1 - \psi_i \right) f_i + \frac{p_i}{\eta_i} (g_i)_{p_i} \right] \left( \frac{p_i \beta_i}{\eta_i} \right), \quad u_i^* = \sum_{i=1}^{n} \left[ \beta_i (g_i)_{n} \right], \quad u_i^* = \sum_{i=1}^{n} \frac{\beta_i (g_i)_{n}}{\eta_i}
\]
Where \( \eta_i, \beta_i; i = 1, 2 \) are elasticities of price and advertising with respect to demand respectively. It is observed from equations (4.3.35) and (4.3.36) that price and advertising elasticity directly influence the optimal price and advertising expenditure policies.

From the necessary optimality conditions for adjoint variables, 
\[
d\lambda_i(t)/dt = -\partial H^* /\partial x_i(t), \quad d\lambda_2(t)/dt = -\partial H^* /\partial x_2(t)
\]
we have adjoint variables \( \lambda_i(t) \) and \( \lambda_2(t) \) that satisfies the following differential equations respectively
\[
\dot{\lambda}_i(t) = r \lambda_i - \frac{\partial H}{\partial x_i} = r \lambda_i - \left\{-c_{t_i} (1-\psi_i) f_i + (p_{t_i} - c_{t_i} + \lambda_2)(1-\psi_i)(f_i)_{t_i} \right\}
\]
(4.3.37)
\[
\dot{\lambda}_2(t) = r \lambda_2 - \frac{\partial H}{\partial x_2} = r \lambda_2 - \left\{(p_{t_2} - c_{t_2} + \lambda_2)(g_i)_{t_2} - (c_2)_{t_2} g_i \right\}
\]
(4.3.38)

With transversality conditions, \( \lambda_i(T) = 0 \) and \( \lambda_2(T) = 0 \). On solving the equations (4.3.37) and (4.3.38), we get
\[
\lambda_i(t) = e^{\alpha t} \int_t^T e^{-\alpha s} \left\{(p_{t_i} - c_{t_i} + \lambda_2)(1-\psi_i)(f_i)_{t_i} + (p_{t_2} - c_{t_2} + \lambda_2)(g_i)_{t_2} - (c_2)_{t_2} g_i \right\} ds
\]
(4.3.39)
\[
\lambda_2(t) = e^{\alpha t} \int_t^T e^{-\alpha s} \left\{(p_{t_2} - c_{t_2} + \lambda_2)(g_i)_{t_2} - (c_2)_{t_2} g_i \right\} ds
\]
(4.3.40)

**Stage One Optimal Control Problem when** \( t \in [0, T] \)

In this section, optimal prices \( p_i^* \) and optimal advertising expenditures \( u_i^* \) from the stage one formulated in (4.3.29) are discussed. Here we use \( J_2 \) as a salvage value. The stage one optimal control problem is
\[
\max_{\bar{p}_i \in P_{t_i}^{(\alpha)} \text{ and } u_i} J = \int_0^T e^{-\alpha t} \left[ \left( \sum_{i=1}^{m} \left[ p_{t_i}(t) - c_{t_i}(t) \right] \hat{x}_i(t) \right) - \left( \alpha u_i(t) + \beta_i \right) \right] dt + J_2
\]
(4.3.41)

Subject to
\[
\dot{x}_i(t) = \left( a_{t_i} + b_{t_i} \frac{x_i(t)}{x_i} \right) y_{u_i}(t) / x_i(t) \hat{x}_i(t) + c_{t_i} \hat{x}_i(t) e^{-\alpha t} \quad \forall t, \quad x_i(0) = 0
\]
(4.3.42)

To solve the above problem, Pontraygin Maximum Principle has been applied. Therefore we define the Hamiltonian as
\[
H_i = \left( \sum_{i=1}^{m} \left[ p_{t_i}(t) - c_{t_i}(t) + \mu_i(t) \right] \hat{x}_i(t) \right) - \left( \alpha u_i(t) + \beta_i \right)
\]
(4.3.43)
Confining our interest to interior solutions, we have the first order necessary conditions $(H_2)_n = 0$ and $(H_2)_m = 0$. Therefore optimality conditions yield the optimal prices $p^*_i$ and advertising $u^*_i$.

\[ p^*_i = \left( \frac{\eta_i}{\eta_i - u_i} \right) (c_i - \mu_i) \] \hspace{1cm} (4.3.44)

\[ u^*_i = \beta_i (p_i - c_i + \mu_i) f_i \] \hspace{1cm} (4.3.45)

where $\mu_i (t)$ is the current value adjoint variable, which satisfies the differential equation

\[ \dot{\mu}_i = r \mu_i - \frac{\partial H_2}{\partial x_i} = r (p_i - c_i + \mu_i) (f_i)_u + (c'_i)_u f_i \] \hspace{1cm} (4.3.46)

with the transversality condition, $\mu_i (T) = \frac{\partial J_2}{\partial x_i}$, the solution of the adjoint differential equation is

\[ \mu_i (t) = e^{-r(T-t)} \mu_i (T) + \int_t^T e^{-rs} \left( (f_i)_u - f_i (c'_i)_u \right) ds \] \hspace{1cm} (4.3.47)

The theoretical results of advertising expenditure and price polices are almost identical to those of Padmanabhan and Bass (1993)

We state some immediate consequences of the optimality conditions (4.3.12) and (4.3.13); (4.3.35) and (4.3.36) for segment specific advertising and single advertising channel model. The impact of technological substitution on the price of the first generation product depends on $\psi$ (substitutability between the two products) and the ratio of sensitivity of sales of the two products to price of the first generation product; the effect of technological substitution alone on the price of the first generation product directly; when $\psi$ approaches one, then the optimal price gets down to marginal cost. In other way, as substitution increases the firm starts following a policy that closely resembles the myopic policy with respect to the first generation product.
Proposition 1: With experience curve effects alone, the price of second
generation product monotonically decreases over time.

Proposition 2: If \( g_{p_u} = 0 \), i.e. if sales of the second generation product are
independent of the price of the first generation product, then technological
substitution has no effect on the price of the first generation product. In case
of advertising, if \( g_{p_u} = 0 \) and \( g_{u_{i}} = 0 \) then advertising expenditure for the first
generation product depends upon technological substitution and as
 technological substitution increases, the advertising expenditure for the first
generation product decreases over time.

Proposition 3: If \( g_{p_u} > 0 \) and \( g_{u_{i}} = 0 \) then the effect of technological
substitution in the price problem is to cause the first generation product to be
priced higher and advertising policy recommends that advertising expenditure
of first generation product increases after introduction of the more advanced
version.

Proposition 4: If \( g_{p_u} < 0 \) and \( g_{u_{i}} = 0 \) then the effect of technological
substitution reduces the price and advertising expenditure of the first
generation product.

We have also seen that the technological substitution does not have a direct
effect on the price and advertising expenditure of the second generation
product but has an indirect effect through its impact on the adjoint variables.

4.4. Numerical Illustration

In order to demonstrate the numerical results of the above problem, the
undiscounted continuous optimal problem (4.3.4) is transformed into
equivalent discrete problem (Rosen, 1968) that is solved to present numerical
solution. The discrete optimal control can be written as follows:

\[
\max_{p_{u_{i}}, p_{u_{j}}, u_{i}, u_{j}} \sum_{j=1}^{T} \sum_{i=1}^{n} \left( p_{u_{i}}(j) - c_{i}(j) \right) f_{i} \left( x_{u_{i}}(j), u_{i}(j), p_{u_{i}}(j) \right) + \left( p_{u_{j}}(j) - c_{j}(j) \right) g_{i} \left( x_{u_{i}}(j), x_{u_{j}}(j), u_{i}(j), u_{j}(j), p_{u_{i}}(j) \right) - u_{i}(j) - u_{j}(j) \right)
\]
Subject to

\[
x_i(j+1) = x_i(j) + (1 - \psi_i) f_i(x_i(j), u_i(j), p_i(j))
\]

\[
x_2(j+1) = x_2(j) + g_i(x_i(j), u_i(j), p_i(j), x_2(j), u_2(j), p_2(j))
\]

Similar discrete optimal control problem can be written for single advertising channel. This discrete optimal control problem is solved by using Differential Evolution.

We assume that the duration of all the testing periods are equal, the number of market segments is 4 and the parameters for the model are: \(c_{1i}, c_{2i}, a_{1i}, a_{2i}, b_{1i}, b_{2i}, \psi_i, \bar{X}_{1i}, \bar{X}_{2i}, \kappa_{1i}, \kappa_{2i}\) for the \(i^{th}\) segment \((i=1,\ldots,M)\) for 1\(^{st}\) and 2\(^{nd}\) generation shown in Table 4.1. Suppose the total advertising expenditure for all segments are $40,000 and unit cost of production $500. The parameters of DE are given in Table 4.2. The optimal advertising expenditure and price values for each segment is shown in Table 4.3 and their corresponding total profit is $13489.64.

**Table 4.1: Values of the parameter used to solve the problem**

<table>
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<tr>
<th>Segment</th>
<th>(a_{1i})</th>
<th>(b_{1i})</th>
<th>(a_{2i})</th>
<th>(b_{2i})</th>
<th>(\psi_i)</th>
<th>(\kappa_{1i})</th>
<th>(\kappa_{2i})</th>
<th>(\bar{X}_{1i})</th>
<th>(\bar{X}_{2i})</th>
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<td>.0042</td>
<td>.031</td>
<td>.0042</td>
<td>.02</td>
<td>.011</td>
<td>.011</td>
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<td>.035</td>
<td>.0046</td>
<td>.03</td>
<td>.012</td>
<td>.013</td>
<td>1500</td>
<td>1400</td>
</tr>
<tr>
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<td>.0046</td>
<td>.036</td>
<td>.0043</td>
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<td>.032</td>
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<td>.014</td>
<td>.012</td>
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<td>1700</td>
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</table>

**Table 4.2: Parameter of the DE**

<table>
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<td>Roulette Wheel</td>
</tr>
<tr>
<td>Scaling Factor(F)</td>
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</tr>
<tr>
<td>Crossover Probability(Cr)</td>
<td>.9</td>
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</table>
In case of single advertising media with a fixed segment spectrum, we assume total advertising expenditure for all segments are $10,000 and unit cost of production $500. The optimal advertising expenditure and price for each segment and their corresponding total profit is shown in Table 4.4 and the value of $\gamma_1$ and $\gamma_2$ for each segment is 0.1, 0.3, 0.1, 0.6; 0.1, 0.1, 0.3, 0.6 respectively and their corresponding total profit is $1345.811.
Table 4.4: The optimal advertising expenditure and price for each segment in single advertising channel

<table>
<thead>
<tr>
<th></th>
<th>$T_1$</th>
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<th>$T_3$</th>
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### 4.4 CONCLUSION

In this paper, we presented a diffusion model for the sales of a two generation product in the segmented market and discussed the optimal advertising expenditure and price policies. We assume that firm has defined its target market in a segmented consumer population and it wants to plan the advertising process for the different segments with the objective of maximizing profit. We have introduced market segmentation concept in the Diffusion Model and discussed its optimal control formulation and solution has been discussed using maximum principle approach. Evolution of segmented sales...
rate is developed under the assumption that practitioner may choose independently the advertising intensity directed to each segment. Further, we describe the advertising process in each segment through single advertising channel with a fixed segment spectrum. We have extended the Padmanabhan and Bass model (1993) by adding advertising expenditure as a control variable. The results discussed here are consistent with the work of Dockner and Jorgenson (1988) and Padmanabhan and Bass (1993) for single generation and successive generation product respectively. The optimal advertising expenditure and price can decrease and increase in many conditions. We have also studied the impact of cannibalization on price and advertising expenditure of the first generation product. The theoretical results obtained here confirm that the optimality conditions as described in literature for a single generation product can also hold for two generation product. The present chapter has been discussed under the assumption that firm has only two generation product in the segmented market. A natural extension to the analysis developed here is the consideration of \( n \)-generation product under dynamic market conditions.