Chapter-2

Optimal Production and Inventory Policies for Inventory-Level-Dependent Demand in Segmented Market

The inventory control problem of retail stores is complicated because inventory and demand are not independent of each other (Whitin 1957)). It has been observed that a large display of items attracts more customers. So, a retailer may display a large number of items to increase the demand. Increase in inventory display gives customers a wider selection choice and increase the probability of sale. For example demand of women’s dresses, decorating items, greeting cards, gift packs, sports clothes etc is proportional to the inventory on display (Wolfe (1968)). This is assumed that presence of retail inventory has motivating effect on the customer. The retailers keep high levels of inventory, as long as item is profitable and the demand increase with inventory level. Many authors noted that the inventory levels have stimulating effect on the demand of many retail items. Many marketing researchers/practitioners recognized the demand is not an exogenous variable; it is endogenous and is a function of the inventory level (Chang and Dye (2000), Kar et al. (2001), Chung (2003), Sana and Chaudhary (2003), Zhou and Yang (2005), Alfares (2007), Goyal and Chang (2009), Pando et al. (2012), Rathod and Bhathawala (2012)).

The area of marketing research in inventory theory has been receiving considerable attention to the inventory models with inventory-level-dependent demand. There are two different types of inventory control models: First, models in which demand is dependent on the initial-inventory-level (Urban (1995), Su et al. 1996, Reddy and Sharma (2001), Dana and petruzzi 2001);


Second, models in which demand is a function of instantaneous-inventory-level (Jorgensen and Kort (2002), Das and Maiti (2003), Zhon and Yang (2003), Balkhi and Benkherouf (2004), Pando et al. (2012)). Let us consider an example, where demand is a linear function of inventory level \( D = aI + b \); \( D \) is demand and \( I \) is inventory; \( a \) and \( b \) are constant. If \( I \) is initial-inventory-level then \( D \) is constant and if \( I \) is instantaneous-inventory-level then \( D \) changes with time. In certain situations these two types of models are equivalent. Many authors discussed optimal policies for demand is linear function of inventory level (Roy and Maiti (1998), Mandal and Maiti (1999), Chung et al. (2000), Kar et al. (2001), Jorgensen and Kort (2002), Ouyang et al. (2003), Das and Maiti (2003), Sana and Chaudhri (2003)). In these situations, we develop a model where demand is dependent on initial-inventory-level and show how second type model (i.e. demand depends on instantaneous-inventory-level) can be expressed as an equivalent form of first type model. Pando et al. (2012) explains an inventory model with both demand and holding cost per unit time dependent on the stock level. This chapter discusses the case where holding cost is a function of inventory, and demand depends on time and instantaneous-inventory-level.

In recent years, there has been a growth of interest in market segmentation application to scientific inventory control models (Duran et al. (2007), Chen and Li (2009)). Segmented customers place demand continuously over time with rates that vary from segment to segment. In response to segmented customers demand, the firm must decide on how much inventory to stock and when to replenish this stock by production. So the firms define their target market in segmented consumer population and develop production-inventory plan to attack each segment.

Optimal control theory application to production-inventory system raised by several authors (axsater (1985), Wikner (1994), Sethi and Thompson (2000), Hedjar et al. (2004), Benhadid et al. (2008)). In this chapter an inventory control model with the objective to minimize the total cost is developed. In this inventory control model the rate of inventory depends on production and total
demand from each segment. This inventory system evolves over time and so we call it dynamic inventory system. This dynamic inventory system can be solved by many optimization techniques. We use optimal control theory to solve and study this dynamic inventory system; and we discuss optimal production and inventory policies for this dynamic system.

This chapter is divided into two sections: section 2.1 and section 2.2. Section 2.1 discusses the optimal control policies for the single-item. In this section; first we discuss the case when production and inventory are at one location, and demand is coming from each segment. Second part of this section gives the optimal control policies for single source production with multi destination inventory and demand; this multi destination inventory is at each segment of market. Section 2.2 gives optimal production and inventory policies for the multi-item. Same as earlier section, first we discuss optimal policies for single source production and inventory with multi destination demand that vary from segment to segment. The later part of this section gives optimal production and inventory policies for single source production with multi destination inventory and demand, this inventory is at each market segment. Numerical problem is solved by Lingo.

2.1 OPTIMAL CONTROL POLICIES OF A PRODUCTION AND INVENTORY SYSTEM FOR SINGLE PRODUCT IN SEGMENTED MARKET

The inventory theory has tended to concentrate on the question, how much and when to add to the inventory of a product. A natural way of constructing a single product problem is to consider customers who demand other items beside this item. An evaluation of this must await the development of multi product theory. It is reasonable to anticipate that single product analysis is probably no worse than single product results in multi product situations. Single product inventory systems are in some way part of larger systems (i.e. multi product inventory system). In this section, we discuss optimal control policies of inventory-production system of single product for two cases; first we discuss the system of single source inventory and then inventory holding at multiple locations.
Notations

$T$ : Length of planning period,

$P(t)$ : Production rate at time $t$,

$I(t)$ : Inventory level at time $t$ (single source inventory),

$I_i(t)$ : Inventory level at time $t$ in $i^{th}$ segment,

$D_i(t)$ : Demand rate at time $t$ in $i^{th}$ segment,

$h(I(t))$ : Holding cost rate (single source inventory),

$h_i(I_i(t))$ : Holding cost rate for $i^{th}$ segment,

$c$ : The unit production cost rate,

$\theta(t,I(t))$ : Deterioration rate at time $t$ (single source inventory),

$\theta_i(t,I_i(t))$ : The deterioration rate at time $t$ in $i^{th}$ segment,

$k(p(i))$ : Cost rate corresponding to the production rate,

$r_i$ : The revenue rate per unit sale in $i^{th}$ segment,

$\rho$ : Constant non-negative discount rate.

Assumptions

1. The time horizon is finite.
2. Production is function of time.
3. Demand is function of time and inventory.
4. The holding cost rate is function of inventory level.
5. Production cost rate depends on the production rate.
6. The functions $h(I_i(t))$ (in case of single source $h(I(t))$) are convex.
7. All functions are non negative and differentiable.
8. The firm may choose independent inventory directed to each segment.
This allows us to derive the most general and robust conclusions. Further, we will consider more specific cases for which we obtain some important results.

2.1.1 Single Source Production and Inventory with Multi Destination Demand Problem

Many manufacturing enterprises use a production–inventory system to manage fluctuations in consumers demand for the product. Such a system consists of a manufacturing plant and a finished goods warehouse to store those products which manufactured but not immediately sold. Here, we assume that once a product is made and put inventory into single warehouse, the demand for single product comes from each segment. Therefore, the inventory evolution in segmented market is described by the following differential equation:

$$\frac{d}{dt} I(t) = P(t) \cdot \sum_{i=1}^{n} D_i(t, I(t))$$  \quad (2.1.1)

The optimal control problem is defined to be admissible production rate which minimize the total cost is given by:

$$\min_{P(t), I(t)} J(P, I) = \int_{t_0}^{T} \left( k(P(t)) + h(I(t)) \right) dt$$ \quad (2.1.2)

Subject to

$$\frac{d}{dt} I(t) = P(t) \cdot \sum_{i=1}^{n} D_i(t, I(t)) \quad I(0) = I_0 \text{ & } I(T) = I_f$$

This optimal control problem has one control variable (rate of production) and one state variable (inventory states). The total demand occurs at rate $\sum_{i=1}^{n} D_i(t, I(t))$ and production occurs at controllable rate $P(t)$. The constraints $P(t) \geq \sum_{i=1}^{n} D_i(t, I(t)) \quad \text{and } I(0) = I_0 \geq 0$ ensure that shortages are not allowed.

Using the maximum principle, the necessary conditions for $(P^*, I^*)$ to be an optimal solution of above problem are that there should exist a piecewise
continuously differentiable function $\lambda$ and piecewise continuous function $\mu$, called the adjoint and Lagrange multiplier functions respectively, such that

$$H(t,I',P',\lambda) \geq H(t,I',P,\lambda), \text{ for all } P(t) \geq \sum_{i=1}^{n} D_i(t,I(t))$$  \hspace{1cm} (2.1.3)

$$\frac{d}{dt} \lambda(t) = -\frac{\partial}{\partial I} L(t,I,P,\lambda,\mu)$$  \hspace{1cm} (2.1.4)

$$I(0) = I_0, \lambda(T) = \beta$$  \hspace{1cm} (2.1.5)

$$\frac{\partial}{\partial P} L(t,I,P,\lambda,\mu) = 0$$  \hspace{1cm} (2.1.6)

$$P(t) - \sum_{i=1}^{n} D_i(t,I(t)) \geq 0, \mu(t) \geq 0, \mu(t) \left[ P(t) - \sum_{i=1}^{n} D_i(t,I(t)) \right] = 0$$  \hspace{1cm} (2.1.7)

Where, $H(t,I,P,\lambda) \text{ and } L(t,I,P,\lambda,\mu)$ are Hamiltonian function and Lagrangian function respectively. In the present problem Hamiltonian function and Lagrangian function are defined as

$$H(t,I,P,\lambda) = e^{-\rho t} \left\{ K(P(t)) + h(I(t)) \right\} + \lambda(t) \left[ P(t) - \sum_{i=1}^{n} D_i(t,I(t)) \right]$$  \hspace{1cm} (2.1.8)

$$L(t,I,P,\lambda,\mu) = e^{-\rho t} \left\{ K(P(t)) + h(I(t)) \right\} + (\lambda(t) + \mu(t)) \left[ P(t) - \sum_{i=1}^{n} D_i(t,I(t)) \right]$$  \hspace{1cm} (2.1.9)

From equation (2.1.4) and (2.1.6), we have following equations respectively

$$\frac{d}{dt} \lambda(t) = e^{-\rho t} \frac{d}{dt} h(I(t)) + (\lambda(t) + \mu(t)) \frac{\partial}{\partial I} \left( \sum_{i=1}^{n} D_i(t,I(t)) \right)$$  \hspace{1cm} (2.1.10)

$$\lambda(t) + \mu(t) = e^{-\rho t} \frac{d}{dp} K(P(t))$$  \hspace{1cm} (2.1.11)

Now, consider equation (2.1.7). Then for any $t$, we have either

$$P(t) - \sum_{i=1}^{n} D_i(t,I(t)) = 0 \text{ or } P(t) - \sum_{i=1}^{n} D_i(t,I(t)) > 0.$$  \hspace{1cm}

**Case 1**

Let $S$ is a subset of planning period $[0,T]$, when $P(t) - \sum_{i=1}^{n} D_i(t,I(t)) = 0$. Then $dI(t)/dt = 0 \text{ on } S$. In this case $I^*$ is constant on $S$ and the optimal production rate is given by the following equation.
Optimal Production and Inventory Policies for Inventory-Level-Dependent-Demand in Segmented Market

\[ P'(t) = \sum_{i=1}^{n} D_i(t, I(t)) \forall t \in S \]  
(2.1.12)

By using equation (2.1.10) and (2.1.11), we have

\[ \frac{d}{dt} \lambda(t) = e^{-\rho t} \left( \frac{d}{dt} h(I(t)) + \frac{d}{dP} K(P(t)) \frac{\partial}{\partial I} \sum_{i=1}^{n} D_i(t, I(t)) \right) \]  
(2.1.13)

Solving the above equation, we get an explicit form of the adjoint function \( \lambda(t) \). Using this \( \lambda(t) \) and equation (2.1.10), we can obtain the value of Lagrange multiplier \( \mu(t) \).

Case 2

\[ P(t) - \sum_{i=1}^{n} D_i(t, I(t)) > 0 \forall t \in [0, T] \setminus S. \]  

Then \( \mu(t) = 0 \forall t \in [0, T] \setminus S \). In this case, the equation (2.1.10) and (2.1.11) becomes

\[ \frac{d}{dt} \lambda(t) = e^{-\rho t} \frac{d}{dt} h(I(t)) + \frac{d}{dP} K(P(t)) \lambda(t) \]  
(2.1.14)

\[ \lambda(t) = e^{-\rho t} \frac{d}{dP} K(P(t)) \]  
(2.1.15)

Combining these equations with the state equation, we have the following second order differential equation:

\[ \frac{d}{dt} P(t) \frac{d^2}{dP^2} K(P) - \left( \rho + \frac{\partial}{\partial I} \sum_{i=1}^{n} D_i(t, I(t)) \right) \frac{d}{dP} K(P) = \frac{\partial h(t, I(t))}{\partial I} \]  
(2.1.16)

And \( I(0) = I_0, dK(P(T))/dP = \beta e^{\rho T} \).

For illustration purpose, let us assume the following exogenous functions

\[ K(P(t)) = kp^2(t)/2, \ h(t, I(t)) = h I^2(t)/2 \] and \[ D_i(t, I(t)) = a_i(t) + b_i \alpha_i I(t), \] where \( k, h, \alpha_i, b_i \) are positive constants. Here \( \alpha_i \) shows the spectrum effect of single inventory affecting the demand of \( i^{th} \) segment.

For these functions the necessary conditions for \( (P', I') \) to be an optimal solution of problem (2.1.2) with equation (2.1.1) becomes
Contribution to Some Optimization Problems for Inventory System and Marketing

\[ \frac{d^2}{dt^2} I(t) - \rho \frac{d}{dt} I(t) - \left( \frac{h}{k} + \rho + \sum_{i=1}^{n} b_i \alpha_i \right) I(t) = \eta(t) \quad (2.1.17) \]

with \( I(0) = I_0, dK(P(T))/dP = \beta e^{\omega t} \). Where \( \eta(t) = \left( \rho + \sum_{i=1}^{n} b_i \alpha_i \right) \sum_{i=1}^{n} a_i(t) - \sum_{i=1}^{n} \dot{a}_i(t) \). This problem is a two-point boundary value problem.

**Proposition:** The optimal solution of \((p^*, I^*)\) to the problem is given by

\[ I^*(t) = c_1 e^{\omega t} + c_2 e^{\omega^{*} t} + Q(t) \quad (2.1.18) \]

and the corresponding \( p^* \) is given by

\[ p^*(t) = c_1 \left( m_1 + \sum_{i=1}^{n} b_i \alpha_i \right) e^{\omega t} + c_2 \left( m_2 + \sum_{i=1}^{n} b_i \alpha_i \right) e^{\omega^{*} t} + \frac{d}{dt} Q(t) + \left( \sum_{i=1}^{n} b_i \alpha_i \right) Q(t) + \sum_{i=1}^{n} a_i(t) \quad (2.1.19) \]

Where the constants \( c_1, c_2, m_1, \text{and} \ m_2 \) are given in the proof, and \( Q(t) \) is a particular solution of the equation (2.1.17).

**Proof:** The solution of the two point boundary value problem (2.1.17) is given by a standard method. Its characteristic equation \( m^2 - \rho m - \left( \frac{h}{k} + \left( \rho + \sum_{i=1}^{n} b_i \alpha_i \right) \sum_{i=1}^{n} b_i \alpha_i \right) = 0 \), has two real roots of opposite sign, given by

\[ m_1 = \frac{1}{2} \left( \rho - \sqrt{\rho^2 + 4 \left( \frac{h}{k} + \left( \rho + \sum_{i=1}^{n} b_i \alpha_i \right) \sum_{i=1}^{n} b_i \alpha_i \right) } \right) < 0, \]

\[ m_2 = \frac{1}{2} \left( \rho + \sqrt{\rho^2 + 4 \left( \frac{h}{k} + \left( \rho + \sum_{i=1}^{n} b_i \alpha_i \right) \sum_{i=1}^{n} b_i \alpha_i \right) } \right) > 0 \]

and therefore \( I^*(t) \) is given by (2.1.18), where \( Q(t) \) is the particular solution. Then initial and terminal conditions used to determine the values of constant \( a_1 \) and \( a_2 \) are as follows

\[ c_1 + c_2 + Q(0) = I_0, \]

\[ c_1 (m_1) e^{\omega t} + c_2 (m_2) e^{\omega^{*} t} + \frac{d}{dt} Q(T) + \sum_{i=1}^{n} b_i \alpha_i Q(T) + \sum_{i=1}^{n} a_i(T) = 0 \]
Putting \( r_1 = I_0 - Q (0) \) and \( r_2 = - \left( \frac{d}{dt} Q (T) + \sum_{i=1}^{n} b_i \alpha_i Q (T) + \sum_{i=1}^{n} a_i (T) \right) \), we obtain the following system of two linear equations with two unknowns

\[
\begin{align*}
    c_1 + c_2 &= r_1, \\
    c_1 (m_i) e^{\alpha_i} + c_2 (m_i) e^{\alpha_i} &= r_2
\end{align*}
\]  

(2.1.20)

The value of \( P^* \) is deduced using the value of \( I^* \) and the state equation.

### 2.1.2 Single Source Production with Multi Destination Inventory and Demand Problem

In this section, we assume that product is being made at one place and its inventory is held into multiple locations. This inventory is held at different market segments i.e. different demand segments. We can say that firm may choose independently the inventory directed to each segment. The optimal control problem is defined to be admissible production rate which minimizes the total cost given by:

\[
\begin{align*}
\text{Min } \int_0^T J = e^{\gamma_0} \left( \sum_{i=1}^{n} h_i (I_i (t)) + K (P (t)) \right) dt
\end{align*}
\]  

(2.1.21)

Subject to

\[
\frac{d}{dt} I_i (t) = \gamma_i P (t) - D_i (t, I_i (t))
\]  

(2.1.22)

Here, we assume that \( \gamma_i > 0 \) and \( \sum_{i=1}^{n} \gamma_i = 1 \) with the conditions \( I_i (0) = I_i^0 \) and \( \gamma_i P (t) \geq D_i (t, I_i (t)) \). We call \( \gamma_i \) as segment production spectrum and \( \gamma_i P (t) \) define the relative segment production rate towards \( i^{th} \) segment.

This is an optimal control problem with production rate as one control variable and stock of inventory in \( n \) segments as \( n \) state variables.

To solve this optimal control problem, the Hamiltonian and Lagrangian are defined as

\[
\begin{align*}
H &= e^{\gamma_0} \sum_{i=1}^{n} \left[ h_i (I_i (t)) + K (P (t)) + \lambda_i (t) (\gamma_i P (t) - D_i (t, I_i (t))) \right] \\
L (t, I, P, \lambda, \mu) &= -e^{\gamma_0} \sum_{i=1}^{n} \left[ h_i (I_i (t)) + K (P (t)) + \left( \lambda_i (t) + \mu_i (t) \right) \left( \gamma_i P (t) - D_i (t, I_i (t)) \right) \right]
\end{align*}
\]  

(2.1.23) \hspace{1cm} (2.1.24)
Equation (2.1.4), (2.1.6) and (2.1.22) yield

\[
\frac{d}{dt} \lambda_i(t) = - \frac{\partial L(I_i(t))}{\partial I_i} \tag{2.1.25}
\]

\[
\sum_{i=1}^{n} (\lambda_i(t) + \mu_i(t)) \gamma_i = e^{-r \tau} \frac{d}{dp} K(P(t)) \tag{2.1.26}
\]

In the next section of the paper, we consider only case when \( \gamma_i P(t) - D_i(t, I_i(t)) > 0 \).

**Case 2**

\( \gamma_i P(t) - \sum_{i=1}^{n} D_i(t, I_i(t)) > 0 \forall t \in [0, T] \setminus S \). Then \( \mu_i(t) = 0, \forall t \in [0, T] \setminus S \). In this case, the equation (2.1.25) and (2.1.26) becomes

\[
\frac{d}{dt} \lambda_i(t) = - \frac{\partial L(I_i(t))}{\partial I_i} \tag{2.1.27}
\]

\[
\sum_{i=1}^{n} \lambda_i(t) \gamma_i = e^{-r \tau} \frac{d}{dp} K(P(t)) \tag{2.1.28}
\]

Combining above equations with the state equation, we have the following second order differential equation:

\[
\frac{d}{dt} \left( P(t) \frac{d^2}{dp^2} K(P(t)) - \rho \frac{d}{dp} K(P(t)) \right) = \sum_{i=1}^{n} \gamma_i \left( \frac{\partial h_i(t, I_i(t))}{\partial I_i} + \lambda_i \frac{\partial}{\partial I_i} D_i(t, I_i(t)) \right) \tag{2.1.29}
\]

And \( I_i(0) = I_i^*, \sum_{i=1}^{n} \beta_i \gamma_i = e^{-r \tau} \frac{d}{dp} K\left(P(T)\right)/dp, \lambda_i(T) = \beta_i \). For illustration purpose, let us assume the following exogenous functions \( K(P(t)) = kP^2(t)/2, h_i(t, I_i(t)) = h_i I_i^2(t)/2 \) and \( D_i(t, I_i(t)) = g_i(t) + q_i I_i(t) \), where \( k, h_i, q_i \) are positive constants.

For these functions the necessary conditions for \( (P^*, I_i^*) \) to be an optimal solution of problem (2.1.19) with equation (2.1.18) becomes

\[
I_i^*(t) + (q_i - \rho) I_i^*(t) - \rho q_i I_i(t) = \eta_i(t) \tag{2.1.29}
\]

With \( I_i(0) = I_i^*, \sum_{i=1}^{n} \beta_i \gamma_i = e^{-r \tau} \frac{d}{dp} K\left(P(T)\right)/dp, \lambda_i(T) = \beta_i \).

Where \( \eta_i(t) = (\rho - 2q_i) g_i(t) + (\gamma_i/k) \sum_{i=1}^{n} (h_i I_i(t) + \lambda_i q_i) - d g_i(t)/dt \). This problem is also a system of two-point boundary value problems.

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The above system of two point boundary value problem (2.1.29) is solved by same method that we used to solve (2.1.17).

2.2 OPTIMAL CONTROL POLICY OF A PRODUCTION AND INVENTORY SYSTEM FOR MULTI PRODUCT IN SEGMENTED MARKET

Many of the classical inventory models concern with single-item. In real life single-item inventory models seldom occur. The multi-item inventory system are more realistic than the single-item inventory system. The analysis of single-item inventory system is almost similar to the multi-item inventory system. The results of single-item and multi-item inventory system are almost parallel. Single product inventory systems are in some way part of larger systems (i.e. multi-product inventory system). In this section, we use market segmentation approach in multi-product inventory system with inventory-level-dependent demand. The objective is to make use of optimal control theory to solve the production–inventory problem and develop an optimal production policy that minimizes the total cost associated with inventory and production rate in segmented market. First, we consider a single production and inventory problem with multi-destination demand that vary from segment to segment. Further, we described a single source production–multi destination demand and inventory problem under the assumption that firm may choose independently the inventory directed to each segment. The optimal control is applied to study and solve the proposed problem.

Notations

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>Length of planning period,</td>
</tr>
<tr>
<td>$P_j(t)$</td>
<td>Production rate for $j^{th}$ product,</td>
</tr>
<tr>
<td>$I_j(t)$</td>
<td>Inventory level for $j^{th}$ product,</td>
</tr>
<tr>
<td>$I_{ij}(t)$</td>
<td>Inventory level for $j^{th}$ product in $i^{th}$ segment,</td>
</tr>
<tr>
<td>$D_{ij}(t, I_{ij}(t))$</td>
<td>Demand rate for $j^{th}$ product in $i^{th}$ segment,</td>
</tr>
<tr>
<td>$h_j(I_j(t))$</td>
<td>Holding cost rate for $j^{th}$ product,</td>
</tr>
<tr>
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<td>Holding cost rate for $j^{th}$ product in $i^{th}$ segment,</td>
</tr>
<tr>
<td>$K_j(P_j(t))$</td>
<td>Cost rate corresponding to the production rate for $j^{th}$ product,</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Constant non-negative discount rate.</td>
</tr>
</tbody>
</table>
Assumptions

1. The time horizon is finite.
2. Production is function of time.
3. Demand is function of time and inventory.
4. The holding cost rate is function of inventory level.
5. Production cost rate depends on the production rate.
6. The function \( h(I_i(t)) \) (in case of single source \( h(I(t)) \)) is convex.
7. All functions are non-negative and differentiable functions.
8. The firm may choose independent inventory directed to each segment.

This allows us to derive the most general and robust conclusions. Further, we will consider more specific cases for which we obtain some important results.

2.2.1 Single Source Production and Inventory with Multi Destination Demand Problem

Many manufacturing enterprises use a production–inventory system to manage fluctuations in consumers demand for the product. Such a system consists of a manufacturing plant and a finished goods warehouse to store those products which are manufactured but not immediately sold. Here, we assume that once a product is made and put as inventory into single warehouse, the demand for single product comes from each segment. The optimal control problem is defined to be admissible production rates which minimize the total cost given by:

\[
\min_{P(t), I(t)} J = \int_{0}^{T} \left[ \sum_{j=1}^{m} \left( K_j(P_j(t)) + h_j(I_j(t)) \right) \right] dt
\]

Subject to

\[
\frac{d}{dt} I_j(t) = P_j(t) - \sum_{i=1}^{m} D_i(t, I_j(t)), \quad I_j(0) = I_{j0}
\]

This optimal control problem has \( m \) control variable as production rate with \( m \) state variable as inventory states. Since total demand occurs at rate \( \sum_{i=1}^{m} D_i(t, I_j(t)) \) and production
occurs at controllable rate \( P_j(t) \) for all products. The constraints \( P_j(t) \geq \sum_{i=1}^{n} D_y(t, I_j(t)) \) and \( I_j(0) = I_{j,0} \geq 0 \) ensure that shortage are not allowed.

Using the maximum principle, the necessary conditions for \((P', I')\) to be an optimal solution of above problem are that there should exist a piecewise continuously differentiable function \( \lambda \) and piecewise continuous function \( \mu \), called the adjoint and Lagrange multiplier function, respectively such that

\[
H(t, I', P', \lambda) \geq H(t, I', P, \lambda), \text{ for all } P_j(t) \geq \sum_{i=1}^{n} D_y(t, I_j(t)) \tag{2.2.3}
\]

\[
\frac{d}{dt} \lambda_j(t) = -\frac{\partial}{\partial I_j} L(t, I, P, \lambda, \mu) \tag{2.2.4}
\]

\[
I_j(0) = I_{j,0}, \lambda_j(T) = 0 \tag{2.2.5}
\]

\[
\frac{\partial}{\partial P_j} L(t, I, P, \lambda, \mu) = 0 \tag{2.2.6}
\]

\[
P_j(t) - \sum_{i=1}^{n} D_y(t, I_j(t)) \geq 0, \quad \mu_j(t) \geq 0, \quad \mu_j(t) \left[ P_j(t) - \sum_{i=1}^{n} D_y(t, I_j(t)) \right] = 0 \tag{2.2.7}
\]

Where, \( H(t, I, P, \lambda) \) and \( L(t, I, P, \lambda, \mu) \) are Hamiltonian function and Lagrangian function respectively. In the present problem Hamiltonian function and Lagrangian function are defined as

\[
H(t, I, P, \lambda) = \sum_{j=1}^{m} \left[ -K_j \left( P_j(t) \right) - h_j \left( I_j(t) \right) + \lambda_j(t) \left( P_j(t) - \sum_{i=1}^{n} D_y(t, I_j(t)) \right) \right] \tag{2.2.8}
\]

\[
L(t, I, P, \lambda, \mu) = \sum_{j=1}^{m} \left[ -K_j \left( P_j(t) \right) - h_j \left( I_j(t) \right) + \lambda_j(t) + \mu_j(t) \left( P_j(t) - \sum_{i=1}^{n} D_y(t, I_j(t)) \right) \right] \tag{2.2.9}
\]

From equation (2.2.4) and (2.2.6), we have following equations respectively

\[
\frac{d}{dt} \lambda_j(t) = \rho \lambda_j(t) + \frac{\partial h_j(t, I_j(t))}{\partial I_j} + \lambda_j(t) + \mu_j(t) \left( \frac{\partial}{\partial I_j} \left( \sum_{i=1}^{n} D_y(t, I_j(t)) \right) \right) \tag{2.2.10}
\]

\[
\lambda_j(t) + \mu_j(t) = \frac{d}{dP_j} K_j \left( P_j(t) \right) \tag{2.2.11}
\]

Now, consider equation (2.2.7). Then for any \( t \), we have either \( P_j(t) - \sum_{i=1}^{n} D_y(t, I_j(t)) = 0 \) or \( P_j(t) - \sum_{i=1}^{n} D_y(t, I_j(t)) > 0 \)
Case 1

Let $S$ is a subset of planning period $[0, T]$, when $P_j(t) - \sum_{i=1}^{n} D_{ij}(t, I_j(t)) = 0$. Then $\frac{dP_j(t)}{dt} = 0$ on $S$. In this case $I_j^*$ is constant on $S$ and the optimal production rate is given by the following equation

$$P_j^*(t) = \sum_{i=1}^{n} D_{ij}(t, I_j(t)) \forall t \in S$$  \hspace{1cm} (2.2.12)

Using equation (2.2.10) and (2.2.11), we have

$$\frac{d}{dt} \lambda_j(t) = \rho \lambda_j(t) + \left[ \frac{\partial h_j(I_j(t))}{\partial I_j} + \lambda_j(t) \left( \frac{\partial}{\partial I_j} \sum_{i=1}^{n} D_{ij}(t, I_j(t)) \right) \right]$$  \hspace{1cm} (2.2.13)

Solving the above equation, we get an explicit form of the adjoint function $\lambda_j(t)$. Using this $\lambda_j(t)$ and equation (2.2.10), we can obtain the value of Lagrange multiplier $\mu_j(t)$.

Case 2

$$P_j(t) - \sum_{i=1}^{n} D_{ij}(t, I_j(t)) > 0 \forall t \in [0, T] \setminus S$$. Then $\mu_j(t) = 0 \forall t \in [0, T] \setminus S$. In this case, the equation (2.2.10) and (2.2.11) becomes

$$\frac{d}{dt} \lambda_j(t) = \rho \lambda_j(t) + \left[ \frac{\partial h_j(I_j(t))}{\partial I_j} + \lambda_j(t) \left( \frac{\partial}{\partial I_j} \sum_{i=1}^{n} D_{ij}(t, I_j(t)) \right) \right]$$  \hspace{1cm} (2.2.14)

$$\lambda_j(t) = c_j + \frac{d}{dP_j} K_j(P_j(t))$$  \hspace{1cm} (2.2.15)

Combining these equations with the state equation, we have the following second order differential equation:

$$\frac{d}{dt} P_j(t) \frac{d^2}{dP_j^2} K_j(P_j) - \left[ \rho + \frac{\partial}{\partial I_j} \sum_{i=1}^{n} D_{ij}(t, I_j(t)) \right] \frac{d}{dP_j} K_j(P_j) = \frac{\partial h_j(t, I_j(t))}{\partial I_j}$$  \hspace{1cm} (2.2.16)

And $I_j(0) = I_{j_0}$, $dK_j(P_j(T))/dP_j = \beta e^{\sigma T}$. For illustration purpose, let us assume the following forms the exogenous functions $K_j(P_j) = k_j P_j^2 / 2$, $h_j(t, I_j(t)) = h_j I_j^2(t) / 2$, and $D_{ij}(t, I_j(t)) = a_{ij}(t) + b_{ij}(t) \alpha_{ij} I_j(t)$, where $k_j, h_j, \alpha_{ij}, b_{ij}$ are positive constants.
For these functions the necessary conditions for \((P^*_j, I^*_j)\) to be an optimal solution of problem (2.2.1) with equation (2.2.2) becomes

\[
\frac{d^2}{dt^2} I_j (t) - \rho \frac{d}{dt} I_j (t) - \left( \frac{h_j}{k_j} + \left( \rho + \sum_{i=1}^{n} b_i \alpha_i \right) \sum_{i=1}^{n} b_i \alpha_i \right) I_j (t) = \eta_j (t) \tag{2.2.17}
\]

with \(I_j (0) = I_{j0}, dK_j (P_j (t)) / dP_j = \beta e^{\alpha P_j} \) and \(\eta_j (t) = \left( \rho + \sum_{i=1}^{n} b_i \alpha_i \right) \sum_{i=1}^{n} a_i (t) - \sum_{i=1}^{n} \dot{a}_i (t)\).

This problem is a two-point boundary value problem.

**Proposition:** The optimal solution of \((P^*_j, I^*_j)\) to the problem is given by

\[
I^*_j (t) = c_1 e^{m_1 t} + c_2 e^{m_2 t} + Q_j (t), \tag{2.2.18}
\]

and the corresponding \(P^*_j\) is given by

\[
P^*_j (t) = c_1 \left( m_j + \sum_{i=1}^{n} b_i \alpha_i \right) e^{m_1 t} + c_2 \left( m_j + \sum_{i=1}^{n} b_i \alpha_i \right) e^{m_2 t} + \frac{d}{dt} Q_j (t) + \left( \sum_{i=1}^{n} b_i \alpha_i \right) Q_j (t) + \sum_{i=1}^{n} a_i (t) \tag{2.2.19}
\]

Where the constants \(c_1, c_2, m_{j1}, \text{and } m_{j2}\) are given in the proof, and \(Q_j (t)\) is a particular solution of the equation (2.2.17).

**Proof:** The solution of the two point boundary value problem (2.2.17) is given by standard method. Its characteristic equation \(m_j^2 - \rho m_j - \left( \frac{h_j}{k_j} + \left( \rho + \sum_{i=1}^{n} b_i \alpha_i \right) \sum_{i=1}^{n} b_i \alpha_i \right) = 0\),

has two real roots of opposite sign, given by

\[
m_{j1} = \frac{1}{2} \left( \rho - \sqrt{\rho^2 + 4 \left( \frac{h_j}{k_j} + \left( \rho + \sum_{i=1}^{n} b_i \alpha_i \right) \sum_{i=1}^{n} b_i \alpha_i \right)} \right) > 0,
\]

\[
m_{j2} = \frac{1}{2} \left( \rho + \sqrt{\rho^2 + 4 \left( \frac{h_j}{k_j} + \left( \rho + \sum_{i=1}^{n} b_i \alpha_i \right) \sum_{i=1}^{n} b_i \alpha_i \right)} \right) < 0,
\]

and therefore \(I^*_j (t)\) is given by (2.2.18), where \(Q_j (t)\) is the particular solution. Then initial and terminal conditions used to determine the values of constant \(a_{j1}\) and \(a_{j2}\) are as follows.
Contribution to Some Optimization Problems for Inventory System and Marketing

Department of Operational Research, University of Delhi

$$c_{ij} + c_{2j} + Q_j(0) = I_{j0},$$

$$c_{ij}(m_{ij})e^{m_{ij}t} + c_{2j}(m_{2j})e^{m_{2j}t} + \left(\frac{d}{dt}Q_j(T) + \sum_{i=1}^{n} b_i r_i Q_i(T) + \sum_{i=1}^{n} a_i T\right) = 0$$

Putting $$r_{ij} = I_{j0} - Q_j(0)$$ and $$r_{2j} = -\left(\frac{d}{dt}Q_j(T) + \sum_{i=1}^{n} b_i r_i Q_i(T) + \sum_{i=1}^{n} a_i T\right),$$ we obtain the following system of two linear equations with two unknowns

$$c_{ij} + c_{2j} = r_i,$$

$$c_{ij}(m_{ij})e^{m_{ij}t} + c_{2j}(m_{2j})e^{m_{2j}t} = r_{2j}$$

(2.2.20)

The value of $$P_j^*$$ is deduced using the value of $$I_j^*$$ and the state equation.

2.2.2 Single Source Production with Multi Destination Inventory and Demand Problem

In this section, we assume that product is being made at one place and its inventory is held into multiple locations. This inventory is held at different market segments i.e. different demand segments. We can say that firm may choose independently the inventory directed to each segment. The optimal control problem is defined to be admissible production rates which minimize the total cost which is give by:

$$\text{Min } J = \int_{0}^{T} \left[ e^{m_{ij}t} \sum_{j=1}^{n} \left[ K_j \left(P_j(t)\right) + \sum_{i=1}^{n} h_{ij} \left(I_{ij}(t)\right) \right] dt \right]$$

(2.2.21)

Subject to

$$\frac{d}{dt} I_{ij}(t) = \gamma_{ij} P_j(t) - D_{ij}(t, I_{ij}(t))$$

(2.2.22)

Here $$\gamma_{ij} > 0, \sum_{i=1}^{n} \gamma_{ij} = 1$$ with the conditions $$I_{ij}(0) = I_{ij}^0$$ and $$\gamma_{ij} P_j(t) \geq D_{ij}(t, I_{ij}(t))$$. We call $$\gamma_{ij}$$ as segment production spectrum and $$\gamma_{ij} P_j(t)$$ defines the relative segment production rate of $$i^{th}$$ product towards $$j^{th}$$ segment.

This optimal control problem has production rate as one control variable with stock of inventory in $$n$$ segments as $$n$$ state variables.
To solve this optimal control problem, the following Hamiltonian and Lagrangian are defined as

\[
H(t, I, P, \lambda) = \sum_{j=1}^{n} \sum_{i=1}^{m} \left[ -K_j \left( P_j(t) \right) - h_j \left( I_j(t) \right) + \lambda_j(t) \left( \gamma_j P_j(t) - D_j(t, I_j(t)) \right) \right] \quad (2.2.23)
\]

\[
L(t, I, P, \lambda, \mu) = \sum_{j=1}^{n} \sum_{i=1}^{m} \left[ -K_j \left( P_j(t) \right) - h_j \left( I_j(t) \right) + \lambda_j(t) + \mu_j(t) \right] \left( \gamma_j P_j(t) - D_j(t, I_j(t)) \right) \quad (2.2.24)
\]

Equation (2.2.4), (2.2.6) and (2.2.22) yield

\[
\frac{d}{dt} \lambda_j(t) = \rho \lambda_j(t) - \left\{ - \frac{\partial h_j \left( I_j(t) \right)}{\partial I_j} - \left( \lambda_j(t) + \mu_j(t) \right) \frac{\partial D_j \left( t, I_j(t) \right)}{\partial I_j} \right\} \quad (2.2.25)
\]

\[
\sum_{j=1}^{n} \lambda_j(t) + \mu_j(t) \gamma_j = \frac{d}{dP_j} K_j \left( P_j(t) \right) \quad (2.2.26)
\]

In the next section of the paper, we consider only case when \( \gamma_j P_j(t) - D_j \left( t, I_j(t) \right) > 0 \ \forall \ i, j \).

Case 2

\( \gamma_j P_j(t) - D_j \left( t, I_j(t) \right) > 0 \ \forall \ t \in [0, T] \setminus S \). Then \( \mu_j(t) = 0 \ \forall \ t \in [0, T] \setminus S \). In this case, the equation (2.2.25) and (2.2.26) becomes

\[
\frac{d}{dt} \lambda_j(t) = \rho \lambda_j(t) + \left\{ - \frac{\partial h_j \left( I_j(t) \right)}{\partial I_j} - \lambda_j(t) \frac{\partial D_j \left( t, I_j(t) \right)}{\partial I_j} \right\} \quad (2.2.27)
\]

\[
\sum_{j=1}^{n} \gamma_j \lambda_j(t) = \frac{d}{dP_j} K_j \left( P_j(t) \right) \quad (2.2.28)
\]

Combining above equations with the state equation, we have the following second order differential equation:

\[
\frac{d}{dt} P_j(t) \frac{d^2}{dP_j^2} K_j \left( P_j \right) - \rho \frac{d}{dP_j} K_j \left( P_j \right) = \sum_{j=1}^{n} \gamma_j \left( \frac{\partial h_j \left( t, I_j(t) \right)}{\partial I_j} + \lambda_j \frac{\partial}{\partial I_j} D_j \left( t, I_j(t) \right) \right) \quad (2.2.29)
\]

And \( I_j(0) = I_j^0, \sum_{j=1}^{n} \gamma_j \lambda_j(T) = e^{-\rho T} \frac{dK_j \left( P_j(T) \right)}{dP_j}, \lambda_j(T) = \beta_j \). For illustration purpose, let us assume the following exogenous functions \( K_j \left( P_j \right) = k_j P_j^2 / 2, h_j \left( t, I_j(t) \right) = h_j I_j(t) \), \( D_j \left( t, I_j(t) \right) = a_j(t) + b_j I_j(t) \) and \( h_j \left( t, I_j(t) \right) = h_j I_j(t)^2 / 2 \) where \( k_j, h_j, b_j \) are positive constants.
For these functions the necessary conditions for \( (P^*_j, I^*_j) \) to be an optimal solution of problem (2.2.19) with equation (2.2.18) becomes
\[
I^*_j(t) + (b_j - \rho)I^*_j(t) - \rho b_j I^*_j(t) = \eta_j(t) \quad \forall i, j
\]
with \( I^*_j(0) = I^*_j \), \( \sum_{i,j}^n \beta_{ij} = e^{-\rho} dK_j \left( P_j(T) \right)/dP_j \), because of \( \lambda_j(T) = \beta_j \forall i \).

Where, \( \eta_j(t) = \rho a_j(t) + (\gamma_j/k_j) \sum_{i,j}^n \beta_{ij} \left( h_{ij} + \lambda_{ij} b_{ij} \right) - da_j(t)/dt \). This problem is also a system of two-point boundary value problems.

The above system of two point boundary value problem (2.2.30) is solved by same method that we used to solve (2.2.17).

### 2.3 NUMERICAL ILLUSTRATION

In order to demonstrate the numerical results of the above problem, the discounted continuous optimal problem (2.2.21) subject to (2.2.22) is transferred into equivalent discrete problem (Rosen 1968) that is solved to present numerical solution. The discrete optimal control can be written as follows:
\[
J = \sum_{t=0}^{T} \left[ \sum_{i,j}^n \left( K_j(P_j(k-1)) + \sum_{i,j}^n h_{ij}(I_j(k-1)) \left( \frac{1}{(1+\rho)^{k-2}} \right) \right) \right]
\]
Subject to
\[
I_j(k) = I_j(k-1) + \gamma_j P_j(k-1) - D_j(k-1, I_j(k-1)) \quad \forall i, j
\]

Similar discrete optimal control problem can be written for single source production –multi destination and inventory control problem. These discrete optimal control problems are solved by using Lingo11. We assume that the duration of all the time periods is equal. The number of market segments \( M \) is 4 and the number of products is 3. The value of parameters are \( \alpha_1=0.236, .252, .290, .222; \alpha_2=0.281, .231, .225, .263; \alpha_3=0.210, .264, .290, .236; \alpha_4=125, 153, 115, 134; a_1=152, 163, 154, 182; a_2=141, 127, 162, 152; b_1=0.35, 0.40, 0.30, 0.25; b_2=0.25, 0.35, 0.25, 0.40; b_3=0.20, 0.40, 0.35, 0.25 for segments \( i=1 \) to 4; \( \rho =0.095; \ k_j =2; \ P_j =10000 \); for all the three products. In case of single source production-multi destination demand and inventory, the number of market segments \( M \) is 4 and the number of products is 3. The values of additional parameters for each segment are shown in Table 2.1.
The optimal production rate and inventory for every product for each segment is shown in Table 2.2. The optimal value of total profit for all products is $989045.

**Table 2.1: Parameters**

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<th>Segment</th>
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<tr>
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**Table 2.2: The Optimal Production Rate, Inventory Rate and Demand Rate for Single Source Production-Multi Destination Demand and Inventory Problem**

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<th></th>
<th>$T_1$</th>
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2.4 CONCLUSION

The concept of market segmentation was developed in economic theory to show how a firm selling a homogenous product in a market characterized by heterogeneous demand could minimize the cost. In this chapter, we have introduced market segmentation concept in the inventory system for single and multi product under single and multi source inventory system. For single and multi product, optimal control model is formulated: (1) single source production and inventory with multi destination demand, and (2) single source production with multi destination inventory and demand, in this case inventory is at each market segment. We used maximum principle to determine the optimal production and inventory rate policies that minimize the cost. For illustration, we took demand as linear function of instantaneous-inventory. To show the numerical results of the above problem, the discounted continuous optimal problem is transferred into equivalent discrete problem (Rosan (1968)) that is solved using Lingo 11 to present numerical solution. The resulting solution yield good insight on how production planning task can be carried out in segmented market environment. A natural extension to the analysis developed here is that items can be deteriorating and price factor could be incorporated as variable for demand.