CHAPTER –2

UNDER-WATER EXPLOSION GIVING RISE TO A SPHERICAL GAS BUBBLE

The basic differential equation giving the growth of the radius $R(t)$ of a spherical gas bubble with the time was given by Rayleigh [37] when the bubble expanded adiabatically in an incompressible, inviscid fluid. Rayleigh’s equation is derived by taking a spherically symmetric flow about the bubble centre which is irrotational and then applying the Bernoulli’s equation at the bubble surface.

2.1 Analysis of the Problem

Let us consider an explosion taking place underwater which was initially at rest. This problem was first considered by Rayleigh [37]. As a result spherical gas bubble will be formed, which expands adiabatically. We shall find the radius of the gas bubble at a given instant after the explosion. Initially, when $t = 0$, let $R_0$ be the radius of the gas bubble. At a given instant ‘t’, after the explosion, we shall compute ‘R’, the radius of the gas bubble. When the gas bubble is formed, after the explosion takes place, let ‘O’ be its centre. The gas bubble expands obeying the adiabatic law of expansion, namely

$$pv^\gamma = \text{constant} \quad (1)$$

where ‘p’ is the pressure and ‘V’ the volume of the gas bubble, constant $\gamma$ is the ratio of the specific heats of the gas. At the time ‘t’ when the bubble is of radius ‘R’, let $r \geq R$, be the radius of a concentric spherical surface in the water. Since
water is an incompressible fluid, the equation of continuity applied to the two concentric surfaces is

\[ 4\pi R^2 \dot{R} = 4\pi r^2 v_r \]  

(2)

where \( v_r \) is the velocity of fluid (in radial direction) on the spherical surface of radius ‘r’. Hence, we have

\[ R^2 \dot{R} = r^2 v_r \]  

(3)

Therefore

\[ V_r = \frac{R^2 \dot{R}}{r^2} \]  

(4)

Consider a point ‘P’ on the surface of the sphere of radius ‘r’ whose spherical polar coordinates are \( (r, \theta, \psi) \).

Hence

\[ V_r = \left( -\frac{\partial \phi}{\partial r} \right) \]  

(5)

where \( \phi \) is the velocity potential, such that

\[ \phi = \phi (r, t) \]  

(6)

Thus, by equation (4) and (5), we have

\[ V_r = \frac{R^2 \dot{R}}{r^2} = -\frac{\partial \phi}{\partial r} \]  

(7)

Integrating equation (7), we get
\[ \phi = \frac{R^2 \dot{R}}{r} \]  

We shall consider the Bernoulli’s equation at \((r, \theta, \psi)\) for an unsteady, incompressible flow under no body forces.

Since the flow is unsteady,

\[ \frac{\partial \phi}{\partial t} \neq 0 \]  

For an incompressible flow,

\[ \int \frac{dp}{\rho} = \frac{p}{\rho} \]  

When the body forces are absent

\[ F = 0 \]  

Hence, the Bernoulli’s equation (Chorlton[9]) takes the form,

\[ \frac{p}{\rho} + \frac{1}{2} V_r^2 - \left[ \frac{\partial \phi}{\partial t} \right] = f(t) \]  

Substituting, values of \( V_r^2 \) and \( \phi \) in equation (12) and let \( p_\infty \) be the pressure at \( r = \infty \), we have

\[ \frac{p - p_\infty}{\rho} + \frac{1}{2} \left[ \frac{R^2 \dot{R}}{r^2} \right]^2 - \frac{\partial}{\partial t} \left[ \frac{R^2 \dot{R}}{r} \right] = 0, \]  

\[ \frac{p - p_\infty}{\rho} + \frac{R^4 \dot{R}^2}{2r^4} - \frac{1}{r} \frac{\partial}{\partial t} (R^2 \dot{R}) = 0 \]
Putting \( r = R \) and \( p = p_g \) (pressure of gas inside the bubble) in equation (14), we get

\[
\frac{p_g - p_\infty}{\rho} + \frac{1}{2} \dot{R}^2 - \frac{1}{R} (R^2 \ddot{R} + 2R \dot{R}^2) = 0
\]  

(15)

or

\[
\frac{p_g - p_\infty}{\rho} + \frac{1}{2} \dot{R}^2 - R \ddot{R} - 2 \dot{R}^2 = 0
\]

or

\[
\frac{p_g - p_\infty}{\rho} = R \ddot{R} + \frac{3}{2} \dot{R}^2
\]

or

\[
p_g - p_\infty = \rho \left( \frac{3}{2} \dot{R}^2 + R \ddot{R} \right)
\]  

(16)

is the pressure of the gas bubble.

Initially when \( t = 0, R = R_0 \), let ‘P’ be the pressure in the gas bubble. Neglecting, the pressure \( p_\infty \) at \( r = \infty \), by adiabatic law, we have

\[ PV^\gamma = \text{constant} \]

\[
\frac{p_g}{P} = \left( \frac{R_0}{R} \right)^{3\gamma}
\]

or

\[
p_g = P \left( \frac{R_0}{R} \right)^{3\gamma}
\]  

(17)

Hence, comparing equation (16) and (17), we get

\[
\rho \left( \frac{3}{2} \dot{R}^2 + R \ddot{R} \right) = P \left( \frac{R_0}{R} \right)^{3\gamma}
\]
\[
\frac{P \left( \frac{R_0}{R} \right)^{3\gamma}}{\rho} = \frac{3}{2} \ddot{R}^2 + R\dddot{R}
\]  \hspace{1cm} (18)

But, we have

\[
\ddot{R} = \frac{d^2R}{dt^2} = \frac{d}{dR} \left( \frac{dR}{dt} \right) \frac{dR}{dt}
\]

\[
= \dot{R} \frac{d\dot{R}}{dR} = \frac{1}{2} \ddot{R} \frac{d\dot{R}}{dR} = \frac{d}{dR} \left( \frac{1}{2} \dot{R}^2 \right)
\]  \hspace{1cm} (19)

Substituting the value of \( \dot{R} \) in equation (18), we get

\[
\frac{3}{2} \ddot{R}^2 + R \frac{d}{dR} \left( \frac{1}{2} \dot{R}^2 \right) = \frac{P}{\rho} \left( \frac{R_0}{R} \right)^{3\gamma}
\]  \hspace{1cm} (20)

Multiplying equation (20) throughout by \( 2/R \), we get

\[
\frac{d}{dR} \dot{R}^2 + \frac{3}{R} \ddot{R}^2 = \frac{2PR_0^{3\gamma}}{\rho R^{3\gamma+1}}
\]  \hspace{1cm} (21)

which is linear in \( \dot{R}^2 \).

We have \( I.F. = e^{\int \frac{3}{R} dR} = e^{3\log R} = e^{\ln R^3} = R^3 \)

Thus \( R^3 \) is an integrating factor of the equation (21), therefore multiplying by it throughout gives
\[ 3 R^2 \dot{R}^2 + R^3 \frac{d}{dR} \left( \dot{R}^2 \right) = \frac{2 P R_0^{3\gamma}}{\rho R^{3\gamma-1}} R^3 \]

\[
\frac{d}{dR} \left( R^3 \dot{R}^2 \right) = \frac{2 P R_0^{3\gamma}}{\rho R^{3\gamma-2}} 
\]

(22)

We now integrate equation (22) w.r.t. \( R \) we have

Since

\[
R^3 \dot{R}^2 = \frac{2 P R_0^{3\gamma}}{\rho} \int_{R_0}^{R} R^{-(3\gamma-2)} dR = \frac{2 P R_0^{3\gamma}}{\rho} \left[ \frac{R^{-3\gamma+3}}{R_0} \right]_R^{R_0} 
\]

(23)

Hence, on integrating equation (22), we get

\[
R^3 \dot{R}^2 = \frac{2}{3} \frac{R_0^{3\gamma} P}{\rho (\gamma-1)} \left[ R_0^{-3\gamma+3} - R^{-3\gamma+3} \right] 
\]

(24)

or

\[
\dot{R}^2 = \frac{2}{3} \frac{R_0^{3\gamma} P}{(\gamma-1) \rho R^3} \left[ R_0^{-3\gamma+3} - R^{-3\gamma+3} \right] 
\]

\[
= \frac{2}{3} \frac{P}{(\gamma-1) \rho} \left[ \frac{R_0^{3\gamma}}{R^3} R_0^{-3\gamma+3} - \frac{R_0^{3\gamma}}{R^3} R_0^{-3\gamma+3} \right] 
\]

(25)

or

\[
\dot{R}^2 = \frac{2}{3} \frac{P}{(\gamma-1) \rho} \left[ \left( \frac{R_0}{R} \right)^3 \left( \frac{R_0}{R} \right)^{3\gamma} \right] 
\]

In the special case, when \( \gamma = 4/3 \), a typical value for most gases, the solution may be completed i.e.
\[ \dot{R}^2 = \frac{2P}{\rho} \left[ \left( \frac{R_0}{R} \right)^3 - \left( \frac{R_0}{R} \right)^4 \right] \]

or \[ \dot{R}^2 = \frac{2P}{\rho} \left( \frac{R_0}{R} \right)^3 \left( 1 - \frac{R_0}{R} \right) \]

or \[ \dot{R}^2 = \frac{2P}{\rho} \left( \frac{R_0}{R} \right)^3 \left( \frac{R-R_0}{R} \right) \] (26)

From equation (26), we have

\[ \frac{dR}{dt} = \left[ \frac{2P}{\rho} \left( \frac{R_0}{R} \right)^3 - \left( \frac{R_0}{R} \right)^4 \right]^{-1/2} \]

or \[ \frac{dR}{dt} = \left[ \frac{2P}{\rho} \left( \frac{R^3}{R_0^3} - \frac{R^4}{R_0^4} \right) \right]^{-1/2} \]

or \[ \frac{dR}{dt} = \left[ \frac{2P}{\rho} \frac{R^3}{R_0^3} (R - R_0) \right]^{1/2} \]

or \[ \frac{dR}{dt} = \frac{1}{R^2} \left[ \frac{2P R^3 (R - R_0)}{\rho} \right]^{1/2} \]

or \[ dt = \frac{R^2 dR}{\left[ \left( \frac{2P R^3}{\rho} \right) (R - R_0) \right]^{1/2}} \] (27)

Substituting \( R = R_0 + x \), equation (27) becomes
\[
dt = \frac{(R_0^2 + x^2 + 2R_0x)}{\left(\frac{2PR_0^3}{\rho}\right)^{1/2}} \frac{dx}{x^{1/2}} \quad (28)
\]

or
\[
dt = \left[\frac{\rho}{2 \ PR_0^3}\right]^{1/2} \frac{(R_0^2 + x^2 + 2R_0x)}{x^{1/2}} \ dx \quad (29)
\]

Integrating equation (29), we have
\[
t = \left[\frac{\rho}{2 \ PR_0^3}\right]^{1/2} \left(2x^{1/2}R_0^2 + \frac{2}{5}x^{5/2} + \frac{4}{3} R_0^3 x^{3/2}\right)
\]

or
\[
t = \left[\frac{2\rho x}{PR_0^3}\right]^{1/2} \left(\frac{1}{5}x^2 + \frac{2}{3} R_0 x + R_0^2\right)
\]

Rayleigh’s equation i.e. equation (16) was modified by Plesset by considering viscosity of liquid.