CHAPTER-6

CHAOS IN A CLASS OF NONLINEAR DIFFERENTIAL SYSTEMS

Chaos may develop in a nonlinear ordinary differential system by the formation and break up of a homoclinic orbit. Mel’nikov has given an asymptotic method to study this development of chaos. We discuss a special class of the nonlinear differential systems in which it is very simple to apply Mel’nikov’s method and to demonstrate that, if a homoclinic orbit is formed in this system, it will break up to give rise to chaos as a parameter is varied. It is found that the nonlinear differential equations, modelling the expansion (Rayleigh [37] and Plesset [33]) and translation (Chakraborty [6] and Chakraborty and Tuteja [8]) of a bubble in a liquid in the presence of a sound field of small amplitude, can be transformed and shown to belong to this special class of non-linear ordinary differential systems.

6.1 Mathematical Formulation

We consider a differential system (Drazin [13]) with a periodic perturbation given by the equation

$$\frac{dx}{dt} = F(x) + \varepsilon f(x,t)$$  \hspace{1cm} (1)

where $\varepsilon$ is small. Adopting the notations as in [33], we have
\[ x = [x, y]^T \in \mathbb{R}^2, \quad F : \mathbb{R}^2 \to \mathbb{R}^2 \text{ with } F = \left[ \frac{\partial H}{\partial y}, -\frac{\partial H}{\partial x} \right]^T \]  

(2)

for some function \( H(x, y) \), \( f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2 \) with \( f(x, t + T) = f(x, t) \) for all \( x \) and \( t \).

Here \( F \) and \( f \) are taken as well behaved.

When \( \varepsilon = 0 \), we get the basic system.

\[ \frac{dx}{dt} = F(x), \]  

(3)

which is autonomous and Hamiltonian. We shall assume that the basic system (3) has a saddle point \( X_0 \), which is connected to itself by a homoclinic orbit. If \( \{ q_0(t) \} \) be the homoclinic orbit then \( q_0 \) satisfies the equation (3) and

\[ q_0(t) \to X_0 \text{ as } t \to \pm \infty \]

For small \( \varepsilon \), the homoclinic orbit \( \{ q_0(t) \} \) breaks up and chaos develops (Drazin [13]) if the Melnikov function \( M(t_0) \), defined by

\[ M(t_0) = \int_{-\infty}^{\infty} F(q_0(t-t_0)) \times f(q_0(t-t_0), t) dt, \]  

(4)

has a simple zero for some \( t_0 \). This condition for the occurrence of chaos, if a homoclinic orbit is present can be shown to be satisfied for a special class of differential systems.

6.2 A Special Class of Differential Systems

We consider a special class of differential equations
\[
\frac{d^2x}{dt^2} = -\frac{dV(x)}{dx} + \varepsilon g(x) \cos \omega t \tag{5}
\]

Writing \( y = \frac{dx}{dt} \), the equation (5) can be seen to be equivalent to the differential system (1) which can be written as

\[
\begin{bmatrix}
\frac{dx}{dt} \\
\frac{dy}{dt}
\end{bmatrix} =
\begin{bmatrix}
y \\
-\frac{dV}{dx} + \varepsilon g(x) \cos \omega t
\end{bmatrix} \tag{6}
\]

so that (cf. (1)),

\[
F = \begin{bmatrix} y, -\frac{dV(x)}{dx} \end{bmatrix}^T \quad \text{and} \quad f = [0, g(x) \cos \omega t]^T \tag{7}
\]

where \( V(x) \) is some function of \( x \).

Equation (2), in view of (7), shows

\[
H = \frac{y^2}{2} + V(x) + c \tag{8}
\]

where \( c \) is a constant, for the basic differential system (3) in the absence of perturbation \((\varepsilon = 0)\). Using equation (4), we find that Mel’nikov function \( M(t_0) \) now becomes

\[
M(t_0) = -\int_{-\infty}^{\infty} [y(t-t_0)g(x(t-t_0)) \cos \omega t] dt \tag{9}
\]

where \( q_0(t) = (x(t), y(t)) \) defines the homoclinic orbit in the basic system joining a saddle point with itself. Taking \( t - t_0 = T \), we get from (9)
\[ M(t_0) = -A\cos \omega t_0 + B\sin \omega t_0 \]  

(10)

where

\[ A = \int_{-\infty}^{\infty} y(t)g(x(t))\cos \omega t dt \quad \text{and} \quad B = \int_{-\infty}^{\infty} y(t)g(x(t))\sin \omega t dt \]  

(11)

\[ M(t_0) \] vanishes when

\[ \tan \omega t_0 = \frac{A}{B} \]  

(12)

provided \( B \neq 0 \).

Since \( \tan \omega t_0 \) changes from \(-\infty\) to \(\infty\) as \( \omega t_0 \) changes from \(-\frac{\pi}{2}\) to \(\frac{\pi}{2}\), equation (12) can be satisfied for some \( t_0 \) lying between \(-\frac{\pi}{\omega}\) and \(\frac{\pi}{\omega}\). Also, if \( B = 0 \), the equation (10) shows that \( M(t_0) = 0 \) when

\[ \omega t_0 = n\frac{\pi}{2}, \]

where \( n \) is any odd integer. In either case, in view of equation (10), \( M(t_0) \) vanishes, and Mel’nikov’s theory shows that the homoclinic orbit breaks up and chaos develops.
6.3 Some Differential Systems that belong to the Special Class after Suitable Transformation

We now show that, with proper transformation of variables, the nonlinear system of differential equations modelling bubble dynamics can be transformed so that they are of the form (6).

A bubble expands adiabatically in a liquid of density $\rho$ in which there is a sound field. The pressure $p_e$, due to the sound field, at a large distance from the bubble is given by

$$p_e = p_{e_0} (1 + \varepsilon \cos \omega t),$$

where $\varepsilon$ is a small constant and $p_{e_0}$ is also a constant. The radius $R$ of a bubble is given by Rayleigh-Plesset equation (Rayleigh [5] and Plesset [4]).

$$R \frac{d^2 R}{dt^2} + \frac{3}{2} \left( \frac{dR}{dt} \right)^2 + \frac{1}{\rho} \left\{ p_{e_0} (1 + \varepsilon \cos \omega t) - p_{g_0} \left( \frac{R_0}{R} \right)^{3\gamma} + \frac{2\sigma}{R} \right\} = 0$$

where $\sigma$ is the surface tension coefficient and $\gamma$ is the ratio of the two specific heats of the gas. $R_0$ and $p_{g_0}$ are the initial values of bubble radius and gas pressure inside the bubble.

Taking

$$r = R^{5/2} \text{ and } r_0 = R_0^{5/2}$$

We can transform equation (13) to

$$\frac{d^2 r}{dt^2} + \frac{5}{2\rho} r^{1/5} \left\{ p_{e_0} (1 + \varepsilon \cos \omega t) - p_{g_0} \left( \frac{r_0}{r} \right)^{6\gamma/5} + 2\sigma r^{-2/5} \right\} = 0$$

(15)
If we put \( r = x \), we find that equation (15) is of the form (5).

Similarly, if the bubble, in addition to expanding, is also translating with speed \( U \), the bubble motion is governed by the equations (Chakraborty [6] and Chakraborty and Tuteja [8])

\[
R \frac{d^2 R}{dt^2} + \frac{3}{2} \left( \frac{dR}{dt} \right)^2 - \frac{U^2}{4} + \frac{1}{\rho} \left( p_e - p_g \left( \frac{R_0}{R} \right)^{3\gamma} + \frac{2\sigma}{R} \right) = 0
\]

and

\[
UR^3 = U_0 R_0^3 = k,
\]

where \( k \) is a constant, and where notation is same as in (15) and \( U_0 \) is speed of gas bubble initially. Making the substitutions (14), we find (16), in view of (17) becomes

\[
\frac{d^2 r}{dt^2} - \frac{5}{2} k^2 r^{-11/5} + \frac{5}{2\rho} r^{1/5} \left( p_e - p_g \left( \frac{r_0}{r} \right)^{6\gamma/5} + 2\sigma r^{-2/5} \right) = 0
\]

Equation (18) is again of the form (5).

6.4 Concluding Remarks

In this paper, we show that a class of differential systems exists to which Mel’nikov’s asymptotic method may be readily applied to give the result that if a homoclinic orbit exists, it breaks up to give chaos if a small parameter varies. The differential equations governing the dynamics of a bubble which expands in a liquid with or without translation through this liquid, are shown to belong to this class.