CHAPTER 6

Stochastic Modeling of Nonlinear Circuits

6.1 Overview of the Chapter

In this chapter, we propose a general model for a nonlinear circuit, in which, the circuit parameters (e.g. resistance and capacitance) are subject to random fluctuations due to noise, which vary with time. The fluctuating amplitudes of these parameters are assumed to be Ornstein-Uhlenbeck (O.U.) processes and not the white noise owing to temporal correlations. The nonlinear circuit is represented by a system of nonlinear differential equations depending upon a set of parameters that fluctuate slowly with time. To model these fluctuations, we use the theory of Ito’s stochastic differential equations (SDEs). Then the driving force of the circuit dynamics is in accordance with the general perturbation theory decomposed into the sum of a strong linear component and a weak nonlinear component by the introduction of a small perturbation parameter. The circuit states are expanded in the powers of this small perturbation parameter and recursive solutions to the various approximates obtained. Finally, the approximate expressions for the output states are obtained as stochastic integrals with respect to Brownian motion processes. The proposed method is applied to a half-wave rectifier circuit which is built out of a diode, a resistor and a capacitor. The diode is represented by a nonlinear voltage-current equation, and resistance and capacitance are subject to random fluctuations due to noise, which vary slowly with time. The results, obtained using the proposed method, are compared with those obtained via the conventional perturbation-based deterministic differential equations model for a nonlinear circuit. Hence, the noise
process component, present at the output, is obtained. The material in this chapter has been summarized from our work in [6].

6.2 Introduction

In an electrical/electronic circuit two types of noise can exist: external and internal noise. External noise components are elements picked up by circuit devices from the external environment through electromagnetic interference (EMI). The effect of such noise components can be avoided by isolating a circuit from EMI. Internal noise (e.g. thermal, shot or flicker) is generated internally within the electronic devices. The internal noise is referred to as an electrical or electronic noise. In this chapter, we consider only the effect of the internal noise (electrical noise). The internal noise is generated because of the fact that an electrical charge flows in discrete form. Thus, an electrical noise which is associated with the fundamental processes in an electronic circuit cannot be avoided. Hence, it represents a fundamental limit on the performance of the circuit [56]. The electrical noise processes exhibit chaotic or random behavior in an analog circuit. Therefore, for efficient analog circuit designing/modeling, precise prediction of the noise effect on the signals is very crucial. It is to be noted that capacitance can vary due to vibration of capacitor plates, also at higher temperature the plates may undergo slight expansion and the distance between plates may vary due to stress and temperature. In such cases, it is better to model a capacitor as a deterministic component plus a small random component, likewise, it is known that resistance changes with temperature. If temperature variation is rapid then resistance value will also fluctuate rapidly around its mean value.

The existing perturbation-based models [1-5] for nonlinear circuits, present different methods for analyzing nonlinear behavior of circuit components. Moreover, these perturbation-based models for the nonlinear circuits omit the effect of electrical noise that exists in the circuits, and deteriorate the signal through the circuits. The proposed model takes into account a) nonlinear behavior, and b) noise effects in a circuit.

In this work, we make use of perturbation theory and Ito’s stochastic differential equations (SDEs) [57, 58]. This chapter has been organized in eight sections. Section 6.3 presents the advantages of using O.U. process to model the noise perturbation of circuit elements. In Section 6.4, we develop a general perturbation-
based stochastic model for nonlinear circuits. In Section 6.5, the proposed model has been applied to a half-wave rectifier circuit. Section 6.6 presents perturbation-based deterministic modeling of the half-wave rectifier circuit. Section 6.7 presents simulations and discussions. In Section 6.8, the autocorrelation function of the random component in the output voltage is presented. In Section 6.9, conclusions are presented.


Advantages of using the Ornstein-Uhlenbeck (O.U.) process to model noisy perturbations of circuit elements are as follows:

The one-dimensional O.U. process can be represented by a stochastic integral of the following form:

\[ x(t) = \sigma \int_0^t e^{-\gamma(t-\tau)} dB(\tau) \]

This process has continuous paths and an autocorrelation function of the form

\[ \lim_{t \to \infty} E(x(t)x(t+\tau)) = R(\tau) = \frac{\sigma^2}{2\gamma} e^{-\gamma|\tau|} \]

Since circuit parameters exhibit slow fluctuations, a process such as O.U. having continuous paths is a good approximation for representing this. The white noise processes are not good models for representing the parameter fluctuations since their sample paths are discontinuous. Brownian motion \( B_t \) has continuous path but is highly non-stationary and hence is not a good model. For Brownian motion,

\[ \lim_{t \to \infty} E(B(t)B(t+\tau)) = \infty \]

and hence it is not a very good model. If \( \gamma \) is large, then the O.U. process has a very short-range correlation and vice versa. So the correlation range can also be controlled by adjusting \( \gamma \). Moreover, the O.U. process is a Gaussian process which is representative of the fact that components fluctuations are caused by several effects like heating up of a resistor due to random electron motion. The aggregate of several small independent effects is Gaussian by the central limit theorem and this is well represented by the O.U. process. The O.U. process when \( \gamma = 0 \) is Brownian
motion and if $\sigma = \gamma$, then in the limit $\gamma \to \infty$, the O.U. process becomes the white noise or the formal derivative of Brownian motion and hence the O.U. process can be used to handle the two extremes too.

6.4 The Proposed Perturbation-Based SDE Modeling of Nonlinear Circuits

The proposed method is a time-domain approach to analyze the effect of noise on the non-random signals in weakly nonlinear circuits. The proposed model takes into account the following two things in a circuit: a) the circuit devices’ nonlinear behaviour, and b) the fluctuating components of the circuit parameters (e.g. resistance) due to noise. In the presented model both the above-said components are assumed to be of the same order of smallness. Hence, for both the components, the same small perturbation parameter $\epsilon$ is introduced as a multiplier, during the application of perturbation theory in the model. As the fluctuating component of a circuit parameter is represented by the O.U. process, we can vary the fluctuating amplitude of a circuit parameter, easily, by incorporating a variation in the O.U. process parameters (e.g. variation in $\sigma$ results in amplitude variation of the random fluctuations). It should be noted that the algorithm of the proposed model provides the opportunity to incorporate such variations in the O.U. process parameters.

The proposed method is general in the sense that any nonlinear circuit can be represented by a system of nonlinear differential equations depending upon a set of parameters that fluctuate slowly with time. These fluctuations are modeled using SDEs to which perturbation theory is applied. This model is excited by a non-random input although the circuit parameters satisfy O.U. stochastic differential equations.

A nonlinear circuit can be described by the following system of differential equation:

$$\frac{d\mathbf{x}_i(t)}{dt} = f_i(\mathbf{x}(t), \theta(t)) + v_i(t); \quad 1 \leq i \leq n$$  \hspace{1cm} (6.1)

where

$$f_i(\mathbf{x}(t), \theta(t)) = \sum_{j=1}^{n} a_{ij}(\theta)x_j + \epsilon g_i(\mathbf{x}, \theta)$$  \hspace{1cm} (6.2)

where $\mathbf{x}(t)$ is a state vector and $x_j$ is component of the state vector. The matrix $a_{ij}(\theta)$ is obtained by setting up the KCL and KVL equations of the nonlinear
circuit and extracting out the linear part, as will be demonstrated in the application example section. After extracting out the linear part in the circuit dynamics, what remains is the nonlinear part, which is precisely \( g_i(x, \theta) \). This nonlinear term involves the small perturbation parameter \( \epsilon \). \( \theta(t) \) denotes circuit parameter (e.g. resistance and capacitance) vector, which gets perturbed due to noise. These slowly varying fluctuations are represented by O.U. processes as follows:

\[
\theta(t) = \theta_0 + \epsilon \phi(t)
\]  

(6.3)

where \( \phi(t) \) is a vector valued O.U. process, which is given by the following equation:

\[
d\phi(t) = -\Gamma \phi(t) dt + \sigma dB(t)
\]  

(6.4)

or, in component form,

\[
d\phi_i(t) = -\sum_{j=1}^{p} \gamma_{ij} \phi_j(t) dt + \sum_{j=1}^{d} \sigma_{ij} dB_j(t)\; 1 \leq i \leq p
\]  

(6.5)

where \( \gamma \) is the damping matrix and \( \sigma \) is the fluctuation matrix.

We note that the same tag \( \epsilon \) has been attached to the nonlinearity as well as to the random perturbation of the circuit elements (i.e. the parameter vector fluctuations \( \phi(t) \)). More generally, we could have different tags, say \( \epsilon_1 \) and \( \epsilon_2 \) where \( \epsilon_1 \) represents the magnitude of the nonlinearity while \( \epsilon_2 \) represents the magnitude of the parameter fluctuations. Then, we would have to expand the state vector as a double perturbation series and equating coefficients of \( \epsilon_1^{r_1} \epsilon_2^{r_2} \) for each ordered integer pair \( (r_1, r_2) \). The computations then become more cumbersome and we have to thus stick to the simpler formulation of having one tag \( \epsilon \).

Applying perturbation theory, now, we expand the state vector in powers of \( \epsilon \) and retain the terms only up to \( O(\epsilon^2) \).

\[
x_i(t) = x_i^{(0)}(t) + \epsilon x_i^{(1)}(t) + O(\epsilon^2)
\]  

(6.6)

so that

\[
x_i(t) = x_i^{(0)}(t) + x_i^{(1)}(t)
\]  

(6.7)

will be the output vector, after putting \( \epsilon = 1 \), in the end having done the calculations.

Actually, \( \epsilon = 1 \), and \( x_i^{(1)}(t) \) is small compared to \( x_i^{(0)}(t) \). However, to keep track of the quantities that are of the first order of smallness, we attach the tag \( \epsilon \) to \( x_i^{(1)}(t) \)
and later on, in the calculations, set $\epsilon = 1$. This is the way in which perturbation theory is traditionally applied to quantum mechanical systems.

Now, plugging (6.2) into (6.1) and putting the values of $x_i(t)$ and $\theta(t)$ from the equations (6.7) and (6.3), respectively, we have the following equation:

$$
\frac{d}{dt}(x_i^{(0)} + \epsilon x_i^{(1)}) = \sum_{j=1}^{n} a_{ij}(\theta_0 + \epsilon \phi(t))(x_j^{(0)}(t) + \epsilon x_j^{(1)}(t))
$$

$$
+ \epsilon g_i(x^{(0)} + \epsilon x^{(1)}, \theta_0 + \epsilon \phi) + O(\epsilon^2) + v_i(t)
$$

(6.8)

Now equating coefficients of $\epsilon^0$ we have:

$$
\frac{dx_i^{(0)}}{dt} = \sum_{j} a_{ij}(\theta_0)x_j^{(0)} + v_i(t)
$$

(6.9)

The following equation results on comparing the coefficients of $\epsilon^1$ in (6.8):

$$
\frac{dx_i^{(1)}}{dt} = \sum_{j,k} \frac{\partial a_{ij}(\theta_0)}{\partial \theta_k} x_j^{(0)}(t)\phi_k(t) + \sum_{j} a_{ij}(\theta_0)x_j^{(1)}(t) + g_i(x^{(0)}(t), \theta_0(t))
$$

(6.10)

$$
\phi(t) = \int_{-\infty}^{t} \{exp(-\Gamma(t - \tau))\} \Sigma dB(\tau)
$$

(6.11)

$$
E[\phi(t)] = 0
$$

For $u \geq 0$,

$$
E[\phi(t+u)\phi^T(t)] = \int_{-\infty}^{t} \{exp(-\Gamma(t+u - \tau))\} \Sigma \Sigma^T exp(-\Gamma^T(t-\tau)) d\tau
$$

(6.12)

$$
= \int_{0}^{\infty} \{exp(-\Gamma(u + s))\} \Sigma \Sigma^T exp(-\Gamma^T s) ds
$$

(6.13)

$$
x^{(0)}(t) = \int_{0}^{t} \{exp(A(\theta_0)(t - \tau))\} v(\tau) d\tau
$$

(6.14)

where $A(\theta_0) = (a_{ij}(\theta_0))$

$$
x_i^{(1)}(t) = \sum_{j} \int_{0}^{t} \{exp(A(\theta_0)(t - \tau))\}_{ij}
$$

$$
\times \left[ \sum_{m,k} \frac{\partial a_{jm}(\theta_0)}{\partial \theta_k} x_m^{(0)}(\tau)\phi_k(\tau) + g_j(x^{(0)}(\tau), \theta_0(\tau)) \right] d\tau
$$

(6.15)
\[
= \sum_{j,m,k} \int_0^t \left[ \exp(A(\theta_0)(t - \tau)) \right]_{ij} \frac{\partial a_{jm}(\theta_0)}{\partial \theta_k} x_m^{(0)}(\tau) d\tau
\]

\[+ \sum_j \int_0^t \left( \exp(A(\theta_0)(t - \tau)) \right)_{ij} g_j(x^{(0)}(\tau), \theta_0(\tau)) d\tau \tag{6.16} \]

### 6.5 Application Example

In this section, the diode circuit is modeled in the form of perturbation-based SDEs, following the proposed method. Approximate solution for the output voltage is obtained in the form of stochastic integrals with respect to the Brownian motion processes.

Let

\[ [B_1(t), B_2(t)] \]

be independent Brownian motion processes. Then, the two independent O.U. processes are given as follows:

\[
dw_1 = -\gamma_1 w_1 dt + \sigma_1 dB_1 \tag{6.17} \\
dw_2 = -\gamma_2 w_2 dt + \sigma_2 dB_2 \tag{6.18} 
\]

The solution and simulation procedure for the above O.U. processes have been given in appendix B. The random fluctuations in resistance and capacitance in the circuit are represented by \( w_1 \) and \( w_2 \), respectively.

We consider a half-wave rectifier circuit with a capacitor filter, shown Fig. 6.1, where \( R = 1000 \) Ohms, \( C = 30 \mu F \), \( I_0 = 2 \times 10^{-9} A \). Let

\[ v_i = Input \ voltage, \ v_o = Output \ voltage, \ x(t) = Diode \ current \]

Then

\[ v_D = v_i - v_o \]

\[ x(t) = f(v_D(t)) = f(v_i - v_o) \tag{6.19} \]

\[ = a(v_i - v_o) + \epsilon g(v_i - v_o) \tag{6.20} \]

Equation (6.20) represents diode voltage-current equation, where the linear and nonlinear parts of the voltage-current equation are separated and a small perturbation parameter \( \epsilon \) is incorporated into the nonlinear part.
Here

\[ f(v_D) = I_0((\exp(v_D/V_T)) - 1) \]  \hspace{1cm} (6.21)

\[ a = f'(0) = \frac{I_0}{V_T} \]  \hspace{1cm} (6.22)

\[ g(v_D) = \frac{f(v_D) - av_D}{\epsilon} \equiv I_0((\exp(v_D/V_T)) - 1 - v_D/V_T) \]  \hspace{1cm} (6.23)

\[ = I_0 \sum_{n=2}^{\infty} \left( \frac{v_D}{V_T} \right)^n \frac{1}{n!} \]  \hspace{1cm} (6.24)

Now, applying K.C.L. in half-wave rectifier circuit of Fig. 6.1, we get

\[ x(t) = i_D = \frac{v_0}{R} + \frac{d(Cv_o)}{dt} \]  \hspace{1cm} (6.25)

Let

\[ Cv_o(t) = \xi(t) \]  \hspace{1cm} (6.26)

\[ v_o(t) = \frac{\xi(t)}{C} = \frac{\xi(t)}{C_0 + \epsilon \omega_2(t)} \]  \hspace{1cm} (6.27)

Let

\[ \xi = \xi_0 + \epsilon \xi_1 + O(\epsilon^2) \]  \hspace{1cm} (6.28)

Then

\[ v_o(t) = \frac{\xi_0 + \epsilon \xi_1 + O(\epsilon^2)}{C_0 + \epsilon \omega_2(t)} \]
\[
\frac{d\xi(t)}{dt} = f(v_i(t) - v_o(t)) - \frac{v_o(t)}{R}
\]  
(6.31)

Now, putting values of \(\xi(t), v_o(t)\) and \(\frac{v_o}{R}\) from (6.28), (6.29) and (6.30), respectively, into (6.31) we have

\[
\frac{d(\xi_0 + \epsilon\xi_1)}{dt} = f(v_i - \frac{\xi_0}{C_0} - \epsilon(\frac{\xi_1}{C_0} - \frac{\xi_0 w_2}{C_0^2} + O(\epsilon^2))
\]

\[
-\left(\frac{\xi_0 + \epsilon\xi_1}{R_0 C_0}\right) + \frac{\epsilon}{(R_0 C_0)^2}\xi_0 (R_0 w_2 + C_0 w_1) + O(\epsilon^2)
\]

\[
= a(v_i - \frac{\xi_0}{C_0}) - a\epsilon(\frac{\xi_1}{C_0} - \frac{\xi_0 w_2}{C_0^2}) + \epsilon g(v_i - \frac{\xi_0}{C_0}) + O(\epsilon^2)
\]

\[
-\left(\frac{\xi_0 + \epsilon\xi_1}{R_0 C_0}\right) + \frac{\epsilon}{(R_0 C_0)^2}\xi_0 (R_0 w_2 + C_0 w_1) + O(\epsilon^2)
\]  
(6.32)

Now, equating the coefficients of \(\epsilon^0\) in equation (6.32) we get

\[
\frac{d\xi_0}{dt} = a(v_i - \frac{\xi_0}{C_0} - \frac{\xi_0}{R_0 C_0})
\]  
(6.33)
or
\[
\frac{d\xi_0}{dt} = -\left(\frac{a}{C_0} + \frac{1}{R_0C_0}\right)\xi_0 + av_i \quad (6.34)
\]

The solution of \(\xi_0(t)\), from the above equation, is as follows:
\[
\xi_0(t) = a \int_0^t e^{-(t-\tau)(\frac{a}{\xi_0} + \frac{1}{R_0C_0})}v_i(\tau)d\tau \quad (6.35)
\]

Now, equating the coefficients of \(\epsilon^1\) in (6.32), we get
\[
\frac{d\xi_1(t)}{dt} = -a\left(\frac{\xi_1}{C_0} - \frac{\xi_0}{C_0^2}\right) + g\left(v_i - \frac{\xi_0}{C_0}\right) - \frac{\xi_1}{R_0C_0} + \frac{\xi_0}{(R_0C_0)^2}\left(R_0w_2 + C_0w_1\right) \quad (6.36)
\]

or
\[
\frac{d\xi_1(t)}{dt} = -\xi_1\left(\frac{a}{C_0} + \frac{1}{R_0C_0}\right) + g\left(v_i - \frac{\xi_0}{C_0}\right) + \frac{aw_2\xi_0}{C_0^2} + \frac{\xi_0}{(R_0C_0)^2}\left(R_0w_2 + C_0w_1\right) \quad (6.37)
\]

The solution of \(\xi_1(t)\) from the above equation is as follows:
\[
\xi_1(t) = \int_0^t e^{-(t-\tau)(\frac{a}{\xi_0} + \frac{1}{R_0C_0})}\left(g\left(v_i - \frac{\xi_0}{C_0}\right) + \frac{aw_2\xi_0}{C_0^2} + \frac{\xi_0}{(R_0C_0)^2}\left(R_0w_2 + C_0w_1\right)\right)d\tau \quad (6.38)
\]

### 6.6 Perturbation-Based Deterministic Modeling of Half-Wave Rectifier Circuit

This section presents perturbation-based deterministic modeling of half-wave rectifier circuit. This model has been described in Appendix C.

Applying K.C.L. in half-wave rectifier circuit of Fig. 6.1, we get
\[
x(t) = i_D = \frac{v_0}{R} + \frac{d(cv_o)}{dt} \quad (6.39)
\]

Let
\[
Cv_o(t) = \xi(t) \quad (6.40)
\]

\[
v_o(t) = \frac{\xi(t)}{C}
\]

Let
\[
\xi = \xi_0 + \epsilon\xi_1 + O(\epsilon^2) \quad (6.41)
\]

Then
\[
v_o(t) = \frac{\xi_0 + \epsilon\xi_1 + O(\epsilon^2)}{C} \quad (6.42)
\]
The above equation is obtained by expanding $\xi$ in powers of $\epsilon$ (applying perturbation theory) and retaining the terms only up to $O(\epsilon)$, so that

$$v_o(t) = \frac{\xi_0 + \xi_1}{C} \quad (6.43)$$

will be the output after putting $\epsilon = 1$, in the end after having done the calculations.

Putting value of $x(t)$ from (6.20) into (6.39), we have

$$\frac{d(Cv_o(t))}{dt} = a(v_i - v_o) + \epsilon g(v_i - v_o) - \frac{v_o}{R} \quad (6.44)$$

Now, putting $\xi(t)$ in the place of $Cv_o(t)$, and then, putting values of $\xi(t)$ and $v_o(t)$ from (6.41) and (6.42), respectively, into (6.44) we have

$$\frac{d(\xi_0 + \epsilon \xi_1)}{dt} = a(v_i - \frac{\xi_0 + \epsilon \xi_1}{C}) + \epsilon g(v_i - \frac{\xi_0 + \xi_1 + O(\epsilon^2)}{C}) - \frac{(\xi_0 + \epsilon \xi_1)}{RC} \quad (6.45)$$

Now, equating the coefficients of $\epsilon^0$ in (6.45) we get

$$\frac{d\xi_0}{dt} = a(v_i - \frac{\xi_0}{C}) - \frac{\xi_0}{RC} \quad (6.46)$$

or

$$\frac{d\xi_0}{dt} = -\left(\frac{a}{C} + \frac{1}{RC}\right)\xi_0 + av_i \quad (6.47)$$

The solution of $\xi_0(t)$, from the above equation is as follows:

$$\xi_0(t) = a \int_0^t e^{-((t-\tau)(\frac{a}{C} + \frac{1}{RC}))}v_i(\tau)d\tau \quad (6.48)$$

Now, equating the coefficients of $\epsilon^1$ in (6.45) we get

$$\frac{d\xi_1}{dt} = -a\left(\frac{\xi_1}{C_0} - \frac{\xi_0 w_2}{C_0^2}\right) + g(v_i - \frac{\xi_0}{C}) - \frac{\xi_1}{RC} \quad (6.49)$$

or

$$\frac{d\xi_1}{dt} = -\xi_1\left(\frac{a}{C} + \frac{1}{RC}\right) + g(v_i - \frac{\xi_0}{C}) \quad (6.50)$$

Solution of $\xi_1(t)$ from the above equation is as follows:

$$\xi_1(t) = \int_0^t e^{-((t-\tau)(\frac{a}{C} + \frac{1}{RC}))}g(v_i - \frac{\xi_0}{C})d\tau \quad (6.51)$$

Now, we have obtained zeroth-order ($\xi_0(t)$) and the first-order ($\xi_1(t)$) approximations for $\xi(t)$. The output voltage up to the first-order of approximation is obtained using equation (6.43).
6.7 Simulations and Discussions

Simulations have been done in MATLAB. MATLAB scripts for the simulations are given in appendix E. The input and output waveforms are plotted for the half-wave rectifier circuit modeled using the following two types of method: a) Modeling using perturbation-based deterministic differential equations method (in Fig. 6.2) and b) Modeling using the proposed, perturbation-based stochastic differential equations method (in Fig. 6.3). During simulations, the O.U. processes \( w_1 \) and \( w_2 \) values (in discretized form) are obtained, using the method presented in appendix B. The noise voltage (in time domain), present in the approximated output signal, is presented in Fig. 6.4. In Fig. 6.4, the noise voltage is obtained by subtracting the output signal obtained using the perturbation-based deterministic differential equations model, from that obtained using the proposed method.

\[
\text{Noise voltage} = a - b
\]

where

\( a \) = Approx. value of output voltage using the proposed method

\( b \) = Approx. value of output voltage using the perturbation-based deterministic differential equations method

The graph of the noise voltage is not like Brownian motion. It is less kinky and smoother than Brownian motion, because it involves weighted integrals of white noise process as (6.38) shows.

Fig. 6.4 also presents pdf of the noise voltage present at the output, due to randomly fluctuating circuit parameters due to noise.

6.8 Autocorrelation Function of Random Component Present in the Output Voltage

From (6.26) we have \( C_{v_o}(t) = \xi(t) \). \( \xi(t) \) has been expanded in perturbation terms and calculations up to first-order of approximation are done. Moreover, zeroth-order approximated value of \( \xi(t) \), i.e., \( \xi_0(t) \) does not contain any random component due to noise. Therefore, we consider only first-order approximated value of \( \xi(t) \), i.e., \( \xi_1(t) \), in order to calculate autocorrelation function of the output voltage.

From (6.38),
Figure 6.2: Plots for output waveforms when the circuit is modeled by the perturbation-based deterministic differential equations method.
Figure 6.3: Plots for output waveforms when the circuit is modeled by the perturbation-based stochastic differential equations method
Figure 6.4: Plots for the noise voltage at the output and its pdf.
Figure 6.5: Plot of autocorrelation function of the noise component of output voltage
\[ \xi_1(t) = \int_0^t e^{-\lambda(t-\tau)}(g(v_i(\tau)) - \frac{\xi_0(\tau)}{C_0}) + \frac{aw_2(\tau)\xi_0(\tau)}{C_0^2} \]
\[ + \frac{\xi_0(\tau)}{(R_0C_0)^2}(R_0w_2(\tau) + C_0w_1(\tau))d\tau \] (6.52)

where
\[ \lambda = \frac{a}{C_0} + \frac{1}{R_0C_0} \] (6.53)

Now,
\[ v_o(t) = \frac{\xi(t)}{C'} \] (6.54)

Using (6.29) and retaining the terms up to only first-order of approximation we have
\[ v_o(t) = \frac{\xi_0}{C_0} + \epsilon(\frac{\xi_1}{C_0} - \frac{\xi_0w_2}{C_0^2}) \] (6.55)

Random component of \( v_o(t) \) from the above equation is as follows:
\[ \tilde{v}_o(t) = \frac{\xi_1}{C_0} - \frac{\xi_0w_2}{C_0^2} \] (6.56)

Where random component of \( \xi_1(t) \) is given as follows:
\[ \tilde{\xi}_1(t) = \int_0^t e^{-\lambda(t-\tau)}(\frac{a\xi_0(\tau)w_2(\tau)}{C_0^2} + \frac{\xi_0(\tau)w_2(\tau)}{R_0C_0^2} + \frac{\xi_0(\tau)w_1(\tau)}{R_0^2C_0})d\tau \] (6.57)

Now,
\[
E[\tilde{\xi}_1(t_1)\tilde{\xi}_1(t_2)] = \int_0^{t_1} \int_0^{t_2} e^{-\lambda(t_1+\tau_1-\tau_2-\tau_2)}(\frac{a\xi_0(\tau_1)w_2(\tau_1)}{C_0^2} + \frac{\xi_0(\tau_1)w_2(\tau_1)}{R_0C_0^2} + \frac{\xi_0(\tau_1)w_1(\tau_1)}{R_0^2C_0})
\times(\frac{a\xi_0(\tau_2)w_2(\tau_2)}{C_0^2} + \frac{\xi_0(\tau_2)w_2(\tau_2)}{R_0C_0^2} + \frac{\xi_0(\tau_2)w_1(\tau_2)}{R_0^2C_0})d\tau_1d\tau_2
\] (6.58)

or
\[
E[\tilde{\xi}_1(t_1)\tilde{\xi}_1(t_2)] = \int_0^{t_1} \int_0^{t_2} e^{-\lambda(t_1+\tau_1-\tau_2-\tau_2)}E[(\frac{a}{C_0^2} + \frac{1}{R_0C_0^2})\xi_0(\tau_1)w_2(\tau_1) + \frac{\xi_0(\tau_1)w_1(\tau_1)}{R_0^2C_0})
\times((\frac{a}{C_0^2} + \frac{1}{R_0C_0^2})\xi_0(\tau_2)w_2(\tau_2) + \frac{\xi_0(\tau_2)w_1(\tau_2)}{R_0^2C_0})]d\tau_1d\tau_2
\] (6.59)

or
\[
E[\tilde{\xi}_1(t_1)\tilde{\xi}_1(t_2)] = \int_0^{t_1} \int_0^{t_2} e^{-\lambda(t_1+\tau_1-\tau_2-\tau_2)}[(\frac{a}{C_0^2} + \frac{1}{R_0C_0^2})^2\xi_0(\tau_1)\xi_0(\tau_2)E(w_2(\tau_1)w_2(\tau_2))
\]
From (6.67), it is clear that random component of $\xi$ is given as follows:

\[ E[w_1(t_1)w_1(t_2)] = \sigma_1^2 \int_{-\infty}^{t_1+t_2} e^{-\gamma_1(t_1+t_2-2\tau)} d\tau = \frac{\sigma_1^2}{2\gamma_1} e^{-\gamma_1|t_1-t_2|} \]  

Similarly,

\[ E[w_2(t_1)w_2(t_2)] = \frac{\sigma_2^2}{2\gamma_2} e^{-\gamma_2|t_1-t_2|} \]  

Putting values of $E[w_1(t_1)w_1(t_2)]$ and $E[w_2(t_1)w_2(t_2)]$ from (6.63) and (6.64), respectively, into (6.60) gives

\[ E[\dot{\xi}_1(t_1)\dot{\xi}_1(t_2)] = \int_0^{t_1} \int_0^{t_2} e^{-\lambda(t_1+t_2-\tau_1-\tau_2)} \xi_0(\tau_1)\xi_0(\tau_2) \left[ \left( \frac{a}{C_0^2} + \frac{1}{R_0 C_0^2} \right)^2 \frac{\sigma_1^2}{2\gamma_1} e^{-\gamma_1|\tau_1-\tau_2|} \right] d\tau_1 d\tau_2 \]  

From (6.67), it is clear that random component of $\xi$ is a non-stationary process.

Now time-averaged autocorrelation of noise component, $\tilde{v}_o(t)$ (6.56), of the output voltage $v_o(t)$ is given as follows:

\[ E[\tilde{v}_o(t_1)\tilde{v}_o(t_2)] = \frac{1}{C_0^2} E\{(C_0\dot{\xi}_1(t_1) - \dot{\xi}_0(t_1)w_2(t_1))(C_0\dot{\xi}_1(t_2) - \dot{\xi}_0(t_2)w_2(t_2))\} \]

\[ = \frac{1}{C_0^2} E\{\dot{\xi}_1(t_1)\dot{\xi}_1(t_2)\} + \xi_0(t_1)\xi_0(t_2)E\{w_2(t_1)w_2(t_2)\} \]

\[-C_0\dot{\xi}_0(t_1)E\{\xi_1(t_2)w_2(t_1)\} - C_0\dot{\xi}_0(t_2)E\{\xi_1(t_1)w_2(t_2)\} \]  

(6.68)
\[ E\{\tilde{\xi}_1(t_1),\tilde{\xi}_1(t_2)\} \text{ is given in (6.67),} \]

\[ E\{w_2(t_1)w_2(t_2)\} = \frac{\sigma^2}{2\gamma_2} e^{-\gamma_2|t_1-t_2|} \]  

(6.69)

\[ E\{\tilde{\xi}_1(t_1)w_2(t_2)\} = \int_0^{t_1} e^{-\lambda(t_1-\tau)} \left( \frac{a}{C_0^2} + \frac{1}{R_0C_0^2} \right) \xi_0(\tau) E\{w_2(\tau)w_2(t_2)\} d\tau \]

\[ = \left( \frac{a}{C_0^2} + \frac{1}{R_0C_0^2} \right) \frac{\sigma^2}{2\gamma_2} \int_0^{t_1} \xi_0(\tau)e^{-\lambda(t_1-\tau)}e^{-\gamma_2|\tau-t_2|} d\tau \]  

(6.70)

Fig. 6.5 presents the plot of autocorrelation function of the noise component of the output voltage, which is given in (6.68).

\section*{6.9 Conclusions}

We have presented a general perturbation-based stochastic model for a nonlinear circuit, in which, the circuit parameters (resistance and capacitance) are subject to random fluctuations due to noise, which vary with time. The fluctuating amplitudes of these parameters are assumed to be O.U. processes and not the white noise owing to temporal correlations. Then, following the proposed method, the diode circuit is modeled in the form of perturbation-based SDEs. Approximate solution for the output voltage is obtained in the form of stochastic integrals with respect to the Brownian motion processes. The results, obtained using the proposed method, are compared with those obtained via the conventional perturbation-based deterministic differential equations model for a nonlinear circuit. The noise voltage (in time domain at the output) is obtained by subtracting the output signal obtained using perturbation-based deterministic model from that obtained using the proposed method. Hence, the proposed method is a time-domain approach to analyze effect of noise on non-random signals in weakly nonlinear circuits.