Chapter 4

Upper (lower) almost cl-supercontinuous multifunctions

4.1 Introduction

In this chapter we extend the notion of almost cl-supercontinuity (≡ almost clopen continuity) of functions([12] [41]) to the framework of multifunctions. The basic properties of almost cl-supercontinuous multifunctions are studied and their place in the hierarchy of variants of continuity of multifunctions that already exist in the mathematical literature is elaborated. The chapter is organized as follows. In Section 4.1 we define the notions of upper and lower almost cl-supercontinuous multifunctions and discuss their interrelations with other strong variants of continuity of multifunctions that already exist in the literature. It turns out that the class of upper (lower) almost cl-supercontinuous multifunctions properly contains the class of upper (lower) cl-supercontinuous multifunctions[26] and so includes all upper (lower) (almost) perfectly continuous multifunctions[27] and is strictly contained in the class of upper (lower)
(almost) \(z\)-supercontinuous multifunctions([4] [47]) which in turn is properly contained in the class of upper (lower) (almost) \(D_\delta\)-supercontinuous multifunctions([5] [47]) as well as in the class of upper(lower)(almost) \(D^*\)-supercontinuous multifunctions([7] [70]). Examples are included to reflect upon the distinctiveness of the notions so introduced from the other variants of continuity of multifunctions that already exist in the mathematical literature. Section 4.2 deals with characterizations, and basic properties of upper almost cl-supercontinuous multifunctions. It turns out that upper almost cl-supercontinuity of multifunctions is preserved under the shrinking and expansion of range, composition of multifunctions, union of multifunctions, restriction to a subspace, and the passage to the graph multifunction. Moreover, we prove that the graph of an upper almost cl-supercontinuous multifunction with closed values into a regular space is strongly cl-closed with respect to \(X\). Moreover, it is shown that an upper almost cl-supercontinuous multifunction maps mildly compact sets to mildly compact sets. In Section 4.3 we study the properties of lower almost cl-supercontinuous multifunctions, wherein characterizations of lower almost cl-supercontinuity are obtained. It is shown that lower almost cl-supercontinuity is preserved under the shrinking and expansion of range, union of multifunctions, restriction to a subspace and passage to the graph multifunction.

**Definition 4.1.1.** We say that a multifunction \(\varphi : X \rightrightarrows Y\) from a topological space \(X\) into a topological space \(Y\) is

(i) **upper almost cl-supercontinuous** if for each \(x \in X\) and each regular open set \(V\) in \(Y\) containing \(\varphi(x)\), there exists a clopen set \(U\) in \(X\) containing \(x\) such that \(\varphi(U) \subset V\); and

(ii) **lower almost cl-supercontinuous** if for each \(x \in X\) and each regular open set \(V\) in \(Y\) with \(V \cap \varphi(x) \neq \emptyset\), there exists a clopen set \(U\) in \(X\) containing \(x\) such that \(\varphi(z) \cap V \neq \emptyset\) for each \(z \in U\).
The following diagram well illustrates the interrelations that exist among variants of continuity of multifunctions defined in Definitions 1.2.4 and upper(lower) almost cl-supercontinuity of multifunctions.

However, none of the above implications is reversible as is well reflected either by examples in ([3] [4] [5] [21] [26]) or the following observations/examples

**Example 4.1.2.** If $X$ is discrete, then every multifunction $\varphi : X \rightarrow Y$ is strongly continuous.

**Example 4.1.3.** If $X$ is endowed with a partition topology, then every upper (lower) semi-continuous multifunction $\varphi : X \rightarrow Y$ is upper (lower) perfectly continuous.

**Example 4.1.4.** If $X$ is endowed with a partition topology, then every upper (lower) almost continuous multifunction $\varphi : X \rightarrow Y$ is upper (lower) almost perfectly continuous.

**Example 4.1.5.** If $X$ is a zero dimensional space, then every upper (lower) semicontinuous multifunction $\varphi : X \rightarrow Y$ is upper (lower) cl-supercontinuous.
Example 4.1.6. If $X$ is a zero dimensional space, then every upper (lower) almost continuous multifunction $\varphi : X \rightarrow Y$ is upper (lower) almost cl-supercontinuous.

Example 4.1.7. If $X$ is endowed with an almost partition (extremally disconnected) topology, then every upper (lower) completely continuous multifunction $\varphi : X \rightarrow Y$ is upper (lower) perfectly continuous.

Example 4.1.8. If $X$ is endowed with an almost partition topology (extremally disconnected) topology, then every upper (lower) almost completely continuous multifunction $\varphi : X \rightarrow Y$ is upper (lower) almost perfectly continuous.

Example 4.1.9. If $X$ is endowed with a $\delta$-partition topology, then every upper (lower) super-continuous multifunction $\varphi : X \rightarrow Y$ is upper (lower) perfectly continuous.

Example 4.1.10. If $X$ is endowed with a $\delta$-partition topology, then every lower $\delta$-continuous multifunction $\varphi : X \rightarrow Y$ is lower $\delta$-perfectly continuous.

Example 4.1.11. Let $X = Q$ be the rationals with the subspace topology it inherits as a subspace of $\mathbb{R}$. Then any upper (lower) semicontinuous multifunction $\varphi : X \rightarrow Y$ is upper (lower) cl-supercontinuous but not necessarily upper (lower) perfectly continuous.

Example 4.1.12. Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be an upper (lower) semicontinuous multifunction. Then $\varphi$ is upper (lower) $z$-supercontinuous but not necessarily upper (lower) cl-supercontinuous.

Example 4.1.13. Let $X$ be a completely regular non-zero dimensional space and let $\varphi : X \rightarrow Y$ be upper (lower) almost continuous multifunction. Then $\varphi$ is upper (lower) almost $z$-supercontinuous but not necessarily upper (lower) almost cl-supercontinuous.

Example 4.1.14. Let $X$ be endowed with a zero dimensional topology which is not a partition topology. Then every upper (lower) almost continuous multifunction $\varphi : X \rightarrow Y$ is upper
(lower) almost cl-supercontinuous but not necessarily upper (lower) almost perfectly continuous.

Example 4.1.15. Let $X$ be equipped with a nondiscrete partition topology. Then every upper (lower) semicontinuous multifunction $\varphi : X \to Y$ is upper (lower) perfectly continuous but not strongly continuous.

4.2 Properties of upper almost cl-supercontinuous multifunctions

Let $X$ be a topological space and let $A \subseteq X$. A point $x \in X$ is called a \textit{cl-adherent point} [75] of $A$ if every clopen set containing $x$ intersects $A$. Let $[A]_{cl}$ denote the set of all cl-adherent points of $A$. A point $x \in X$ is called \textit{cl-interior point} of $A$ if there exists a clopen set containing $x$ and contained in $A$. Let $int_{cl}(A)$ denote the set of cl-interior points of the set $A$. A set $A$ is cl-closed if $A = [A]_{cl}$ and a set $B$ is cl-open if $B = int_{cl}(B)$.

Theorem 4.2.1. \textit{For a multifunction $\varphi : X \to Y$, the following statements are equivalent}

(a) $\varphi$ is upper almost cl-supercontinuous.

(b) $\varphi^{-1}_{-}(V)$ is a cl-open set in $X$ for each regular open set $V$ in $Y$.

(c) $\varphi_{+}^{-1}(B)$ is a cl-closed set in $X$ for each regular closed set $B$ in $Y$.

(d) $\varphi^{-1}_{-}(G) \subseteq int_{cl}(\varphi^{-1}_{+}(G))$ for each regular open subset $G$ of $Y$.

(e) $[\varphi_{+}^{-1}(K)]_{cl} \subseteq \varphi^{-1}_{+}(K)$ for each regular closed subset $K$ of $Y$.

Proof. (a) $\Rightarrow$ (b). Let $V$ be a regular open subset of $Y$. To show that $\varphi^{-1}_{-}(V)$ is cl-open in $X$, let $x \in \varphi^{-1}_{-}(V)$. Then $\varphi(x) \subseteq V$. Since $\varphi$ is upper almost cl-supercontinuous, therefore, there exists a clopen set $H$ containing $x$ such that $\varphi(H) \subseteq V$. Hence $x \in H \subseteq \varphi^{-1}_{-}(V)$ and so $\varphi^{-1}_{-}(V)$
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is a cl-open set in $X$.

(b) $\Rightarrow$ (c). Let $B$ be a regular closed subset of $Y$. Then $Y - B$ is a regular open subset of $Y$. In view of (b), $\varphi_{-1}(Y - B) = X - \varphi_{+1}(B)$ is cl-open in $X$ and so $\varphi_{+1}(B)$ is a cl-closed set in $X$.

(c) $\Rightarrow$ (d). Let $G$ be a regular open subset of $Y$. Then $Y - G$ is a regular closed set in $Y$. In view of (c), $\varphi_{+1}(Y - G) = Y - \varphi_{-1}(G)$ is a cl-closed set in $X$ and so $\varphi_{-1}(G)$ is cl-open in $X$.

Therefore $\varphi_{-1}(G) \subset \text{int}_{cl}(\varphi_{-1}(G))$.

(d) $\Rightarrow$ (e). Let $K$ be a regular closed subset of $Y$. Then $Y - K$ is a regular open subset of $Y$. In view of (d), $\varphi_{+1}(Y - K) \subset \text{int}_{cl}(\varphi_{-1}(Y - K))$ and so $X - \varphi_{+1}(K) \subset \text{int}_{cl}(X - \varphi_{+1}(K))$ and so $[\varphi_{+1}(K)]_{cl} \subset \varphi_{+1}(K)$.

(e) $\Rightarrow$ (a). Let $x \in X$ and let $V$ be a regular open set in $Y$ containing $\varphi(x)$. Then $Y - V$ is a regular closed set in $Y$. In view of (e), $[\varphi_{+1}(Y - V)]_{cl} \subset \varphi_{+1}(Y - V)$. This implies that $\varphi_{+1}(Y - V) = X - \varphi_{-1}(V)$ is a cl-closed set not containing $x$, and so $\varphi_{-1}(V)$ is a cl-open set containing $x$. Thus there exist a clopen set $U$ containing $x$ such that $U \subset \varphi_{-1}(V)$ and so $\varphi(U) \subset V$. Whence $\varphi$ is upper almost cl-supercontinuous.

In the forthcoming result we formulate sufficient conditions for the preservation of upper almost cl-supercontinuity of multifunctions under the shrinking and expansion of range. First we quote the following definition from [41].

Definition 4.2.2. ([41]) A subset $S$ of a space $X$ is said to be $\delta$-embedded in $X$ if every regular open set in $S$ is the intersection of a regular open set in $X$ with $S$; or equivalently every regular closed set in $S$ is the intersection of a regular closed set in $X$ with $S$.

Theorem 4.2.3. Let $\varphi : X \rightarrow Y$ be an upper almost $\text{cl}$-supercontinuous multifunction Then the following statements are true.

(a) If $\varphi(X)$ is $\delta$-embedded in $Y$, then the multifunction $\varphi : X \rightarrow \varphi(X)$ is upper almost $\text{cl}$-
supercontinuous.

(b) Let $Z$ be a superspace of $Y$ such that the intersection with $Y$ of every regular open set in $Z$ is a regular open set in $Y$. Then the multifunction $\psi : X \to Z$ defined by $\psi(x) = \varphi(x)$ for $x \in X$ is upper almost cl-supercontinuous.

(c) Let $A$ be a subset of $X$, then the restriction $\varphi|_A : A \to Y$ is upper almost cl-supercontinuous.

Proof. (a) Let $G$ be a regular open set in $\varphi(X)$. Since $\varphi(X)$ is $\delta$-embedded in $Y$, there exists a regular open set $G_1$ in $Y$ such that $G = G_1 \cap \varphi(X)$. Again, since $\varphi$ is upper almost cl-supercontinuous, $\varphi^{-1}(G) = \varphi^{-1}(G_1 \cap \varphi(X)) = \varphi^{-1}(G_1) \cap X = \varphi^{-1}(G_1)$ is cl-open and hence $\varphi : X \to \varphi(X)$ is upper almost cl-supercontinuous.

(b) Let $W$ be a regular open set in $Z$. By hypothesis on $Z$, $W \cap Y$ is a regular open set in $Y$. Since $\varphi : X \to Y$ is upper almost cl-supercontinuous, $\varphi^{-1}(W \cap Y)$ is cl-open in $X$. Now since $\psi^{-1}(W) = \varphi^{-1}(W \cap Y)$, $\psi$ is upper almost cl-supercontinuous.

(c) Let $V$ be a regular open set in $Y$. Since $\varphi$ is upper almost cl-supercontinuous, in view of Theorem 4.2.1, $\varphi^{-1}(V)$ is cl-open in $X$. Then $(\varphi|_A)^{-1}(V) = A \cap \varphi^{-1}(V)$, which is cl-open in $A$ and so $\varphi|_A$ is almost upper cl-supercontinuous.

Theorem 4.2.4. $\varphi : X \to Y$ be a multifunction and let $\Sigma = \{U_\alpha : \alpha \in I\}$ be a cl-open cover of $X$. If for each $\alpha \in I$, $\varphi|_{U_\alpha}$ is upper almost cl-supercontinuous, then $\varphi$ is upper almost cl-supercontinuous.

Proof. Let $V$ be a regular open subset of $Y$. Then since $\varphi|_{U_\alpha} : U_\alpha \to Y$ is upper almost cl-supercontinuous, $(\varphi|_{U_\alpha})^{-1}(V) = U_\alpha \cap \varphi^{-1}(V)$ is cl-open in $U_\alpha$ and hence in $X$. Now since $\varphi^{-1} = \bigcup_{\alpha \in I}(U_\alpha \cap \varphi^{-1}(V))$ and since any union of cl-open sets is cl-open, $\varphi^{-1}(V)$ is cl-open in $X$ and so the multifunction $\varphi$ is upper almost cl-supercontinuous.

Theorem 4.2.5. Let $\varphi : X \to Y$ and $\psi : Y \to Z$ be multifunctions. The following statements
are true.

(a) If \( \varphi \) is upper almost cl-supercontinuous and \( \psi \) is upper almost completely continuous, then the multifunction \( \psi \circ \varphi \) is upper almost cl-supercontinuous.

(b) If \( \varphi \) is upper almost cl-supercontinuous and \( \psi \) is upper completely continuous, then the composition \( \psi \circ \varphi \) is upper cl-supercontinuous.

(c) If \( \varphi \) is upper cl-supercontinuous and \( \psi \) is upper almost continuous, then \( \psi \circ \varphi \) is upper almost cl-supercontinuous.

Proof. (a) Let \( W \) be a regular open set in \( Z \). Since \( \psi \) is upper almost completely continuous, \( \psi^{-1}(W) \) is a regular open set in \( Y \). In view of upper almost cl-supercontinuity of \( \varphi \), \( \varphi^{-1}(\psi^{-1}(W)) = (\psi \circ \varphi)^{-1}(W) \) is cl-open in \( X \) and so the multifunction \( \psi \circ \varphi \) is upper almost cl-supercontinuous.

(b) Let \( W \) be an open set in \( Z \). Since \( \psi \) is upper completely continuous, \( \psi^{-1}(W) \) is a regular open set in \( Y \). Again, since \( \varphi \) is upper almost cl-supercontinuous, \( \varphi^{-1}(\psi^{-1}(W)) = (\psi \circ \varphi)^{-1}(W) \) is cl-open in \( X \). Thus \( \psi \circ \varphi \) is upper cl-supercontinuous.

(c) Let \( W \) be a regular open set in \( Z \). Since \( \psi \) is upper almost continuous, \( \psi^{-1}(W) \) is open in \( Y \). Again in view of upper cl-supercontinuity of \( \varphi \), \( \varphi^{-1}(\psi^{-1}(W)) = (\psi \circ \varphi)^{-1}(W) \) is cl-open in \( X \) and so the \( \psi \circ \varphi \) is upper almost cl-supercontinuous.

Definition 4.2.6. A multifunction \( \varphi : X \to Y \) is said to be upper \( \delta \)-cl supercontinuous if for each \( x \in X \) and each \( \delta \)-open set \( V \) containing \( \varphi(x) \) there exists a clopen set \( U \) containing \( x \) such that \( \varphi(U) \subset V \), or equivalently \( \varphi^{-1}(V) \) is cl-open in \( X \) for every \( \delta \)-open set \( V \) in \( Y \).

We may recall that a space \( X \) is almost zero dimensional ([24] [41]) if for each \( x \in X \) and each regular open set \( G \) containing \( x \) there exists a clopen set \( U \) such that \( x \in U \subset G \); equivalently every regular open set in \( X \) is cl-open.
The following lemma due to Singal and Singal[68] will be useful in the sequel.

**Lemma 4.2.7.** Let \( \{X_\alpha : \alpha \in \Lambda\} \) be a family of spaces and let \( X = \prod X_\alpha \) be the product space. If \( x = (x_\alpha) \) and \( V \) is a regular open subset of \( X \) containing \( x \), then there exists a basic regular open set \( \prod V_\alpha \) such that \( x \in \prod V_\alpha \subset V \), where \( V_\alpha \) is regular open in \( X_\alpha \) for each \( \alpha \in \Lambda \) and \( V_\alpha = X_\alpha \) for all except finitely many \( \alpha_1, ..., \alpha_n \in \Lambda \).

**Theorem 4.2.8.** Let \( \varphi : X \to Y \) be a multifunction and let \( g : X \to X \times Y \) be its graph multifunction defined by \( g(x) = \{x\} \times \varphi(x) \) for every \( x \in X \). If \( g \) is upper almost cl-supercontinuous, then \( \varphi \) is almost upper cl-supercontinuous and the space \( X \) is almost zero dimensional. Further, if \( \varphi \) is upper \( \delta \)-cl-supercontinuous and each \( \varphi(x) \) is nearly compact, then \( g \) is upper almost cl-supercontinuous.

**Proof.** Let \( V \) be a regular open subset of \( Y \). Then \( X \times V \) is a regular open set in \( X \times Y \). Since \( g \) is upper almost cl-supercontinuous, \( g^{-1}(X \times V) = \{x \in X \mid \{x\} \times \varphi(x) \subset X \times V\} = \{x \in X \mid \varphi(x) \subset V\} = \varphi^{-1}(V) \) is cl-open in \( X \) and so \( \varphi \) is upper almost cl-supercontinuous. To prove that the space \( X \) is almost zero dimensional, let \( U \) be a regular open set in \( X \) and let \( x \in U \). Then the set \( U \times Y \) is regular open set in \( X \times Y \) containing \( g(x) \). In view of upper almost cl-supercontinuity of the multifunction \( g \), there exists a clopen set \( W \) containing \( x \) such that \( g(W) \subset U \times Y \). Then \( x \in W \subset U \). This shows that the space \( X \) is almost zero dimensional.

Conversely, suppose that the multifunction \( \varphi \) is upper \( \delta \)-cl-supercontinuous such that each \( \varphi(x) \) is nearly compact and \( X \) is almost zero dimensional. To prove that the multifunction \( g \) is upper almost cl-supercontinuous, let \( x \in X \) and let \( W \) be a regular open set containing \( g(x) = \{x\} \times \varphi(x) \). By Lemma 4.2.7 for each \( y \in \varphi(x) \), there are regular open sets \( U_y \) and \( V_y \) such that \( (x, y) \in U_y \times V_y \subset W \). The collection \( \{V_y : y \in \varphi(x)\} \) is a regular open cover of \( \varphi(x) \). Since \( \varphi(x) \) is nearly compact, there exist finitely many \( y_1, ..., y_n \in \varphi(x) \) such that \( \varphi(x) \subset
Let \( U = \bigcap_{i=1}^{n} U_{y_{i}} \) be a clopen set containing \( x \) such that \( x \in G_{1} \subseteq U \). Again since \( \phi \) is upper \( \delta \)-cl-supercontinuous and since \( \phi(x) \) is contained in the \( \delta \)-open set \( V \), there exists a clopen set \( G_{2} \) such that \( \phi(G_{2}) \subseteq V \). Let \( G = G_{1} \cap G_{2} \). Then \( G \) is a clopen set containing \( x \) such that \( g(G) \subseteq U \times V \subseteq W \). Thus \( g \) is upper almost cl-supercontinuous.

We may recall that a space \( X \) is said to be hyperconnected ([1] [76]) if each non-empty open set in \( X \) is dense in \( X \) or equivalently any two nonempty open sets intersect.

**Theorem 4.2.9.** Let \( \phi : X \to Y \) be an upper almost cl-supercontinuous multifunction from a connected space \( X \) onto \( Y \). Then the space \( Y \) is hyperconnected.

**Proof.** Suppose \( X \) is connected and let if possible, \( V \) be a nonempty proper open subset of \( Y \) which is not dense in \( Y \). Then \( W = V^{o} \) is a nonempty proper regular open subset of \( Y \). Since \( \phi \) is upper almost cl-supercontinuous, \( \phi^{-1}(W) \) is a nonempty proper cl-open subset of \( X \), contradicting the fact that \( X \) is connected.

**Theorem 4.2.10.** If \( \phi : X \to Y \) and \( \psi : X \to Y \) are upper almost cl-supercontinuous multifunctions, then \( \phi \cup \psi : X \to Y \) defined by \( (\phi \cup \psi)(x) = \phi(x) \cup \psi(x) \) for each \( x \in X \), is upper almost cl-supercontinuous.

**Proof.** Let \( U \) be a regular open set in \( Y \). Since \( \phi \) and \( \psi \) are upper almost cl-supercontinuous, \( \phi^{-1}(U) \) and \( \psi^{-1}(U) \) are cl-open sets in \( X \). Since \( (\phi \cup \psi)^{-1}(U) = \phi^{-1}(U) \cap \psi^{-1}(U) \) and since finite intersection of cl-open sets is cl-open, \( (\phi \cup \psi)^{-1}(U) \) is cl-open in \( X \). Thus \( \phi \cup \psi \) is upper almost cl-supercontinuous.

**Definition 4.2.11.** The graph \( \Gamma_{\phi} \) of a multifunction \( \phi : X \to Y \) is said to be strongly cl-closed if for each \( (x, y) \not\in \Gamma_{\phi} \) there exist a clopen set \( U \) containing \( x \) and a regular open set \( V \) containing \( y \) such that \( (U \times V) \cap \Gamma_{\phi} = \emptyset \).
Theorem 4.2.12. If $\varphi : X \to Y$ is upper almost cl-supercontinuous, where $Y$ is a regular space and $\varphi(x)$ is closed for each $x \in X$, then the graph $\Gamma_\varphi$ of $\varphi$ is strongly cl-closed with respect to $X$.

Proof. Let $(x, y) \notin \Gamma_\varphi$. Then $y \notin \varphi(x)$. Since $Y$ is a regular space, there exist disjoint open sets $V_y$ and $V_{\varphi(x)}$ containing $y$ and $\varphi(x)$, respectively. It is easily verified that the sets $V_y$ and $V_{\varphi(x)}$ may be chosen to be regular open. Since $\varphi$ is upper almost cl-supercontinuous, there exists a clopen set $U_x$ containing $x$ such that $\varphi(U_x) \subset V_{\varphi(x)}$. We assert that $(U_x \times V_y) \cap \Gamma_\varphi = /0$. For, if $(h, k) \in (U_x \times V_y) \cap \Gamma_\varphi$, then $h \in \varphi^{-1}(V_{\varphi(x)})$, $k \in V_y$ and $k \in \varphi(h)$. Hence $\varphi(h) \subset V_{\varphi(x)}$ and $k \in \varphi(h) \cap V_y$ which contradicts the fact that $V_y$ and $V_{\varphi(x)}$ are disjoint. Consequently the graph $\Gamma_\varphi$ is strongly cl-closed with respect to $X$. 

We may recall that a set $A$ in a space $X$ is said to be N-closed [11] if each cover of $A$ by regular open sets in $X$ has a finite subcover.

Theorem 4.2.13. If $\varphi : X \to Y$ is a multifunction with strongly cl-closed graph with respect to $X$ and $K \subset Y$ is an N-closed set, then $\varphi_+^{-1}(K)$ is cl-closed in $X$. Further, if in addition $Y$ is nearly compact, then $\varphi$ is upper almost cl-supercontinuous.

Proof. To prove that $\varphi_+^{-1}(K)$ is cl-closed, we shall show that $X - \varphi_+^{-1}(K)$ is cl-open. To this end, let $x \in X - \varphi_+^{-1}(K)$. Then $\varphi(x) \cap K = /0$. Since the graph $\Gamma_\varphi$ is strongly cl-closed with respect to $X$, for each $y \in K$ there exists a clopen set $U_y$ containing $x$ and a regular open set $V_y$ containing $y$ such that $(U_y \times V_y) \cap \Gamma_\varphi = /0$. Now the collection $\Omega = \{V_y \mid y \in K\}$ is a regular open cover of the N-closed set $K$. So there exists a finite subset $\{y_1, \ldots, y_n\}$ of $K$ such that $K \subset \bigcup_{i=1}^n V_{y_i} = V(\text{say})$. Let $U = \bigcap_{i=1}^n U_{y_i}$. Then $U$ is a clopen set containing $x$ and since $\varphi(U) \cap K = /0$, thus $U \subset X - \varphi_+^{-1}(K)$ and so $X - \varphi_+^{-1}(K)$ is cl-open as desired. The last
assertion is immediate in view of Theorem 4.2.1 and the fact that a regular closed subset of a nearly compact space is nearly compact [71].

**Corollary 4.2.14.** If \( \varphi : X \rightarrow Y \) is a multifunction with \( \varphi(X) \subseteq K \), where \( K \) is \( N \)-closed set in \( Y \) and the graph \( \Gamma_\varphi \) of \( \varphi \) is strongly \( cl \)-closed with respect to \( X \), then \( \varphi \) is upper almost \( cl \)-supercontinuous.

**Theorem 4.2.15.** Let \( \varphi : X \rightarrow Y \) be an upper almost \( cl \)-supercontinuous multifunction such that \( \varphi(x) \) is mildly compact for each \( x \in X \). If \( A \) is a mildly compact set in \( X \), then \( \varphi(A) \) is mildly compact.

**Proof.** Let \( \Omega \) be a clopen cover of \( \varphi(A) \). Then \( \Omega \) is also a clopen cover of \( \varphi(a) \) for each \( a \in A \). Since each \( \varphi(a) \) is mildly compact, there exists a finite subset \( \beta_a \subset \Omega \) such that \( \varphi(a) \subseteq \bigcup_{B \in \beta_a} B = V_a \) (say) which being a clopen set is regular open. Since \( \varphi \) is upper almost \( cl \)-supercontinuous, there exists a clopen set \( U_a \) containing \( a \) such that \( \varphi(U_a) \subseteq V_a \) and so \( U_a \subseteq \varphi^{-1}(V_a) \). Let \( Q = \{ U_a \mid a \in A \} \). Then \( Q \) is a clopen covering of \( A \). Since \( A \) is mildly compact, there exists a finite subset \( \{ a_1, \ldots, a_n \} \) of \( A \) such that \( A \subseteq \bigcup_{i=1}^n U_{a_i} \subseteq \bigcup_{i=1}^n \varphi^{-1}(V_{a_i}) \). Therefore \( \varphi(A) \subseteq \varphi(\bigcup_{i=1}^n \varphi^{-1}(V_{a_i})) = \bigcup_{i=1}^n \varphi(\varphi^{-1}(V_{a_i})) \subseteq \bigcup_{i=1}^n V_{a_i} \), where \( V_{a_i} = \bigcup_{B \in \beta_i} B, i = 1, \ldots, n \) and each \( \beta_i \) is finite. Thus \( \varphi(A) \) is mildly compact.

4.3 Properties of lower almost \( cl \)-supercontinuous multifunctions

**Theorem 4.3.1.** For a multifunction \( \varphi : X \rightarrow Y \) from a topological space \( X \) into a topological space \( Y \) the following statements are equivalent.

(a) \( \varphi \) is lower almost \( cl \)-supercontinuous.
(b) \( \varphi_+^{-1}(B) \) is a cl-open set in \( X \) for every regular open set \( B \) in \( Y \).

(c) \( \varphi_-^{-1}(B) \) is a cl-closed set in \( X \) for every regular closed set \( B \) in \( Y \).

(d) For each \( x \in X \) and for each regular open set \( V \) with \( \varphi(x) \cap V \neq \emptyset \) there exists a cl-open set \( U \) containing \( x \) such that \( \varphi(z) \cap V \neq \emptyset \) for each \( z \in U \).

**Proof.** \((a) \implies (b)\). Let \( B \) be a regular open subset of \( Y \). To show that \( \varphi_+^{-1}(B) \) is cl-open in \( X \), let \( x \in \varphi_+^{-1}(B) \). Then \( \varphi(x) \cap B \neq \emptyset \). Since \( \varphi \) is lower almost cl-supercontinuous, there exists a clopen set \( H \) containing \( x \) such that \( \varphi(h) \cap B \neq \emptyset \) for each \( h \in H \). Hence \( x \in H \subset \varphi_+^{-1}(B) \) and so \( \varphi_+^{-1}(B) \) is a cl-open set in \( X \) being a union of clopen sets.

\((b) \implies (c)\). Let \( B \) be a regular closed subset of \( Y \). Then \( Y - B \) is a regular open subset of \( Y \). In view of \((b)\), \( \varphi_+^{-1}(Y - B) \) is a cl-open set in \( X \). Since \( \varphi_+^{-1}(Y - B) = X - \varphi_-^{-1}(B) \), \( \varphi_-^{-1}(B) \) is a cl-closed set in \( X \).

\((c) \implies (d)\). Let \( x \in X \) and let \( V \) be a regular open set in \( Y \) with \( \varphi(x) \cap V \neq \emptyset \). Then \( Y - V \) is a regular closed set in \( Y \) with \( \varphi(x) \not\subseteq Y - V \). So using \((c)\), \( \varphi_-^{-1}(Y - V) = X - \varphi_+^{-1}(V) \) is a cl-closed set in \( X \) not containing \( x \) and so \( \varphi_+^{-1}(V) \) is a cl-open set in \( X \) containing \( x \). Let \( U = \varphi_+^{-1}(V) \). Then \( U \) is a cl-open set containing \( x \) such that \( \varphi(z) \cap V \neq \emptyset \) for each \( z \in U \).

The assertion \((d) \implies (a)\) is trivial, since every cl-open set is the union of clopen sets. \( \square \)

The following theorem embodies a sufficient condition for the preservation of lower cl-supercontinuity of a multifunction under the shrinking of its range.

**Theorem 4.3.2.** If \( \varphi : X \rightharpoonup Y \) is lower almost cl-supercontinuous and \( \varphi(X) \) is \( \delta \)-embedded in \( Y \), then \( \varphi : X \rightharpoonup \varphi(X) \) is lower almost cl-supercontinuous.

**Proof.** Let \( V_1 \) be a regular open set in \( \varphi(X) \). Since \( \varphi(X) \) is \( \delta \)-embedded in \( Y \), there exists a regular open set \( V \) in \( Y \) such that \( V_1 = V \cap \varphi(X) \). Again since \( \varphi : X \rightharpoonup Y \) is lower almost cl-supercontinuous \( \varphi_+^{-1}(V_1) = \varphi_+^{-1}(V \cap \varphi(X)) = \varphi_+^{-1}(V) \) is cl-open in \( X \). \( \square \)
Theorem 4.3.3. Let $\varphi: X \to Y$ and $\psi: Y \to Z$ be multifunctions. Then the following statements are true.

(a) If $\varphi$ is lower almost cl-supercontinuous and $\psi$ is lower completely continuous, then the composition $\psi \circ \varphi$ is lower cl-supercontinuous.

(b) If the multifunction $\varphi$ is lower almost cl-supercontinuous and $\psi$ is lower almost completely continuous, then the multifunction $\psi \circ \varphi$ is lower almost cl-supercontinuous.

(c) If $\varphi$ is lower almost cl-supercontinuous and $\psi$ is lower $\delta$-continuous then the multifunction $\psi \circ \varphi$ is lower almost cl-supercontinuous.

(d) If $\varphi$ is lower almost cl-supercontinuous and $\psi$ is lower supercontinuous, then $\psi \circ \varphi$ is lower cl-supercontinuous.

Proof. (a) Let $W$ be an open set in $Z$. Since $\psi$ is lower completely continuous, $\psi^{-1}_+(W)$ is a regular open set in $Y$. Again, since $\varphi$ is lower almost cl-supercontinuous, $\varphi^{-1}_+(\psi^{-1}_+(W)) = (\psi \circ \varphi)^{-1}_+(W)$ is cl-open in $X$. Thus $\psi \circ \varphi$ is lower cl-supercontinuous.

(b) Let $G$ be a regular open set in $Z$. Since $\psi$ is lower almost completely continuous, $\psi^{-1}_+(G)$ is a regular open set in $Y$. In view of lower almost cl-supercontinuity of $\varphi$, it follows that $\varphi^{-1}_+(\psi^{-1}_+(G))$ is a cl-open set in $X$. Hence the multifunction $\psi \circ \varphi$ is lower almost cl-supercontinuous.

(c) Let $G$ be a regular open set in $Z$. Since $\psi$ is lower $\delta$-continuous, $\psi^{-1}_+(G)$ is a $\delta$-open set in $Y$. Suppose $\psi^{-1}_+(G) = \bigcup_{\alpha \in \Lambda} V_\alpha$, where each $V_\alpha$ is a regular open set in $Y$. Since the multifunction $\varphi$ is lower almost cl-supercontinuous, each $\varphi^{-1}_+(V_\alpha)$ is cl-open set in $X$. Now since $(\psi \circ \varphi)^{-1}_+(G) = \varphi^{-1}_+(\psi^{-1}_+(G)) = \varphi^{-1}_+(\bigcup_{\alpha \in \Lambda} V_\alpha) = \bigcup_{\alpha \in \Lambda} \varphi^{-1}_+(V_\alpha)$ and since any union of cl-open set is cl-open, $\psi \circ \varphi$ is lower almost cl-supercontinuous.

(d) Let $G$ be an open subset of $Z$. Since the multifunction $\psi$ is lower supercontinuous, $\psi^{-1}_+(G)$ is a $\delta$-open set in $Y$. Suppose $\psi^{-1}_+(G) = \bigcup_{\alpha \in \Lambda} V_\alpha$, Where each $V_\alpha$ is a regular open set in $Y$. 
CHAPTER 4. UPPER (LOWER) ALMOST cl-SUPERCONTINUOUS MULTIFUNCTIONS

Since \( \varphi \) is lower almost cl-supercontinuous, each \( \varphi^{-1}_+(V_\alpha) \) is a cl-open set in \( X \). Now since 
\[(\psi \circ \varphi)^{-1}_+(G) = \varphi^{-1}_+(\psi^{-1}_+(G)) = \varphi^{-1}_+(\bigcup_{\alpha \in \Lambda} V_\alpha) = \bigcup_{\alpha \in \Lambda} \varphi^{-1}_+(V_\alpha) \]
and since any union of cl-open sets is cl-open, the composition \( \psi \circ \varphi \) is lower cl-supercontinuous.

In analogy with Theorem 4.2.3(b) the following result embodies a sufficient condition for the preservation of lower almost cl-supercontinuity of a multifunction under the expansion of its range.

**Corollary 4.3.4.** Let \( \varphi : X \rightarrow Y \) be lower almost cl-supercontinuous. If \( Z \) is a space containing \( Y \) as a subspace such that the intersection with \( Y \) of every regular open in \( Z \) is a regular open set in \( Y \), then \( \psi : X \rightarrow Z \) defined by \( \psi(x) = \varphi(x) \) for \( x \in X \) is lower almost cl-supercontinuous.

### 4.4 Change of topology

In this section we study the behaviour of a lower almost cl-supercontinuous multifunction if its domain and/or range are retopologized in an appropriate way. Let \( (X, \tau) \) be a topological space and let \( \beta \) denote the collection of all clopen subsets of \( (X, \tau) \). Since the intersection of two clopen sets is a clopen set, the collection \( \beta \) is a base for a topology \( \tau^* \) on \( X \). Clearly \( \tau^* \subset \tau \) and any topological property which is preserved under continuous bijections is transferred from \( (X, \tau) \) to \( (X, \tau^*) \). Moreover, the space \( (X, \tau) \) is zero dimensional if and only if \( \tau = \tau^* \). The topology \( \tau^* \) has been extensively referred to in the mathematical literature (see [12],[71]).

Throughout the section, the symbol \( \tau^* \) will have the same meaning as in the above paragraph.

**Semiregularization:** Let \( (X, \nu) \) be a topological space. Let \( \beta \) denote the collection of all
ν-regular open sets in X. Then β is a base for a topology νδ on X and νδ ⊂ ν. (X, νδ) is semiregular space and is called semiregularization of (X, ν). Moreover, the space (X, ν) is semi-regular if and only if ν ⊂ νδ.

**Theorem 4.4.1.** For a multifunction φ : ((X, τ) → (Y, ν) the following statements are equivalent.

(a) f : (X, τ) → (Y, ν) is lower almost cl-supercontinuous.

(b) f : (X, τ) → (Y, νδ) is lower cl-supercontinuous.

(c) f : (X, τ*) → (Y, ν) is lower almost continuous.

(d) f : (X, τ*) → (Y, νδ) is lower semi continuous.

**Proof.** (a) ⇒ (b). Let V be an open set in (Y, νδ). Then V = ∪αUα, where Uα is regular open in (Y, ν). By (a) φ−1(Uα) is cl-open in (Y, τ). Therefore φ−1(Uα) = ∪αφ−1(Uα) is cl-open in (X, τ).

(b) ⇒ (c). Let V be a regular open set in (Y, ν). Then V is open in (Y, νδ). By (b) φ−1(V) is cl-open in (X, τ). Therefore φ−1(V) is open in (X, τ*). So φ is lower almost continuous.

(c) ⇒ (d). Let V be an open set in (Y, νδ). Then V = ∪αUα, where Uα is regular open in (Y, ν). By (c) φ−1(Uα) is open in (X, τ*). Therefore φ−1(V) = ∪αφ−1(Uα) is open in (X, τ*).

(d) ⇒ (a). Let V be a regular open set in (Y, ν). Then V is open in (Y, νδ). By (d) φ−1(V) is open in (X, τ*). Moreover φ−1(V) being union of clopen sets is cl-open in (X, τ).  □