Chapter 3

Upper and Lower cl-Supercontinuous Multifunctions

3.1 Introduction

The notion of cl-supercontinuity (≡ clopen continuity) for functions was introduced by Reilly and Vamanamurthy [65] and has been extensively studied by D. Singh [71]. In this chapter we extend the notion of cl-supercontinuity to the realm of multifunctions and elaborate on its properties. The basic properties of upper and lower cl-supercontinuous multifunctions are studied and their place in the hierarchy of strong variants of continuity of multifunctions is discussed. In the process certain results of Singh[71] pertaining to cl-supercontinuous functions and others have been extended to the framework of multifunctions. The chapter is organised as follows: In Section 3.1 we define upper and lower cl-supercontinuous multifunctions and discuss their interrelations with other strong variants of continuity of multifunctions that already exist in the literature. Examples are provided to reflect upon the
distinctiveness of the notions so defined. Section 3.2 deals with characterizations and basic properties of upper cl-supercontinuous multifunctions. It turns out that cl-supercontinuity of multifunctions is preserved under the shrinking and expansion of range, composition of multifunctions, union of multifunctions, restriction to a subspace, and the passage to the graph multifunction. Further, we formulate a sufficient condition for the intersection of two multifunctions to be cl-supercontinuous. Moreover, we prove that the graph of an upper cl-supercontinuous multifunction with closed values into a regular space is cl-closed with respect to X. Furthermore, it is shown that an upper cl-supercontinuous multifunction maps mildly compact sets to compact sets. Finally it is shown that an open, upper cl-supercontinuous nonmingled multifunction with paracompact(para-Lindelöf) values maps cl-paracompact(cl-para-Lindelöf) sets to paracompact(para-Lindelöf) sets. Characterizations and basic properties of lower cl-supercontinuous multifunctions are discussed in Section 3.3. It is shown that lower cl-supercontinuity is preserved under the shrinking and expansion of range, union of multifunctions, restriction to a subspace and passage to the graph multifunction. Further, it is shown that an arbitrary product of multifunctions is lower cl-supercontinuous if and only if each multifunction is lower cl-supercontinuous.

**Definition 3.1.1.** We say that a multifunction \( \varphi : X \rightharpoonup Y \) from a topological space \( X \) into a topological space \( Y \) is

(a) **upper cl-supercontinuous** at \( x \in X \) if for each open set \( V \) with \( \varphi(x) \subset V \), there exists a clopen set \( U \) containing \( x \) such that \( \varphi(U) \subset V \). The multifunction is said to be upper cl-supercontinuous if it is upper cl-supercontinuous at each \( x \in X \).

(b) **lower cl-supercontinuous** at \( x \in X \) if for each open set \( V \) with \( \varphi(x) \cap V \neq \emptyset \), there exists a clopen set \( U \) containing \( x \) such that \( \varphi(z) \cap V \neq \emptyset \) for each \( z \in U \). The multifunction is
said to be lower cl-supercontinuous if it is lower cl-supercontinuous at each \( x \in X \).

The following diagram well illustrates the interrelations that exist among various strong variants of continuity of multifunctions defined in Definition 1.2.4 and 3.1.1.

\[ \text{upper completely continuous} \rightarrow \text{upper perfectly continuous} \rightarrow \text{strongly continuous} \rightarrow \text{lower perfectly continuous} \rightarrow \text{lower completely continuous} \]

\[ \rightarrow \text{upper cl-supercontinuous} \rightarrow \text{upper z-supercontinuous} \rightarrow \text{upper } D_{\theta}^* \text{-supercontinuous} \rightarrow \text{upper strongly } \theta \text{-continuous} \rightarrow \text{upper supercontinuous} \]

\[ \rightarrow \text{lower cl-supercontinuous} \rightarrow \text{lower z-supercontinuous} \rightarrow \text{lower } D_{\theta}^* \text{-supercontinuous} \rightarrow \text{lower strongly } \theta \text{-continuous} \rightarrow \text{lower supercontinuous} \]

\[ \rightarrow \text{lower completely continuous} \rightarrow \text{lower perfectly continuous} \rightarrow \text{lower semicontinuous} \rightarrow \text{lower supercontinuous} \]

However, none of the above implications is reversible as is well illustrated by the examples in the sequel and the examples in ([2], [3], [4], [5], [27], [50]).

**Example 3.1.2.** : Let \( X = \{a, b, c\} \) with the topology \( \mathcal{Z}_X = \{\emptyset, X, \{a\}, \{b, c\}\} \) and let \( Y = \{x, y\} \) with the topology \( \mathcal{Z}_Y = \{\emptyset, Y, \{y\}\} \). Define a multifunction \( \phi : (X, \mathcal{Z}_X) \rightarrow (Y, \mathcal{Z}_Y) \) by \( \phi(a) = \{y\}, \phi(b) = \{x, y\}, \phi(c) = \{x\} \). Then the multifunction is upper perfectly continuous but not lower perfectly continuous. Again, for \( \{x\} \subset Y, \phi^{-1}(\{x\}) = \{c\} \) is not clopen which implies that the multifunction \( \phi \) is not strongly continuous.

**Example 3.1.3.** : Let \( X = \{a, b, c\} \) with the topology \( \mathcal{Z}_X = \{\emptyset, X, \{a\}, \{a, c\}, \{a, b\}\} \) and let \( Y = \{x, y\} \) with the topology \( \mathcal{Z}_Y = \{\emptyset, Y, \{y\}\} \). Define a multifunction \( \phi : (X, \mathcal{Z}_X) \rightarrow (Y, \mathcal{Z}_Y) \) by \( \phi(a) = \{y\}, \phi(b) = \{x, y\}, \phi(c) = \{y\} \). Then clearly \( \phi \) is lower perfectly continu-
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ous. But for \{y\} \subset Y \varphi^{-1}(\{y\}) = \{a\} is not clopen, which implies the multifunction \varphi is not strongly continuous.

Example 3.1.4. : Let \(X = \mathbb{R}\), set of real numbers with upper limit topology \(\mathcal{S}\) and let \(Y\) be same as \(X\) with usual topology \(U\). Define a multifunction \(\varphi : (X, \mathcal{S}) \rightarrow (Y, U)\) by \(\varphi(x) = \{x\}\) for each \(x \in X\). Then clearly \(\varphi\) is upper (lower) cl-supercontinuous. But for \(\varphi^{-1}(a, b) = (a, b) = \varphi^{-1}(a, b)\) is not clopen in \(X\), which implies that \(\varphi\) is not upper (lower) perfectly continuous.

Example 3.1.5. : Let \(X\) be a completely regular space which is not zero dimensional and let \(Y\) be same as \(X\). Then the identity mapping \(\varphi : X \rightarrow Y\) defined by \(\varphi(x) = \{x\}\) for each \(x \in X\), is upper (lower) \(z\)-supercontinuous but not upper (lower) cl-supercontinuous.

3.2 Properties of upper cl-supercontinuous multifunctions

Theorem 3.2.1. For a multifunction \(\varphi : X \rightarrow Y\) from a topological space \(X\) into a topological \(Y\) the following statements are equivalent:

(a) \(\varphi\) is upper cl-supercontinuous.

(b) \(\varphi^{-1}_-(B)\) is a cl-open set in \(X\) for every open set \(B\) in \(Y\).

(c) \(\varphi^{-1}_+(B)\) is a cl-closed set in \(X\) for every closed set \(B\) in \(Y\).

(d) \([\varphi^{-1}_+(B)]_{cl} \subset \varphi^{-1}_-(\overline{B})\) for every subset \(B\) of \(Y\).

Proof. (a) \(\Rightarrow\) (b) Let \(B\) be an open subset of \(Y\). To show that \(\varphi^{-1}_-(B)\) is cl-open in \(X\), let \(x \in \varphi^{-1}_-(B)\). Then \(\varphi(x) \subset B\). Since \(\varphi\) is upper cl-supercontinuous, therefore, there exists a clopen set \(H\) containing \(x\) such that \(\varphi(H) \subset B\). Hence \(x \in H \subset \varphi^{-1}_-(B)\) and so \(\varphi^{-1}_-(B)\) is a cl-open set in \(X\).
(b) ⇒ (c). Let \( B \) be a closed subset of \( Y \). Then \( Y \setminus B \) is an open subset of \( Y \). By (b), \( \varphi_-(Y \setminus B) \) is cl-open set in \( X \). Since \( \varphi_-(Y \setminus B) = X \setminus \varphi_+(B) \), \( \varphi_+(B) \) is a cl-closed set in \( X \).

(c) ⇒ (d). Since \( \overline{B} \) is closed, \( \varphi_-(\overline{B}) \) is a cl-closed set containing \( \varphi_-(B) \). Therefore \( \varphi_-(\overline{B}) \) is cl-open set in \( X \).

(d) ⇒ (a). Let \( x \in X \) and let \( V \) be an open set in \( Y \) such that \( \varphi(x) \subset V \). Then \( \varphi(x) \cap (Y \setminus V) = \emptyset \) and \( (Y \setminus V) = Y \setminus V \). Hence \( \varphi_-(Y \setminus V) \subset \varphi_-(Y \setminus V) = X \setminus \varphi_-(V) \). Since \( \varphi_+(Y \setminus V) \) is cl-closed, its complement \( \varphi_-(V) \) is a cl-open set containing \( x \). So there is a clopen set \( U \) containing \( x \) and contained in \( \varphi_-(V) \), whence \( \varphi(U) \subset V \). Thus \( \varphi \) is upper cl-supercontinuous.

Theorem 3.2.2. If a multifunction \( \varphi : X \rightharpoonup Y \) is upper cl-supercontinuous and \( \varphi(X) \) is endowed with the subspace topology, then, the multifunction \( \varphi : X \rightharpoonup \varphi(X) \) is upper cl-supercontinuous.

Proof. Since \( \varphi \) is upper cl-supercontinuous for every open set \( V \) of \( Y \), \( \varphi_-(V \cap \varphi(X)) = \varphi_-(V) \cap \varphi_-(\varphi(X)) = \varphi_-(V) \cap X = \varphi_-(V) \) is cl-open and hence \( \varphi : X \rightharpoonup \varphi(X) \) is upper cl-supercontinuous.

Theorem 3.2.3. If \( \varphi : X \rightharpoonup Y \) is upper cl-supercontinuous and \( \psi : Y \rightharpoonup Z \) is upper semicontinuous, then \( \psi \circ \varphi \) is upper cl-supercontinuous. In particular, composition of upper cl-supercontinuous multifunctions is upper cl-supercontinuous.

Proof. Let \( W \) be an open set in \( Z \). Since \( \psi \) is upper semicontinuous, \( \psi_-(W) \) is an open set in \( Y \). Again, since \( \varphi \) is upper cl-supercontinuous, \( \varphi_-(\psi_-(W)) = (\psi \circ \varphi)_-(W) \) is a cl-open set in \( X \). Thus \( \psi \circ \varphi : X \rightharpoonup Z \) is upper cl-supercontinuous.

Analogous to Theorem 3.2.2, the following corollary shows that upper cl-supercontinuity of a multifunction remains invariant under extension of its range.
Corollary 3.2.4. Let $\varphi : X \to Y$ be upper cl-supercontinuous. If $Z$ is a space containing $Y$ as a subspace, then $\psi : X \to Z$ defined by $\psi(x) = \varphi(x)$ for each $x \in X$ is upper cl-supercontinuous.

Proof. Let $W$ be an open set in $Z$. Then $W \cap Y$ is an open set in $Y$. Since $\varphi : X \to Y$ is upper cl-supercontinuous, $\varphi^{-1}(W \cap Y)$ is cl-open in $X$. Now $\psi^{-1}(W) = \{x \in X : \psi(x) \subset W\}$ = $\{x \in X : \varphi(x) \subset W \cap Y\}$. Thus $\psi : X \to Z$ is upper cl-supercontinuous. \qed

Theorem 3.2.5. If $\varphi : X \to Y$ and $\psi : X \to Y$ are upper cl-supercontinuous multifunctions, then $\varphi \cup \psi : X \to Y$ defined by $(\varphi \cup \psi)(x) = \varphi(x) \cup \psi(x)$ for each $x \in X$, is upper cl-supercontinuous.

Proof. Let $U$ be an open set in $Y$. Since $\varphi$ and $\psi$ are upper cl-supercontinuous, $\varphi^{-1}(U)$ and $\psi^{-1}(U)$ are cl-open sets in $X$. Since $(\varphi \cup \psi)^{-1}(U) = \varphi^{-1}(U) \cap \psi^{-1}(U)$ and since finite intersection of cl-open sets is cl-open, $(\varphi \cup \psi)^{-1}(U)$ is cl-open in $X$. Thus $\varphi \cup \psi$ is upper cl-supercontinuous. \qed

In general, intersection of two upper cl-supercontinuous multifunctions need not be upper cl-supercontinuous. However, in the following theorem we formulate a sufficient condition for the intersection of two multifunctions to be upper cl-supercontinuous.

Theorem 3.2.6. Let $\varphi : X \to Y$ and $\psi : X \to Y$ be multifunctions from a space $X$ into a Hausdorff space $Y$ such that $\varphi(x)$ is compact for each $x \in X$. Suppose further that

1. $\varphi$ is upper cl-supercontinuous, and
2. the graph $\Gamma_{\psi}$ of $\psi$ is cl-closed with respect to $X$.

Then the multifunction $\varphi \cap \psi$ defined by $(\varphi \cap \psi)(x) = \varphi(x) \cap \psi(x)$ for each $x \in X$, is upper cl-supercontinuous.
Proof. Let \( x_0 \in X \) and \( V \) be an open set containing \( \varphi(x_0) \cap \psi(x_0) \). It suffices to find a clopen set \( U \) containing \( x_0 \) such that \( (\varphi \cap \psi)(U) \subset V \). If \( V \supset \varphi(x_0) \), it follows from upper cl-supercontinuity of \( \varphi \). If not, then consider the set \( K = \varphi(x_0) \setminus V \) which is compact. Now for each \( y \in K \), \( y \in Y \setminus \psi(x_0) \). This implies that \( (x_0, y) \in (X \times Y) \setminus \Gamma_\psi \). Since the graph of \( \psi \) is cl-closed with respect to \( X \), there exist a clopen set \( U \) containing \( x_0 \) and an open set \( V \) containing \( y \) such that \( \Gamma_\psi \cap (U_y \times V_y) = \emptyset \). Therefore, for each \( x \in U_y \), \( \psi(x) \cap V_y = \emptyset \). Since \( K \) is compact, there exist finitely many points \( y_1, y_2, \ldots, y_n \) in \( K \) such that \( K \subset \bigcup_{i=1}^n V_{y_i} \). Let \( W = \bigcup_{i=1}^n V_{y_i} \). Then \( V \cup W \) is an open set containing \( \varphi(x_0) \). Since \( \varphi \) is upper cl-supercontinuous, there exists a clopen set \( U_0 \) containing \( x_0 \) such that \( \varphi(U_0) \subset V \cup W \). Let \( U = U_0 \cap (\bigcap_{i=1}^n U_{y_i}) \). Then \( U \) is a clopen set containing \( x_0 \). Hence for each \( z \in U \), \( \varphi(z) \subset V \cup W \) and \( \psi(z) \cap W = \emptyset \). Therefore, \( (\varphi(z) \cap \psi(z)) \cap W = \emptyset \) for each \( z \in U \). This proves that \( \varphi \cap \psi \) is upper cl-supercontinuous at \( x_0 \).

Corollary 3.2.7. Let \( \psi : X \rightarrow Y \) be a multifunction from a space \( X \) into a compact Hausdorff space \( Y \) such that the graph \( \Gamma_\psi \) of \( \psi \) is cl-closed with respect to \( X \). Then \( \psi \) is upper cl-supercontinuous.

Proof. Let the multifunction \( \varphi : X \rightarrow Y \) be defined by \( \varphi(x) = Y \) for each \( x \in X \). Now an application of Theorem 3.2.6 yields the desired result. \( \square \)

Theorem 3.2.8. Let \( \varphi : X \rightarrow Y \) be any multifunction. Then the following statements are true:

(a) If \( \varphi : X \rightarrow Y \) is upper cl-supercontinuous and \( A \subset X \), then the restriction \( \varphi_A : A \rightarrow Y \) is upper cl-supercontinuous.

(b) If \( \{U_\alpha : \alpha \in \Delta\} \) is a cl-open cover of \( X \) and if for each \( \alpha \), the restriction \( \varphi_\alpha = \varphi_{U_\alpha} : U_\alpha \rightarrow Y \) is upper cl-supercontinuous, then \( \varphi : X \rightarrow Y \) is upper cl-supercontinuous.
Proof. (a) Let $W$ be an open set in $Y$. Since $\varphi : X \to Y$ is upper cl-supercontinuous, $\varphi^{-1}(W)$ is a cl-open set in $X$. Now $(\varphi|_A)^{-1}(W) = \{ x \in A \mid \varphi(x) \subset W \} = A \cap \varphi^{-1}(W)$, which is cl-open in $X$ and so $\varphi|_A$ is upper cl-supercontinuous.

(b) Let $W$ be an open set in $Y$. Since $\varphi_\alpha = \varphi|_{U_\alpha} : U_\alpha \to Y$ is upper cl-supercontinuous, $(\varphi_\alpha)^{-1}(W)$ is a cl-open set in $U_\alpha$ and consequently cl-open in $X$. Since $\varphi^{-1}(W) = \bigcup_{\alpha \in \Delta} (\varphi_\alpha)^{-1}(W)$ and since the union of cl-open set is cl-open, $\varphi^{-1}(W)$ is cl-open set in $X$. In view of Theorem 3.2.1, $\varphi : X \to Y$ is upper cl-supercontinuous.

Theorem 3.2.9. Let $\varphi : X \to Y$ be a multifunction and let $g : X \to X \times Y$ defined by $g(x) = \{(x,y) \in X \times Y \mid y \in \varphi(x)\}$ for each $x \in X$, be the graph multifunction. If $g$ is upper cl-supercontinuous, then $\varphi$ is upper cl-supercontinuous and the space $X$ is zero dimensional.

Furthermore, if in addition $\varphi(x)$ is compact for each $x \in X$ and $X$ is zero dimensional, then $g$ is upper cl-supercontinuous whenever $\varphi$ is.

Proof. Suppose that $g$ is upper cl-supercontinuous. By Theorem 3.2.3, the multifunction $\varphi = p_y \circ g$ is upper cl-supercontinuous, where $p_y : X \times Y \to Y$ denotes the projection mapping. To show that $X$ is zero dimensional, let $U$ be an open set in $X$ and let $x \in U$. Then $U \times Y$ is an open set in $X \times Y$ and $g(x) \subset U \times Y$. Since $g$ is upper cl-supercontinuous, there exists a clopen set $W$ containing $x$ such that $g(W) \subset U \times Y$ and so $W \subset g^{-1}(U \times Y) = U$. Hence $x \in W \subset U$ and thus $X$ is zero dimensional.

Conversely, suppose that $X$ is zero dimensional, the multifunction $\varphi$ is upper cl-supercontinuous and $\varphi(x)$ is compact for each $x \in X$. Let $W$ be an open set containing $g(x) = \{x\} \times \varphi(x)$. Then by Wallace theorem ([22], p.142) there exist open sets $U$ in $X$, $V$ in $Y$ and $g(x) \subset U \times V \subset W$. So $x \in U$ and $\varphi(x) \subset V$. Since $X$ is zero dimensional there exists a clopen set $G_1$ containing $x$ such that $x \in G_1 \subset U$. Again since $\varphi$ is upper cl-supercontinuous, there exists a clopen set $G_2$
containing \( x \) such that \( \varphi(G_2) \subset V \). Let \( G = G_1 \cap G_2 \). Then \( G \) is a clopen set containing \( x \) and it is easily verified that \( g(G) \subset U \times V \subset W \). This proves that \( g \) is upper cl-supercontinuous.

The following theorem gives sufficient conditions for the graph of a multifunction to be cl-closed with respect to \( X \).

**Theorem 3.2.10.** If \( \varphi : X \rightarrow Y \) is upper cl-supercontinuous, where \( Y \) is a regular space and \( \varphi(x) \) is closed for each \( x \in X \), then graph \( \Gamma_\varphi \) of \( \varphi \) is a cl-closed with respect to \( X \).

**Proof.** Let \((x, y) \notin \Gamma_\varphi\). Then \( y \notin \varphi(x) \). Since \( Y \) is regular, there exist disjoint open sets \( V_y \) and \( V_{\varphi(x)} \) containing \( y \) and \( \varphi(x) \), respectively. Since \( \varphi \) is upper cl-supercontinuous, there exists a clopen set \( U_x \) containing \( x \) such that \( \varphi(U_x) \subset V_{\varphi(x)} \). We assert that \( (U_x \times V_y) \cap \Gamma_\varphi = \emptyset \). For, if \((h, k) \in (U_x \times V_y) \cap \Gamma_\varphi\), then \( h \in \varphi^{-1}(V_{\varphi(x)}) \), \( k \in V_y \) and \( k \in \varphi(h) \). Hence \( \varphi(h) \subset V_{\varphi(x)} \) and \( k \in \varphi(h) \cap V_y \) which contradicts the fact that \( V_y \) and \( V_{\varphi(x)} \) are disjoint. Thus the graph \( \Gamma_\varphi \) of \( \varphi \) is cl-closed with respect to \( X \).

The following theorem is a sort of partial converse to Theorem 3.2.10 and shows that the multifunctions which have cl-closed graph with respect to \( X \) have nice properties.

**Theorem 3.2.11.** If \( \varphi : X \rightarrow Y \) is a multifunction with cl-closed graph with respect to \( X \) and \( K \subset Y \) is compact, then \( \varphi_+^{-1}(K) \) is cl-closed in \( X \). Further, if in addition \( Y \) is compact, then \( \varphi \) is upper cl-supercontinuous.

**Proof.** To prove that \( \varphi_+^{-1}(K) \) is cl-closed, we shall show that \( X \setminus \varphi_+^{-1}(K) \) is cl-open. To this end, let \( x \in X \setminus \varphi_+^{-1}(K) \). Then \( \varphi(x) \cap K = \emptyset \). Since \( \Gamma_\varphi \) is cl-closed with respect to \( X \), for each \( y \in K \) there exist clopen set \( U_y \) containing \( x \) and an open set \( V_y \) containing \( y \) such that \( (U_y \times V_y) \cap \Gamma_\varphi = \emptyset \). The collection \( \Omega = \{V_y \mid y \in K\} \) is an open cover of the compact set \( K \). So there exists a finite subset \( \{y_1, \ldots, y_n\} \) of \( K \) such that \( K \subset \bigcup_{i=1}^{n} V_{y_i} = V \) (say). Let
\[ U = \cap_{i=1}^n U_{y_i}. \] Then \( U \) is a clopen set containing \( x \) and \( \varphi(U) \cap K = \emptyset. \) Thus \( U \subset X \setminus \varphi^{-1}(K) \) and so \( X \setminus \varphi^{-1}(K) \) is cl-open as desired. The last assertion is immediate in view of Theorem 3.2.1 and the fact that a closed subset of a compact space is compact.

**Corollary 3.2.12.** If \( \varphi : X \to Y \) is a multifunction with \( \varphi(X) \subset K, \) where \( K \) is compact and the graph \( \Gamma_\varphi \) of \( \varphi \) is cl-closed with respect to \( X, \) then \( \varphi \) is upper cl-supercontinuous.

**Theorem 3.2.13.** Let \( \varphi : X \to Y \) be an upper cl-supercontinuous multifunction such that \( \varphi(x) \) is compact for each \( x \in X. \) If \( A \) is a mildly compact set in \( X, \) then \( \varphi(A) \) is compact.

**Proof.** Let \( \Omega \) be an open cover of \( \varphi(A). \) Then \( \Omega \) is also an open cover of \( \varphi(a) \) for each \( a \in A. \) Since each \( \varphi(a) \) is compact, there exists a finite subset \( \beta_a \subset \Omega \) such that \( \varphi(a) \subset \bigcup_{B \in \beta_a} B = V_a \) (say). Since \( \varphi \) is upper cl-supercontinuous, there exists a clopen set \( U_a \) containing \( a \) such that \( \varphi(U_a) \subset V_a \) and so \( U_a \subset \varphi^{-1}(V_a). \) Let \( Q = \{U_a \mid a \in A\}. \) Then \( Q \) is a clopen covering of \( A. \) Since \( A \) is mildly compact, there exists a finite subset \( \{a_1, \ldots, a_n\} \) of \( A \) such that \( A \subset \bigcup_{i=1}^n U_{a_i} \subset \bigcup_{i=1}^n \varphi^{-1}(V_{a_i}). \) Therefore \( \varphi(A) \subset \varphi\left(\bigcup_{i=1}^n \varphi^{-1}(V_{a_i})\right) = \bigcup_{i=1}^n \varphi\left(\varphi^{-1}(V_{a_i})\right) \subset \bigcup_{i=1}^n V_{a_i}, \) where \( V_{a_i} = \bigcup_{B \in \beta_{a_i}} B, i = 1, \ldots, n \) and each \( \beta_{a_i} \) is finite. Thus \( \varphi(A) \) is compact.

We may recall that a space \( X \) is called a \( P \)-space if every \( G_\delta \) in \( X \) is open in \( X. \)

**Theorem 3.2.14.** Let \( \varphi : X \to Y \) be an open, upper cl-supercontinuous and nonmingled multifunction from a space \( X \) into a \( P \)-space \( Y \) such that \( \varphi(x) \) is para-Lindelöf for each \( x \in X. \) If \( A \) is a cl-para-Lindelöf set in \( X, \) then \( \varphi(A) \) is para-Lindelöf set in \( Y. \) In particular, if \( X \) is cl-para-Lindelöf and \( \varphi \) is onto, then \( Y \) is para-Lindelöf.

**Proof.** Let \( \Psi \) be an open cover of \( \varphi(A). \) Then \( \Psi \) is also an open covering of \( \varphi(x) \) for each \( x \in A. \) Since \( \varphi(x) \) is para-Lindelöf, \( \Psi \) has a locally countable open refinement \( \psi_x \) such that \( \varphi(x) \subset \bigcup \psi_x = V_x \) (say). Since \( \varphi \) is upper cl-supercontinuous, there exists a clopen set \( U_x \)
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containing \( x \) such that \( \varphi(U_x) \subset V_x \). Now \( u = \{U_x \mid x \in A\} \) is a clopen cover of \( A \). Since \( A \) is \( \text{cl-para-Lindelöf} \), \( u \) has a locally countable open refinement \( \Omega = \{W_\alpha \mid \alpha \in \Lambda\} \) such that \( A \subset \bigcup_{\alpha \in \Lambda} W_\alpha \). So for each \( \alpha \in \Lambda \) there exists an \( x_\alpha \in A \) such that \( W_\alpha \subset U_{x_\alpha} \) and hence \( \varphi(W_\alpha) \subset \varphi(U_{x_\alpha}) \subset \bigcup \psi_{x_\alpha} \). Let \( \mathcal{R}_\alpha = \{\varphi(W_\alpha) \cap V \mid V \in \psi_{x_\alpha}\} \) and let \( \mathcal{R} = \{R \mid R \in \mathcal{R}_\alpha, \alpha \in \Lambda\} \).

We shall show that \( \mathcal{R} \) is a locally countable open refinement of \( \Psi \). Since \( \varphi \) is open, \( \varphi(W_\alpha) \) is open and so each \( R \in \mathcal{R} \) is open. Let \( R \in \mathcal{R} \). Then \( R \in \mathcal{R}_\alpha \) for some \( \alpha \in \Lambda \), i.e. \( R = \varphi(W_\alpha) \cap V \subset V \subset U \) for some \( U \in \Psi \). This shows that \( \mathcal{R} \) is an open refinement of \( \Psi \). To show that \( \mathcal{R} \) is locally countable, let \( y \in \varphi(A) \). Then \( y \in \varphi(x) \) for some \( x \in A \). Since \( \Omega \) is locally countable, for this \( x \in A \) we can choose an open neighborhood \( G_x \) of \( x \) which intersects only countably many members \( W_{\alpha_1}, W_{\alpha_2}, \ldots, W_{\alpha_n}, \ldots \) of \( \Psi \). Since \( \varphi \) is a nonmingled open multifunction, it follows that \( H_0 = \varphi(G_x) \) is an open neighborhood of \( y \) which intersects only countably many members \( \varphi(W_{\alpha_1}), \varphi(W_{\alpha_2}) \ldots \varphi(W_{\alpha_n}) \ldots \) of the family \( \{\varphi(W_\alpha) \mid \alpha \in \Lambda\} \). Furthermore each \( \mathcal{R}_{\alpha_k} \) \((k = 1, \ldots, n, \ldots)\) is locally countable, hence there exists an open neighborhood \( H_k \) \((k = 1, \ldots, n, \ldots)\) of \( y \) which intersects only countably many members of \( \mathcal{R}_{\alpha_k} \) \((k = 1, \ldots, n, \ldots)\).

Finally let \( H = \bigcap_{k=1}^\infty H_k \). Since \( Y \) is \( P \)-space, \( H \) is an open neighborhood of \( y \) which intersects at most countably many members of \( \mathcal{R} \), and so \( \mathcal{R} \) is locally countable. Moreover, \( \varphi(A) \subset \varphi(\bigcup_{\alpha \in \Lambda} W_\alpha) \subset \bigcup_{\alpha \in \Lambda} \varphi(W_\alpha) \subset \bigcup_{\alpha \in \Lambda} (\bigcup \mathcal{R}_\alpha) = \bigcup \{R \mid R \in \mathcal{R}\} \). Hence \( \mathcal{R} \) is a locally countable open refinement of \( \Psi \) that covers \( \varphi(A) \). Thus \( \varphi(A) \) is para-Lindelöf. \( \square \)

**Theorem 3.2.15.** Let \( \varphi : X \to Y \) be an open, upper \( \text{cl-}\)supercontinuous nonmingled multifunction from a space \( X \) into a space \( Y \) such that \( \varphi(x) \) is paracompact for each \( x \in X \). If \( A \) is a \( \text{cl-paracompact} \), then \( \varphi(A) \) is paracompact. In particular, if \( X \) is a \( \text{cl-paracompact} \) space and \( \varphi \) is onto, then \( Y \) is paracompact.

**Proof.** Proof of Theorem 3.2.15 is similar (even simpler) to that of Theorem 3.2.14 and hence
3.3 Properties of lower cl-supercontinuous multifunctions

Theorem 3.3.1. For a multifunction \( \varphi : X \rightarrow Y \) from a topological space \( X \) into a topological space \( Y \) the following statements are equivalent.

(a) \( \varphi \) is lower cl-supercontinuous.

(b) \( \varphi_+^{-1}(B) \) is a cl-open set in \( X \) for every open set \( B \) in \( Y \).

(c) \( \varphi_-^{-1}(B) \) is a cl-closed set in \( X \) for every closed set \( B \) in \( Y \).

(d) For each \( x \in X \) and for each open set \( V \) with \( \varphi(x) \cap V \neq \emptyset \) there exists a cl-open set \( U \) containing \( x \) such that \( \varphi(z) \cap V \neq \emptyset \) for each \( z \in U \).

Proof. (a)\(\Rightarrow\)(b). Let \( B \) be an open subset of \( Y \). To show that \( \varphi_+^{-1}(B) \) is cl-open in \( X \), let \( x \in \varphi_+^{-1}(B) \). Then \( \varphi(x) \cap B \neq \emptyset \). Since \( \varphi \) is lower cl-supercontinuous, there exists a clopen set \( H \) containing \( x \) such that \( \varphi(h) \cap B \neq \emptyset \) for each \( h \in H \). Hence \( x \in H \subset \varphi_+^{-1}(B) \) and so \( \varphi_+^{-1}(B) \) is a cl-open set in \( X \) being a union of clopen sets.

(b)\(\Rightarrow\)(c). Let \( B \) be a closed subset of \( Y \). Then \( Y \setminus B \) is an open subset of \( Y \). By (b), \( \varphi_+^{-1}(Y \setminus B) \) is a cl-open set in \( X \). Since \( \varphi_-^{-1}(Y \setminus B) = X \setminus \varphi_+^{-1}(B) \), \( \varphi_-^{-1}(B) \) is a cl-closed set in \( X \).

(c)\(\Rightarrow\)(d). Let \( x \in X \) and let \( V \) be an open set in \( Y \) with \( \varphi(x) \cap V \neq \emptyset \). Then \( Y \setminus V \) is a closed set in \( Y \) with \( \varphi(x) \nsubseteq (Y \setminus V) \). Therefore, By (c), \( \varphi_-^{-1}(Y \setminus V) = X \setminus \varphi_+^{-1}(V) \) is a cl-closed set in \( X \) not containing \( x \) and so \( \varphi_+^{-1}(V) \) is a cl-open set in \( X \) containing \( x \). Let \( U = \varphi_+^{-1}(V) \). Then \( U \) is a cl-open set containing \( x \) such that \( \varphi(z) \cap V \neq \emptyset \) for each \( z \in U \).

The assertion (d)\(\Rightarrow\)(a) is trivial, since every cl-open set is the union of clopen sets. \(\square\)
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Theorem 3.3.2. A multifunction \( \varphi : X \rightarrow Y \) is lower cl-supercontinuous if and only if \( \varphi([A]_{cl}) \subset \overline{\varphi(A)} \) for every subset \( A \) of \( X \).

Proof. Suppose that \( \varphi : X \rightarrow Y \) is lower cl-supercontinuous. Let \( A \) be a subset of \( X \). Then \( \overline{\varphi(A)} \) is a closed subset of \( Y \). By Theorem 3.3.1, \( \varphi^{-1}(\overline{\varphi(A)}) \) is a cl-closed set in \( X \). Since \( A \subset \varphi^{-1}(\overline{\varphi(A)}) \) and since \( [A]_{cl} \subset [\varphi^{-1}(\overline{\varphi(A)})]_{cl} = \varphi^{-1}(\overline{\varphi(A)}) \), \( \varphi([A]_{cl}) \subset \varphi(\varphi^{-1}(\overline{\varphi(A)})) \subset \overline{\varphi(A)} \).

Conversely, suppose that \( \varphi([A]_{cl}) \subset \overline{\varphi(A)} \) for every subset \( A \) of \( X \) and let \( F \) be a closed set in \( Y \). Then \( \varphi^{-1}(F) \) is subset of \( X \). By hypothesis, \( \varphi([\varphi^{-1}(F)]_{cl}) \subset \varphi(\varphi^{-1}(F)) \subset F = F \) and \( \varphi^{-1}(\varphi([\varphi^{-1}(F)]_{cl})) \subset \varphi^{-1}(F) \), which in turn implies that \( [\varphi^{-1}(F)]_{cl} \subset \varphi^{-1}(F) \). Hence \( \varphi^{-1}(F) = [\varphi^{-1}(F)]_{cl} \) and so in view of Theorem 3.3.1, \( \varphi : X \rightarrow Y \) is lower cl-supercontinuous.

\[ \square \]

Theorem 3.3.3. A multifunction \( \varphi : X \rightarrow Y \) is lower cl-supercontinuous if and only if \( [\varphi^{-1}(B)]_{cl} \subset \varphi^{-1}(B) \) for every subset \( B \) of \( Y \).

Proof. Suppose that \( \varphi : X \rightarrow Y \) is lower cl-supercontinuous. Let \( B \subset Y \). Then \( \overline{B} \) is a closed subset of \( Y \). By Theorem 3.3.1, \( \varphi^{-1}(\overline{B}) \) is a cl-closed subset of \( X \). Since, \( \varphi^{-1}(B) \subset \varphi^{-1}(\overline{B}) \), \( [\varphi^{-1}(B)]_{cl} \subset [\varphi^{-1}(\overline{B})]_{cl} = \varphi^{-1}(\overline{B}) \). That is \( [\varphi^{-1}(B)]_{cl} \subset \varphi^{-1}(\overline{B}) \). Conversely, suppose that \( [\varphi^{-1}(B)]_{cl} \subset \varphi^{-1}(\overline{B}) \) for every \( B \subset Y \). Let \( F \) be any closed subset of \( Y \). By hypothesis \( [\varphi^{-1}(F)]_{cl} \subset \varphi^{-1}(\overline{F}) = \varphi^{-1}(F) \). Hence \( [\varphi^{-1}(F)]_{cl} = \varphi^{-1}(F) \) and so in view of Theorem 3.3.1, \( \varphi \) is lower cl-supercontinuous.

\[ \square \]

The following theorem shows that lower cl-supercontinuity of a multifunction remains invariant under the shrinking of its range.

Theorem 3.3.4. If \( \varphi : X \rightarrow Y \) is lower cl-supercontinuous and \( \varphi(X) \) is endowed with subspace topology, then \( \varphi : X \rightarrow \varphi(X) \) is lower cl-supercontinuous.
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Proof. Since $\varphi$ is lower cl-supercontinuous, for every open set $V$ of $Y$, $\varphi^{-1}_+(V \cap \varphi(X)) = \varphi^{-1}_+(V) \cap \varphi^{-1}_+((X)) = \varphi^{-1}_+(V) \cap X = \varphi^{-1}_+(V)$ is cl-open and hence $\varphi : X \to \varphi(X)$ is lower cl-supercontinuous.

\[\text{Theorem 3.3.5.} \quad \text{If } \varphi : X \to Y \text{ is lower cl-supercontinuous and } \psi : Y \to Z \text{ is lower semi-continuous, then } \psi \circ \varphi \text{ is lower cl-supercontinuous. In particular, composition of two lower cl-supercontinuous multifunctions is lower cl-supercontinuous.}\]

Proof. Let $W$ be an open set in $Z$. Since $\psi$ is lower semi continuous, $\psi^{-1}_+(W)$ is an open set in $Y$. Again since $\varphi$ is lower cl-supercontinuous, $\varphi^{-1}_+(\psi^{-1}_+(W))$ is cl-open in $X$, and so $(\psi \circ \varphi)^{-1}_+(W) = \varphi^{-1}_+(\psi^{-1}_+(W))$ is a cl-open set in $X$. Thus $\psi \circ \varphi : X \to Z$ is lower cl-supercontinuous.

Analogous to Theorem 3.3.4 the following corollary shows that lower cl-supercontinuity of a multifunction is preserved under the expansion of its range.

\[\text{Corollary 3.3.6.} \quad \text{Let } \varphi : X \to Y \text{ be lower cl-supercontinuous. If } Z \text{ is a space containing } Y \text{ as a subspace, then } \psi : X \to Z \text{ defined by } \psi(x) = \varphi(x) \text{ for } x \in X \text{ is lower cl-supercontinuous.}\]

Proof. Let $W$ be an open set in $Z$. Then $W \cap Y$ is an open set in $Y$. Since $\varphi : X \to Y$ is lower cl-supercontinuous, $\varphi^{-1}_+(W \cap Y)$ is cl-open set in $X$. Now, $\psi^{-1}_+(W) = \{x \in X : \psi(x) \cap W \neq \emptyset\} = \{x \in X : \varphi(x) \cap (W \cap Y) \neq \emptyset\} = \varphi^{-1}_+(W \cap Y)$. Thus $\psi : X \to Z$ is lower cl-supercontinuous.

\[\text{Theorem 3.3.7.} \quad \text{If } \varphi : X \to Y \text{ and } \psi : X \to Y \text{ are lower cl-supercontinuous multifunctions, then the multifunction } \varphi \cup \psi : X \to Y \text{ defined by } (\varphi \cup \psi)(x) = \varphi(x) \cup \psi(x) \text{ for each } x \in X, \text{ is lower cl-supercontinuous.}\]

Proof. Let $U$ be an open set in $Y$. Then $\varphi^{-1}_+(U)$ and $\psi^{-1}_+(U)$ are cl-open sets in $X$. Since $(\varphi \cup \psi^{-1}_+(U))$. \]
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ψ)^−1(U) = φ)^−1(U) ∪ ψ)^−1(U) and since any union of cl-open sets is cl-open, (φ ∪ ψ)^−1(U) is cl-open in X. Thus φ ∪ ψ is lower cl-supercontinuous.

Theorem 3.3.8. Let φ : X → Y be any multifunction. Then the following statements are true:

(a) If φ : X → Y is lower cl-supercontinuous and A ⊂ X, then the restriction φ|A : A → Y is lower cl-supercontinuous.

(b) If \{U_α : α ∈ Δ\} is a cl-open cover of X and for each α, the restriction φ_α = φ|U_α : U_α → Y is lower cl-supercontinuous, then φ : X → Y is lower cl-supercontinuous.

Proof. (a) Let W be an open set in Y. Since φ : X → Y is lower cl-supercontinuous, φ)^−1(W) is a cl-open set in X. Now, (φ|A)^−1(W) = \{x ∈ A | φ(x) ∩ W ≠ ∅\} = \{x ∈ A | x ∈ φ)^−1(W)\} = A ∩ φ)^−1(W), which is cl-open in X and so φ|A is lower cl-supercontinuous.

(b) Let W be an open set in Y. Since φ_α = φ|U_α : U_α → Y is lower cl-supercontinuous, (φ_α)^−1(W) is a cl-open set in U_α and consequently cl-open in X. Since φ)^−1(W) = ∪_α∈Δ(φ_α)^−1(W) and since any union of cl-open sets is cl-open, φ)^−1(W) is a cl-open set in X, and hence φ : X → Y is lower cl-supercontinuous.

Theorem 3.3.9. Let \{φ_α : X → X_α | α ∈ Λ\} be a family of multifunctions and let φ : X → Π_α∈ΛX_α be defined by φ(x) = Π_α∈Λφ_α(x). Then φ is lower cl-supercontinuous if and only if each φ_α : X → X_α is lower cl-supercontinuous.

Proof. Let φ : X → Π_α∈ΛX_α be lower cl-supercontinuous. Let p_β : Π_α∈ΛX_α → X_β be the projection map onto X_β. Then p_β being a single valued continuous function is lower semicontinuous. By Theorem 3.3.5 φ_β = p_β ∘ φ is lower cl-supercontinuous for each β ∈ Λ.

Conversely, suppose that φ_β = p_β ∘ φ : X → X_β is a lower cl-supercontinuous for each β ∈ Λ. Since the finite intersections and arbitrary union of cl-open sets is cl-open, therefore, in view of
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Theorem 3.3.10. For each \( \alpha \in \Lambda \) let \( \varphi_\alpha : X_\alpha \to Y_\alpha \) be a multifunction and let \( \varphi : \prod_{\alpha \in \Lambda} X_\alpha \to \prod_{\alpha \in \Lambda} Y_\alpha \) be a multifunction defined by \( \varphi(x) = \prod_{\alpha \in \Lambda} \varphi_\alpha(x_\alpha) \) for each \( x = (x_\alpha) \in \prod_{\alpha \in \Lambda} X_\alpha \). Then \( \varphi \) is lower cl-supercontinuous if and only if each \( \varphi_\alpha \) is lower cl-supercontinuous.

Proof. Suppose that \( \varphi : \prod_{\alpha \in \Lambda} X_\alpha \to \prod_{\alpha \in \Lambda} Y_\alpha \) is lower cl-supercontinuous. Let \( U_\beta \) be an open set in \( Y_\beta \). Then \( U_\beta \times \prod_{\alpha \in \Lambda, \alpha \neq \beta} Y_\alpha \) is a subbasic open set in \( \prod_{\alpha \in \Lambda} Y_\alpha \). So in view of Theorem 3.3.1, \( \varphi_\alpha ^{-1}(U_\beta \times \prod_{\alpha \neq \beta} Y_\alpha) \) is a cl-open set in \( \prod_{\alpha \in \Lambda} X_\alpha \). Now it is easily verified that \( \varphi_\alpha ^{-1}(U_\beta \times \prod_{\alpha \neq \beta} Y_\alpha) = (\varphi_\alpha)_+ ^{-1}(U_\beta) \times \prod_{\alpha \neq \beta} X_\alpha \), and so \( (\varphi_\alpha)_+ ^{-1}(U_\beta) \) is cl-open in \( X_\beta \). This proves that each \( \varphi_\alpha \) is lower cl-supercontinuous.

Conversely suppose that \( \varphi_\alpha : X_\alpha \to Y_\alpha \) is lower cl-supercontinuous for each \( \alpha \in \Lambda \) and let \( B = V_{\alpha_1} \times V_{\alpha_2} \times \cdots \times V_{\alpha_N} \times (\prod_{\alpha \neq \alpha_1, \alpha_2, \ldots, \alpha_N} Y_\alpha) \) be a basic open set in \( \prod_{\alpha \in \Lambda} Y_\alpha \). Then \( \varphi_\alpha ^{-1}(V_{\alpha_1} \times \cdots \times V_{\alpha_N} \times (\prod_{\alpha \neq \alpha_1, \alpha_2, \ldots, \alpha_N} Y_\alpha)) = (\varphi_\alpha_1)_+ ^{-1}(V_{\alpha_1}) \times \cdots \times (\varphi_\alpha_N)_+ ^{-1}(V_{\alpha_N}) \times (\prod_{\alpha \neq \alpha_1, \alpha_2, \ldots, \alpha_N} X_\alpha) \).

Since each \( \varphi_\alpha \) is lower cl-supercontinuous, \( \varphi_\alpha ^{-1}(B) \) cl-open in \( \prod_{\alpha \in \Lambda} X_\alpha \) and so \( \varphi \) is lower cl-supercontinuous. \( \square \)

Theorem 3.3.11. Let \( \varphi : X \to Y \) be multifunction and let \( g : X \to X \times Y \) defined by \( g(x) = \{(x, y) \in X \times Y \mid y \in \varphi(x)\} \) for each \( x \in X \) be the graph multifunction. Then \( g \) is lower cl-supercontinuous if and only if \( \varphi \) is lower cl-supercontinuous and the space \( X \) is zero dimensional.

Proof. Suppose that \( g \) is lower cl-supercontinuous. By Theorem 3.3.5 the multifunction \( \varphi = \)
\( p_y \circ g \) is lower cl-supercontinuous. Next we shall show that \( X \) is zero dimensional. Let \( U \) be an open set in \( X \) and let \( x \in U \). Then \( U \times Y \) is an open set in \( X \times Y \) and \( g(x) \cap (U \times Y) \neq \emptyset \). Since \( g \) is lower cl-supercontinuous, there exists a clopen set \( W \) containing \( x \) such that \( g(z) \cap (U \times Y) \neq \emptyset \) for every \( z \in W \) and so \( W \subset g^{-1}(U \times Y) = U \). Hence \( x \in W \subset U \) and \( X \) is zero dimensional.

Conversely, suppose that \( \phi \) is lower cl-supercontinuous. Let \( x \in X \) and let \( W \) be an open set with \( g(x) \cap W \neq \emptyset \). Then there exist open sets \( U \) in \( X \) and \( V \) in \( Y \) such that \( g(x) \cap (U \times V) \neq \emptyset \) and so \( x \in U \) and \( \phi(x) \cap V \neq \emptyset \). Since \( X \) is zero dimensional, there exists a clopen set \( G_1 \) containing \( x \) such that \( x \in G_1 \subset U \). Again since \( \phi \) is lower cl-supercontinuous, there exists a clopen set \( G_2 \) containing \( x \) such that \( \phi(h) \cap V \neq \emptyset \) for each \( h \in G_2 \). Let \( G = G_1 \cap G_2 \). Then \( G \) is a clopen set containing \( x \) and it is easily verified that \( g(h) \cap W \neq \emptyset \) for each \( h \in G \). This proves that \( g \) is lower cl-supercontinuous. \( \Box \)