Chapter 1

Introduction

1.1 Motivation and Introduction

Functions arise everywhere in mathematics and applications of mathematics, and so do the multifunctions. Multifunctions are also called set-valued maps or point to set maps by some authors (see [10]). Multifunctions provide a larger/broader framework which encompasses functions or so called single valued functions. This introductory chapter is divided into three sections. The first section of the chapter is devoted to motivation and to reflect upon the applications of multifunctions to mathematics and other disciplines, while Section 2 is devoted to notations, preliminaries and basic definitions which are used throughout the thesis. Section 3 of this chapter presents a summary of the whole thesis.

Multifunctions arise naturally in many areas of mathematics and applications of mathematics and have wide ranging applications in optimization theory, control theory, game theory, mathematical economics, dynamical systems and differential inclusions. Here we present some examples from([10] [16]) which give sufficient motivation for consideration of multifunc-
tions. Throughout the thesis we essentially follow the notations of Górniewicz [16] regarding multifunctions. Unless otherwise stated, $X$, $Y$, $Z$ will denote topological spaces.

1. (Inverse functions). Let $f : X \to Y$ be a continuous function from $X$ onto $Y$. Then its inverse is a multifunction. $\varphi_f : Y \rightharpoonup X$ defined by $\varphi_f(y) = f^{-1}(y)$, for each $y \in Y$.

2. (Implicit functions). Let $f : X \times Y \to Z$ and $g : X \to Z$ be two continuous functions such that for every $x \in X$, there exists $y \in Y$ such that $f(x,y) = g(x)$. The implicit function(defined by $f$ and $g$) is a multifunction $\varphi : X \rightharpoonup Y$ defined as follows $\varphi(x) = \{y \in Y \mid f(x,y) = g(x)\}$.

3. Let $f : X \times Y \to R$ be a continuous function. Assume that there is $r > 0$ such that for every $x \in X$, there exists $y \in Y$ such that $f(x,y) \leq r$. Then $\varphi_r : X \rightharpoonup Y$, is a multifunction defined by $\varphi_r(x) = \{y \in Y \mid f(x,y) \leq r\}$.

4. (Control theory) To solve
\[
\begin{cases}
  x'(t) = f(t,x(t),u(t)) \\
  x(0) = x_0
\end{cases}
\]
controlled by parameters $u(t)$, where $f : [0,a] \times R^n \times R^m \to R^n$. Define a multifunction $F : [0,a] \times R^n \rightharpoonup R^n$ by $F(t,x) = \{f(t,x,u) : u \in U\}$. Then the solution of the control problem is given by the solution of following differential problem
\[
\begin{cases}
  x'(t) \in F(t,x) \\
  x(0) = x_0
\end{cases}
\]
Thus a control problem is translated to a differential inclusion problem using multifunction.
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5. (Metric projection). Let $A$ be a compact subset of a metric space $(X, d)$. Then for every $x \in X$ there exists $a \in A$ such that $d(a, x) = \text{dist}(x, A)$. We define the metric projection $P : X \to A$ by putting $P(x) = \{ a \in A \mid d(a, x) = \text{dist}(x, A) \}$.

6. (Minimization problem) Let $f : X \times Y \to R$ and $g : X \to R$ be functions such that

$$g(x) = \min_{y \in Y} f(x, y).$$

(1.1)

The solution set $F(x) = \{ y \in Y : f(x, y) = g(x) \}$ of the minimization problem defines a multifunction $F : X \to Y$.

7. Let $f : X \to R$ be a real-valued function. Define multifunction $\phi : X \to R$ by $\phi(x) = f(x) + R_+$ for each $x \in X$. The graph $\Gamma_\phi$ of the multifunction $\phi$ is called the epigraph of the function $f$.

8. Multifunctions also arise naturally in complex analysis and are called multivalued functions therein. For example the multivalued functions $\phi(z) = z^{1/n}, n \in \mathbb{N}$ and $\psi(z) = \log z$, are multifunctions.

The continuity of functions plays a prominent role in topology, analysis and other branches of mathematics. There are many situations/applicaions in geometry, topology, functional analysis, complex analysis where continuity is not sufficient and a condition stronger than continuity is required. For example, the notions of homeomorphisms, isometries, perfect functions (≡ proper mappings), uniformly continuous mappings all exemplify strong forms of continuity and are the building blocks of many coherent theories in mathematics. Compact linear operators provide another example of a strong form of continuity and play central role in theory of integral equations and various problems of functional analysis and partial differential equa-
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tions. Several strong variants of continuity of functions occur in the lore of mathematical literature and applications of mathematics. For example, see ([35] [36] [37] [43] [44] [53] [57] [60] [63] [65] [71]). The class of strongly continuous functions was introduced by N. Levine [53] in 1960, Noiri introduced the notions of strongly \( \theta \)-continuous functions [63] and perfectly continuous functions[63], cl-supercontinuous functions were defined by Reilly and Vamanamurthy [65] and studied in detail by Singh [71]. The class of \( z \)-supercontinuous functions is due to Kohli and Kumar [35]. Similarly, the classes of D-supercontinuous functions and \( D_\delta \)-supercontinuous functions have been initiated by Kohli and Singh([36] [37]).

On the other hand in many applications of mathematics full force of continuity is not required and a condition weaker than continuity is sufficient to meet the demand of a particular situation. For example connected functions, compactness preserving functions, semi-connected functions, connectivity functions are all examples of non-continuous functions and have been a source of considerable mathematical research. In 1875, Darboux has proved that derivative of differentiable function need not be a continuous function but possesses the intermediate value property. Moreover, Bair functions arise as pointwise limits of sequences of continuous functions and need not be continuous. Sequential continuity (k-continuity) is another example of weak form of continuity which coincides with continuity if domain is a sequential space (k-space). Unbounded linear operators (non-continuous functions) occur in many applications notably in connection with differential equations and in quantum mechanics. Lower (upper) semicontinuous functions are important in analysis, calculus of variation, optimization theory and economics. Several weak forms of continuity of functions have been introduced and studied by host of authors. For example, see ([13] [23] [38] [52] [54] [58] [67] [68] [69]). The class of almost continuous functions was introduced by Singal and Singal [68], Fomin introduced the notions of \( \theta \)-continuous functions [13], and D-continuous functions were defined by
Kohli [23]. The class of faintly continuous functions was introduced by Long and Herrington [58] and z-continuous functions were defined by Singal and Niemse [67].

There are situations/applications in mathematics wherein a condition analogous to but independent of continuity is useful. For example, functions of bounded variation and monotone functions are independent of continuity and are important in analysis, statistics, economics and many other disciplines. Mathematics literature is replete with examples of such variants of continuity. See for example([41] [42] [46] [47] [56] [62]). The class of almost cl-supercontinuous functions was defined by Kohli and Singh [41]. Noiri introduced $\delta$-continuous functions [62], and quasi cl-supercontinuous functions have been defined by Kohli and Aggarwal [25]. The class of quasi z-supercontinuous functions was introduced and studied by Kohli, Singh and Kumar [46].

Recently there has been considerable interest in trying to extend the notions and results of weak, strong and other variants of continuity of functions to the realm of multifunctions. For example, see ([2] [3] [4] [5] [6] [19] [20] [21] [50] [51] [73] [80]). The continuity of multifunctions was studied by G.T. Whyburn [79] in 1965. Several strong forms of continuity in the framework of multifunctions have been studied by host of authors. For example, see ([2] [3] [4] [5] [6] [50]). M. Akdağ has studied supercontinuous multifunctions [3], z-supercontinuous multifunctions [4] and $D_\delta$-supercontinuous multifunctions [5].

Similarly several weak variants of continuity have been extended to the framework of multifunctions. For example, see ([19] [20] [21] [51]). The notions of super and faintly continuous multifunctions have been introduced by I. Zorlutuna and Y. Kucuk [80]. Almost and weak continuity for multifunctions were defined by R.E. Smithson [73] and further studied by L. Hola [20].

The main aim of the thesis is to extend and study several strong and other variants of conti-
nuity of functions to the framework of multifunctions. The notions of strong continuity, complete continuity, perfect continuity, cl-supercontinuity, almost cl-supercontinuity and quasi cl-supercontinuity are extended to the framework of multifunctions. In the process of their study we obtain several characterizations of strong continuity of multifunctions, upper and lower perfect continuity of multifunctions, upper and lower (almost) cl-supercontinuity of multifunctions, upper and lower quasi cl-supercontinuity of multifunctions. Moreover, we study basic properties of all these classes of multifunctions and elaborate on their place in the hierarchy of strong forms of continuity of multifunctions and discuss their interrelations and interconnections with other variants which already exist in the mathematical literature.

1.2 Notations, Preliminaries and Basic definitions

Throughout the thesis we essentially follow the notations and terminology of L. Górniewicz [16]. Let $X$ and $Y$ be nonempty sets. Then $\varphi : X \to Y$ is called a multifunction from $X$ into $Y$ if for each $x \in X$, $\varphi(x)$ is a nonempty subset of $Y$. In some applications in optimization theory $\varphi(x)$ may be empty (see [10]). Let $B$ be a subset of $Y$. Then following Górniewicz [16] the set $\varphi^{-1}(B) = \{ x \in X : \varphi(x) \cap B \neq \emptyset \}$ is called large inverse image of $B$ and the set $\varphi^{-1}(B) = \{ x \in X : \varphi(x) \subseteq B \}$ is called small inverse image of $B$. The set $\Gamma_{\varphi} = \{ (x,y) \in X \times Y \mid y \in \varphi(x) \}$ is called the graph of the multifunction. Let $A$ be subset of $X$. Then $\varphi(A) = \cup\{ \varphi(x) : x \in A \}$ is called image of $A$. If $\varphi : X \to Y$ and $\psi : Y \to Z$ are two multifunctions then the composition $\psi \circ \varphi : X \to Z$ of $\varphi$ and $\psi$ is defined by $(\psi \circ \varphi)(x) = \cup \{ \psi(y) \mid y \in \varphi(x) \}$; for each $x \in X$.

\footnote{However, following Górniewicz what we call “large inverse image $\varphi^{-1}(B)$” some authors call it ‘lower inverse image’ and denote it by $\varphi^{-}(B)$; and similarly they call “small inverse image $\varphi^{-1}(B)$” as ‘upper inverse image’ and employ the notation $\varphi^{+}(B)$ for the same.}
1.2.1 Elementary properties of multifunctions

Let \( \varphi : X \to Y \) be a multifunction and let \( K_1, K_2 \) and \( K \) be subsets of \( X \). Then

(i) \( \varphi(K_1 \cup K_2) = \varphi(K_1) \cup \varphi(K_2) \)

(ii) \( \varphi(K_1 \cap K_2) \subset \varphi(K_1) \cap \varphi(K_2) \)

(iii) \( \varphi(X \setminus K) \supset \varphi(X) \setminus \varphi(K) \)

(iv) \( K_1 \subset K_2 \Rightarrow \varphi(K_1) \subset \varphi(K_2) \).

Below we summarize properties of image and inverse image of multifunctions. Let \( \varphi : X \to Y \) be a multifunction and let \( A \subset X, B \subset Y \) and \( B_j \subset Y, j \in J \). Then we have

(1) \( \varphi^{-1}(\varphi(A)) \supset A \)

(2) \( \varphi(\varphi^{-1}(B)) \subset B \)

(3) \( X \setminus \varphi^{-1}(B) = \varphi^{-1}(Y \setminus B) \)

(4) \( \varphi^{-1}(\bigcup_{j \in J} B_j) \supset \bigcup_{j \in J} \varphi^{-1}(B_j) \)

(5) \( \varphi^{-1}(\bigcap_{j \in J} B_j) = \bigcap_{j \in J} \varphi^{-1}(B_j) \)

(6) \( \varphi^{-1}(\varphi(A)) \supset A \)

(7) \( \varphi(\varphi^{-1}(B)) \supset B \cap \varphi(X) \)

(8) \( X \setminus \varphi^{-1}(B) = \varphi^{-1}(Y \setminus B) \)

(9) \( \varphi^{-1}(\bigcup_{j \in J} B_j) = \bigcup_{j \in J} \varphi^{-1}(B_j) \)

(10) \( \varphi^{-1}(\bigcap_{j \in J} B_j) \subset \bigcap_{j \in J} \varphi^{-1}(B_j) \)
For given two multifunctions $\varphi : X \rightarrow Y$ and $\psi : X \rightarrow Y$, the new multifunctions $\varphi \cup \psi : X \rightarrow Y$ and $\varphi \cap \psi : X \rightarrow Y$ are defined by $(\varphi \cup \psi)(x) = \varphi(x) \cup \psi(x)$ and $(\varphi \cap \psi)(x) = \varphi(x) \cap \psi(x)$ respectively for each $x \in X$. Note that the multifunction $\varphi \cap \psi$ is defined only if $\varphi(x) \cap \psi(x) \neq \emptyset$ for each $x \in X$. Let $B \subset Y$;

(i) $(\varphi \cup \psi)^{-1}(B) = \varphi^{-1}(B) \cap \psi^{-1}(B)$

(ii) $(\varphi \cap \psi)^{-1}(B) \supset \varphi^{-1}(B) \cap \psi^{-1}(B)$

(iii) $(\varphi \cup \psi)^{-1}(B) = \varphi^{-1}(B) \cup \psi^{-1}(B)$

(iv) $(\varphi \cap \psi)^{-1}(B) \subset \varphi^{-1}(B) \cap \psi^{-1}(B)$.

Again, if $\varphi : X \rightarrow Y$ and $\psi : Y \rightarrow Z$ are multifunctions, then for $B \subset Z$,

(1) $(\psi \circ \varphi)^{-1}(B) = \varphi^{-1}(\psi^{-1}(B))$

(2) $(\psi \circ \varphi)^{-1}(B) = \varphi^{-1}(\psi^{-1}(B))$.

A multifunction $\varphi : X \rightarrow Y$ from a topological space $X$ into a topological space $Y$ is said to be upper semicontinuous (respectively lower semicontinuous) if $\varphi^{-1}(U)$ (respectively $\varphi^{-1}(U)$) is an open set in $X$ for every open set $U$ in $Y$. A subset $U$ of a topological space $X$ is called a cl-open set if it can be expressed as a union of clopen sets. The complement of a cl-open set will be referred to as a cl-closed set. A subset $A$ of a space $X$ is called regular open if it is the interior of its closure, i.e., $A = \overline{A}^c$. A collection $\beta$ of subsets of a space $X$ is called an open complementary system [17] if $\beta$ consists of open sets such that for each $B \in \beta$, there exist $B_1, B_2, \ldots$, in $\beta$ with $B = \bigcup\{X \setminus B_i : i \in Z^+\}$. A subset $U$ of a space $X$ is called strongly open $F_\sigma$-set[17] if there exists a countable open complementary system $\beta(U)$ with $U \in \beta(U)$. A subset $H$ of a space $X$ is called a regular $G_\delta$-set [59] if $H$ is the intersection of a sequence of
closed sets whose interiors contain $H$, i.e., if $H = \bigcap_{i=1}^{\infty} F_i = \bigcap_{i=1}^{\infty} F_i^c$, where each $F_i$ is a closed subset of $X$. The complement of a regular $G_\delta$-set is called a regular $F_\sigma$-set[59]. Let $X$ be a topological space and let $A \subset X$. A point $x \in X$ is called a $\theta$-adherent point[78] of $A$ if every closed neighbourhood of $x$ intersects $A$. Let $cl_\theta A$ denote the set of all $\theta$-adherent points of $A$. The set $A$ is called $\theta$-closed[78] if $A = cl_\theta A$. The complement of a $\theta$-closed set is referred to as a $\theta$-open set[78]. A point $x \in X$ is said to be a $cl$-adherent point of $A$ if every clopen set containing $x$ intersects $A$. Let $[A]_{cl}$ denote the set of all $cl$-adherent points of $A$. Then a set $A$ is $cl$-closed[71] if and only if $A = [A]_{cl}$. It is easily verified that a subset $A$ of a space $X$ is $cl$-closed if it is the intersection of clopen sets. Next we give the definition of $\delta$-embedded which is analogous to that of the notion of $z$-embedding, utilized and elaborated in ([8], p.77)

**Definition 1.2.2.** A subspace $S$ of a space $X$ is said to be $\delta$-embedded[41] in $X$ if every regular open set in $S$ is the intersection of a regular open set in $X$ with $S$; or equivalently every regular closed set in $S$ is the intersection of a regular closed set in $X$ with $S$.

**Definitions 1.2.3.** A space $X$ is said to be

(i) mildly compact[75] if every clopen cover of $X$ has a finite subcover. In [74] Sostak calls mildly compact spaces as clustered spaces;

(ii) $cl$-paracompact[4](cl-para-Lindelöf[4]) if every clopen cover of $X$ has a locally finite (locally countable) open refinement which covers $X$;

(iii) a $P$-space[15] if every $G_\delta$ set in $X$ is open in $X$; and

(iv) ultra Hausdorff[76] if for every pair of distinct points $x$ and $y$ of $X$, there exist two disjoint clopen sets $U$ and $V$ such that $x \in U$ and $y \in V$. 
Several authors have studied strong and weak variants of continuity of multifunctions. Here we mention the definitions of only those variants of continuity from the literature which are related to our work.

**Definitions 1.2.4.** A multifunction $\varphi : X \rightarrow Y$ from a topological space $X$ into a topological space $Y$ is said to be

1. **upper completely continuous**$^{[29]}$ if $\varphi^{-1}(U)$ is a regular open set in $X$ for every open set $U$ in $Y$;

2. **lower completely continuous**$^{[29]}$ if $\varphi_+(U)$ is a regular open set in $X$ for every open set $U$ in $Y$;

3. **upper almost completely continuous**$^{[29]}$ if $\varphi_{-1}(U)$ is a regular open set in $X$ for every regular open set $U$ in $Y$;

4. **lower almost completely continuous**$^{[29]}$ if $\varphi_+(U)$ is a regular open set in $X$ for every regular open set $U$ in $Y$;

5. **upper $z$-supercontinuous**$^{[4]}$ if for each $x \in X$ and each open set $V$ containing $\varphi(x)$, there exists a cozero set $U$ containing $x$ such that $\varphi(U) \subset V$;

6. **lower $z$-supercontinuous**$^{[4]}$ if for each $x \in X$ and each open set $V$ with $\varphi(x) \cap V \neq \emptyset$, there exists a cozero set $U$ containing $x$ such that $\varphi(z) \cap V \neq \emptyset$ for each $z \in U$;

7. **upper $D_\delta$-supercontinuous**$^{[5]}$ if for each $x \in X$ and each open set $V$ containing $\varphi(x)$, there exists a regular $F_\sigma$-set $U$ containing $x$ such that $\varphi(U) \subset V$;
(8) **lower \( D_\delta \)-supercontinuous**\(^5\) if for each \( x \in X \) and each open set \( V \) with \( \varphi(x) \cap V \neq \emptyset \), there exists a regular \( F_\sigma \)-set \( U \) containing \( x \) such that \( \varphi(z) \cap V \neq \emptyset \) for each \( z \in U \); 

(9) **upper \( D \)-supercontinuous**\(^2\) if for each \( x \in X \) and each open set \( V \) containing \( \varphi(x) \), there exists an open \( F_\sigma \)-set \( U \) containing \( x \) such that \( \varphi(U) \subset V \); 

(10) **lower \( D \)-supercontinuous**\(^2\) if for each \( x \in X \) and each open set \( V \) with \( \varphi(x) \cap V \neq \emptyset \), there exists an open \( F_\sigma \)-set \( U \) containing \( x \) such that \( \varphi(z) \cap V \neq \emptyset \) for each \( z \in U \); 

(11) **upper \( D^* \)-supercontinuous**\(^7\) if for each \( x \in X \) and each open set \( V \) containing \( \varphi(x) \), there exists a strongly open \( F_\sigma \)-set \( U \) containing \( x \) such that \( \varphi(U) \subset V \); 

(12) **lower \( D^* \)-supercontinuous**\(^7\) if for each \( x \in X \) and each open set \( V \) with \( \varphi(x) \cap V \neq \emptyset \), there exists a strongly open \( F_\sigma \)-set \( U \) containing \( x \) such that \( \varphi(z) \cap V \neq \emptyset \) for each \( z \in U \); 

(13) **upper strongly \( \theta \)-continuous**\(^{50}\) if for each \( x \in X \) and each open set \( V \) containing \( \varphi(x) \), there exists a \( \theta \)-open set \( U \) containing \( x \) such that \( \varphi(U) \subset V \); 

(14) **lower strongly \( \theta \)-continuous**\(^{50}\) if for each \( x \in X \) and each open set \( V \) with \( \varphi(x) \cap V \neq \emptyset \), there exists a \( \theta \)-open set \( U \) containing \( x \) such that \( \varphi(z) \cap V \neq \emptyset \) for each \( z \in U \); 

(15) **upper quasi \( z \)-supercontinuous**\(^{[30] [47]}\) if for each \( x \in X \) and each \( \theta \)-open set \( V \) containing \( \varphi(x) \), there exists a cozero set \( U \) containing \( x \) such that \( \varphi(U) \subset V \); 

(16) **lower quasi \( z \)-supercontinuous**\(^{[30] [47]}\) if for each \( x \in X \) and each \( \theta \)-open set \( V \) with \( \varphi(x) \cap V \neq \emptyset \), there exists a cozero set \( U \) containing \( x \) such that \( \varphi(z) \cap V \neq \emptyset \) for each \( z \in U \); and 

(17) **upper (lower) quasi \( \theta \)-continuous (faintly continuous)** if for every \( \theta \)-open set \( V \subset Y \), 

\( \varphi^{-1}(V)(\varphi^{-1}(V)) \) is \( \theta \)-open (open) in \( X \);
(18) **upper supercontinuous**[^3](δ-continuous[^51]) if for each \( x \in X \) and each open (regular open) set \( V \) containing \( \varphi(x) \), there exists a regular open set \( U \) containing \( x \) such that \( \varphi(U) \subset V \);

(19) **lower supercontinuous**[^3](δ-continuous[^51]) if for each \( x \in X \) and each open (regular open) set \( V \) with \( \varphi(x) \cap V \neq \emptyset \), there exists a regular open set \( U \) containing \( x \) such that \( \varphi(z) \cap V \neq \emptyset \) for each \( z \in U \);

(20) **upper almost continuous**[^73] if for each \( x \in X \) and each regular open set \( V \) containing \( \varphi(x) \), there exists an open set \( U \) containing \( x \) such that \( \varphi(U) \subset V \); and

(21) **lower almost continuous**[^73] if for each \( x \in X \) and each regular open set \( V \) with \( \varphi(x) \cap V \neq \emptyset \), there exists an open set \( U \) containing \( x \) such that \( \varphi(z) \cap V \neq \emptyset \) for each \( z \in U \).

### 1.3 Summary of the thesis

The thesis is divided into five chapters. Each chapter is divided into sections and some of the sections are further subdivided into subsections.

**Chapter 1** Sections 1.1 deals with introduction and provides motivation to study multifunctions and briefly outlines applications of multifunctions to various disciplines in mathematics and applications of mathematics. In Section 1.2, necessary definitions and preliminary results used in the thesis are given. Section 1.3 is an attempt to summarise the contents of the thesis and to give a bird’s eye view of the whole thesis.

In **Chapter 2** notions of strong continuity of Levine[^53] and perfect continuity due to Noiri[^63] are extended to the framework of multifunctions and their basic properties are studied. This chapter is divided into six sections. In Section 2.1 we introduce notions of strongly continu-
ous multifunctions, upper and lower perfectly continuous multifunctions, and upper and lower almost perfectly continuous multifunctions, and elaborate on their place in the hierarchy of strong variants of continuity of multifunctions that already exist in the mathematical literature.

In Section 2.2 we study basic properties of strongly continuous multifunctions. In Sections 2.3 we obtain characterizations of upper perfectly continuous multifunctions and study their basic properties. It turns out that upper perfect continuity of multifunctions is preserved under expansion and shrinking of range, composition of multifunctions, union of multifunctions, restriction to a subspace, and passage to the graph multifunction. Furthermore, we prove that the graph of an upper perfectly continuous multifunction with closed values into a regular space is cl-closed with respect to $X$. Moreover, an upper perfectly continuous multifunction maps mildly compact sets to compact sets. Finally it is shown that an open and upper perfectly continuous nonmingled multifunction with para-Lindelöf values maps cl-para-Lindelöf sets to para-Lindelöf sets. Section 2.4 is devoted to the study of basic properties of lower perfectly continuous multifunctions wherein characterizations of lower perfectly continuity are obtained. It is shown that lower perfect continuity of multifunctions is preserved under expansion and shrinking of range, composition, union, restriction to a subspace. Section 2.5 deals with the study of basic properties of upper almost perfectly continuous multifunctions and characterizations of upper almost perfect continuity of multifunctions. Finally basic properties and characterizations of lower almost perfectly continuous multifunctions are discussed in section 2.6.

Chapter 3 is devoted to extending the notion of cl-supercontinuity to the realm of multifunctions. In section 3.2 notions of upper cl-supercontinuous and lower cl-supercontinuous multifunctions are defined and we discuss their interrelations with other variants of continuity of multifunctions that already exist in the lore of mathematical literature. Examples are
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included to reflect upon the distinctiveness of these notions from other variants of continuity of multifunctions. It turns out that the class of upper (lower) cl-supercontinuous multifunc-
tions properly includes the class of upper (lower) perfectly continuous multifunctions and so includes all strongly continuous multifunctions [53] and is strictly contained in the class of up-
ner (lower) z-supercontinuous multifunctions [4]. In Section 3.3 we obtain characterizations and study basic properties of upper cl-supercontinuous multifunctions. The preservation of upper cl-supercontinuity of multifunctions under the shrinking and expansion of range, com-
position, union, restriction to a subspace, and the passage to the graph multifunction is shown. Further, we formulate a sufficient condition for the intersection of two multifunctions to be upper cl-supercontinuous. Moreover, we prove that the graph of an upper cl-supercontinuous multifunction with closed values into a regular space is cl-closed with respect to $X$. Fur-
thermore, an upper cl-supercontinuous multifunction maps mildly compact sets to compact sets. Finally it is shown that an open, upper cl-supercontinuous nonmingled multifunction with paracompact values maps cl-paracompact sets to paracompact sets. Section 3.4 is de-
voted to the study of lower cl-supercontinuous multifunctions, wherein characterizations of lower cl-supercontinuity are obtained. It is shown that a product of multifunctions is lower cl-supercontinuous if and only if each multifunction is lower cl-supercontinuous.

In Chapter 4 we extend the notion of almost cl-supercontinuity to the framework of multifunc-
tions. In Section 4.1 we introduce the notions of upper almost cl-supercontinuous and lower almost cl-supercontinuous multifunctions and discuss the interrelations that exist among them and other strong variants of continuity of multifunctions. It turns out that the class of upper (lower) almost cl-supercontinuous multifunctions properly contains the class of upper (lower) cl-supercontinuous multifunctions [26] and so includes all upper (lower) (almost) perfectly continuous multifunctions [27] and is strictly contained in the class of upper (lower) (almost)
z-supercontinuous multifunctions ([4] [47]) which in turn is properly contained in the class of upper (lower) (almost) $D_\delta$-supercontinuous multifunctions ([5] [47]). Examples are included to reflect upon the distinctiveness of the notions so introduced from the other variants of continuity of multifunctions that already exist in the mathematical literature. In Section 4.2 we obtain characterizations, and study basic properties of upper almost cl-supercontinuous multifunctions. It turns out that upper almost cl-supercontinuity of multifunctions is preserved under shrinking and expansion of range, composition and union of multifunctions, restriction to a subspace, and the passage to the graph multifunction. Moreover, we prove that the graph of an upper almost cl-supercontinuous multifunction with closed values into a regular space is strongly cl-closed with respect to $X$. Furthermore, it is shown that a point mildly compact upper almost cl-supercontinuous multifunction maps mildly compact sets to mildly compact sets. Section 4.3 is devoted to the study of lower almost cl-supercontinuous multifunctions, wherein characterizations of lower almost cl-supercontinuity are obtained. It is shown that lower almost cl-supercontinuity is preserved under the shrinking and expansion of range, union of multifunctions, restriction to a subspace and passage to the graph multifunction. In Section 4.4 we study the behaviour of a lower almost cl-supercontinuous multifunction if its domain and/or range are retopologized. 

In Chapter 5 we extend the notion of quasi cl-supercontinuity [25] of functions to the realm of multifunctions. In section 5.1 we define the notions of upper and lower quasi cl-supercontinuous multifunctions and elaborate upon their place in the hierarchy of variants of continuity of multifunctions that already exist in the literature. It turns out that the class of upper (lower) quasi cl-supercontinuous multifunctions properly contains the class of upper (lower) cl-supercontinuous multifunctions [26] and so includes all upper (lower) perfectly continuous multifunctions [27] and is strictly contained in the class of upper (lower) quasi z-supercontinuous
multifunctions [30] which in turn is properly contained in the class of quasi upper (lower) $D_{\delta}$-supercontinuous multifunctions [30]. Examples are included to reflect upon the distinctiveness of the notions so introduced and other variants of continuity of multifunctions that already exist in the mathematical literature. In Section 5.2 we discuss characterizations and study basic properties of upper quasi cl-supercontinuous multifunctions. It turns out that upper quasi cl-supercontinuity of multifunctions is preserved under the composition, union of multifunctions, restriction to a subspace and passage to the graph multifunction. Moreover, we formulate a sufficient condition for the intersection of two upper quasi cl-supercontinuous multifunctions to be upper quasi cl-supercontinuous. In Section 5.3 we study lower quasi cl-supercontinuous multifunctions, and give their characterizations. It is shown that quasi lower cl-supercontinuity of multifunctions is preserved under the shrinking and expansion of range, union of multifunctions, and under restriction to a subspace. Section 5.4 deals with the behaviour of a lower almost cl-supercontinuous multifunction if its domain and/or range are retopologized.

The present thesis draws its subject matter from the following papers written by the author.

