Abstract of the Thesis

Entitled

ON SOME GENERALIZATIONS OF CONTINUITY OF
MULTIFUNCTIONS

FOR THE AWARD OF THE DEGREE OF
DOCTOR OF PHILOSOPHY

Submitted by
CHAMAN PRAKASH ARYA
Research Scholar
Department of Mathematics
University of Delhi
Delhi-110007

May, 2011
Multifunctions arise naturally in many areas of mathematics and applications of mathematics and have wide ranging applications in optimization theory, control theory, game theory, mathematical economics, dynamical systems and differential inclusions. Recently there has been considerable interest in trying to extend the notions and results of weak, strong and other variants of continuity of functions to the realm of multifunctions. The main aim of the thesis is to extend and study several strong and other variants of continuity of functions to the framework of multifunctions. The notions of strong continuity, complete continuity, perfect continuity, cl-supercontinuity, almost cl-supercontinuity and quasi cl-supercontinuity are extended to the framework of multifunctions. In the process of their study we obtain several characterizations of strong continuity of multifunctions, upper and lower perfect continuity of multifunctions, upper and lower (almost) cl-supercontinuity of multifunctions, upper and lower quasi cl-supercontinuity of multifunctions. Moreover, we study basic properties of all these classes of multifunctions and elaborate on their place in the hierarchy of strong forms of continuity of multifunctions and discuss their interrelations and interconnections with other variants which already exist in the mathematical literature.

The present thesis entitled, "On Some Generalizations of Continuity of Multifunctions" is divided into five chapters followed by references.

Chapter 1 Sections 1.1 deals with introduction and provides motivation to study multifunctions and briefly outlines applications of multifunctions to various disciplines in mathematics and applications of mathematics. In Section 1.2, necessary definitions and preliminaries results used in the thesis are given. Section 1.3 is an attempt to summarise the contents of the thesis and to give a bird's eye view of the whole thesis.

In Chapter 2 notions of strong continuity of Levine (Amer. Math. Monthly 67(1960), 269) and perfect continuity due to Noiri (Indian J. Pure Appl. Math. 15(3) (1984), 241-250) are extended to the framework of multifunctions and their basic properties are studied. This chapter is divided into six sections. In Section 2.1 we introduce notions of strongly continuous multifunctions, upper and lower perfectly continuous multifunctions, and upper and lower almost perfectly continuous multifunctions, and elaborate on their place in the hierarchy of strong variants of continuity of multifunctions that already exist in the mathematical literature. In Section 2.2 we study basic properties of strongly continuous multifunctions. In Sections 2.3 we obtain characterizations of upper perfectly continuous
multifunctions and study their basic properties. It turns out that upper perfect continuity of multifunctions is preserved under expansion and shrinking of range, composition of multifunctions, union of multifunctions, restriction to a subspace, and passage to the graph multifunction. Furthermore, we prove that the graph of an upper perfectly continuous multifunction with closed values into a regular space is cl-closed with respect to $X$. Moreover, an upper perfectly continuous multifunction maps mildly compact sets to compact sets. Finally it is shown that an open and upper perfectly continuous nonmingled multifunction with para-Lindelöf values maps cl-para-Lindelöf sets to para-Lindelöf sets.

Section 2.4 is devoted to the study of basic properties of lower perfectly continuous multifunctions wherein characterizations of lower perfect continuity are obtained. It is shown that lower perfect continuity of multifunctions is preserved under expansion and shrinking of range, composition, union, restriction to a subspace. Section 2.5 deals with the study of basic properties of upper almost perfectly continuous multifunctions and characterizations of upper almost perfect continuity of multifunctions. Finally basic properties and characterizations of lower perfectly continuous multifunctions are discussed in section 2.6.

Chapter 3 is devoted to extending the notion of cl-supercontinuity to the realm of multifunctions. In section 3.2 upper cl-supercontinuous and lower cl-supercontinuous multifunctions are defined and we discuss their interrelations with other variants of continuity of multifunctions that already exist in the lore of mathematical literature. Examples are included to reflect upon the distinctiveness of these notions from other variants of continuity of multifunctions. It turns out that the class of upper (lower) cl-supercontinuous multifunctions properly includes the class of upper (lower) perfectly continuous multifunctions and so includes all strongly continuous multifunctions (Amer. Math. Monthly, 67(1960), 269) and is strictly contained in the class of upper (lower) z-supercontinuous multifunctions (Kyungpook Math. J. 45(2005), 221-230). In Section 3.3 we obtain characterizations and study basic properties of upper cl-supercontinuous multifunctions. The preservation of upper cl-supercontinuity of multifunctions under the shrinking and expansion of range, composition, union, restriction to a subspace, and the passage to the graph multifunction is shown. Further, we formulate a sufficient condition for the intersection of two multifunctions to be upper cl-supercontinuous. Moreover, we prove that the graph of an upper cl-supercontinuous multifunction with closed values into a regular space is cl-closed with respect to $X$. Furthermore, an upper cl-supercontinuous multifunction maps mildly compact sets to compact sets. Finally it is shown that an open, upper cl-supercontinuous nonmingled multifunction with paracompact values maps cl-paracompact sets to paracom-
pact sets. Section 3.4 is devoted to the study of lower cl-supercontinuous multifunctions, wherein characterizations of lower cl-supercontinuity are obtained. It is shown that a product of multifunctions is lower cl-supercontinuous if and only if each multifunction is lower cl-supercontinuous.

In Chapter 4 we extend the notion of almost cl-supercontinuity to the framework of multifunctions. In Section 4.1 we introduce the notions of upper almost cl-supercontinuous and lower almost cl-supercontinuous multifunctions and discuss the interrelations that exist among them and other strong variants of continuity of multifunctions. It turns out that the class of upper (lower) almost cl-supercontinuous multifunctions properly contains the class of upper (lower) cl-supercontinuous multifunctions (Applied General Topology) and so includes all upper (lower) (almost) perfectly continuous multifunctions (Scientific Studies and Research. Series Mathematics and Informatics, Volume 20, No. 1, (2010), 103-118) and is strictly contained in the class of upper (lower) (almost)\( Z \)-supercontinuous multifunctions (Kyungpook Math. J. 45(2005), 221-230 and Appl. Gen. Top. 9(2) (2008), 239-251) which in turn is properly contained in the class of upper (lower) (almost)\( D_\delta \)-supercontinuous multifunctions (Miskolc Mathematical Notes vol.7, No.1(2006), pp. 3-11 and Appl. Gen. Top. 9(2) (2008), 239-251). Examples are included to reflect upon the distinctiveness of the notions so introduced from the other variants of continuity of multifunctions that already exist in the mathematical literature. In Section 4.2 we obtain characterizations, and study basic properties of upper almost cl-supercontinuous multifunctions. It turns out that upper almost cl-supercontinuity of multifunctions is preserved under shrinking and expansion of range, composition and union of multifunctions, restriction to a subspace, and the passage to the graph multifunction. Moreover, we prove that the graph of an upper almost cl-supercontinuous multifunction with closed values into a regular space is strongly cl-closed with respect to \( X \). Furthermore, it is shown that a point mildly compact upper almost cl-supercontinuous multifunction maps mildly compact sets to mildly compact sets. Section 4.3 is devoted to the study of lower almost cl-supercontinuous multifunctions, wherein characterizations of lower almost cl-supercontinuity are obtained. It is shown that lower almost cl-supercontinuity is preserved under the shrinking and expansion of range, union of multifunctions, restriction to a subspace and passage to the graph multifunction. In Section 4.4 we study the behaviour of a lower almost cl-supercontinuous multifunction if its domain and/or range are retopologized.

In Chapter 5 we extend the notion of quasi cl-supercontinuity (Demonstratio Mathematica 44 (3-4) (2011)) of functions to the realm of multifunctions. In section 5.1 we define
the notions of upper and lower quasi cl-supercontinuous multifunctions and elaborate upon their place in the hierarchy of variants of continuity of multifunctions that already exist in the literature. It turns out that the class of upper (lower) quasi cl-supercontinuous multifunctions properly contains the class of upper (lower) cl-supercontinuous multifunctions (Applied General Topology) and so includes all upper (lower) perfectly continuous multifunctions (Scientific Studies and Research. Series Mathematics and Informatics, Volume 20, No. 1, (2010), 103-118) and is strictly contained in the class of upper (lower) quasi z-supercontinuous multifunctions which in turn is properly contained in the class of quasi upper (lower) $D_\delta$-supercontinuous multifunctions. Examples are included to reflect upon the distinctiveness of the notions so introduced and other variants of continuity of multifunctions that already exist in the mathematical literature. In Section 5.2 we discuss characterizations and study basic properties of upper quasi cl-supercontinuous multifunctions. It turns out that upper quasi cl-supercontinuity of multifunctions is preserved under the composition, union of multifunctions, restriction to a subspace and passage to the graph multifunction. Moreover, we formulate a sufficient condition for the intersection of two upper quasi cl-supercontinuous multifunctions to be upper quasi cl-supercontinuous. In Section 5.3 we study lower quasi cl-supercontinuous multifunctions, and give their characterizations. It is shown that quasi lower cl-supercontinuity of multifunctions is preserved under the shrinking and expansion of range, union of multifunctions, and under restriction to a subspace. Section 5.4 deals with the behaviour of a lower almost cl-supercontinuous multifunction if its domain and/or range are retopologized.

The present thesis is based on the following research papers written by the author.


