Chapter 2

Basics and theoretical framework

One of the ubiquitous and most important theoretical models in quantum physics describes an atom interacting with a simple harmonic oscillator. This model opens up many possibilities in the context of quantum computing and information processing. In quantum optics it describes the interaction of an atom with an electromagnetic field mode in a cavity. In this work an atomic system with two accessible energy levels (qubit) is considered throughout. This chapter gives a brief review of superconducting qubits and the importance of studying the interacting system in the ultra strong coupling regime. Also, an overview of the adiabatic approximation and the solution of the Rabi Hamiltonian under this approximation are presented.

2.1 Superconducting qubits

Two-level atoms can exist as superposition of two logical states, which act as the basic unit of quantum computers known as qubits. Even though it is conceptually well-explained using the quantum principle, experimental implementation of quantum computation is still a challenging task due to various factors. Recently, various experiments employing superconducting qubits have been performed. These qubits have the advantage that they can be controlled by various external means such as currents, magnetic fields, and so on. These artificial atoms have many similarities with natural atoms such as both having discrete energy levels and exhibiting quantum coherence. But unlike the natural atoms these artificial atoms may be made to interact strongly with each other,
and also with the environment. These artificial atoms have advantage in that they offer additional flexibility in selecting the control parameters which facilitate experimental tests of fundamental quantum mechanical principles at a macroscopic scale [20]. The tailor-made physical characteristics of the superconducting circuits and their easy adjustability, say by changing external magnetic fields, make them particularly suitable candidates for quantum simulators [21]- [23]. The superconducting circuits based on the Josephson junctions behave, in certain parametric range, as effective two level systems, and can exhibit macroscopic quantum coherence [24, 25]. Because of these reasons studying the qubit in the presence of a bias term becomes important. The effective Hamiltonian of a two-level system in the presence of a static bias in natural units ($\hbar = 1$) is given by,

$$H = -\frac{\Delta}{2} \sigma_x - \frac{\epsilon}{2} \sigma_z,$$

(2.1)

where the qubit is characterized by an energy splitting $\Delta$ as well as an external static bias $\epsilon$ expressed via the spin variables ($\sigma_x, \sigma_z$).

### 2.2 Strongly coupled qubit-oscillator system

In quantum optics a two-level system (qubit) interacting with a harmonic oscillator is the simplest model with various quantum properties. Semiclassical version of this model has been proposed by Rabi and for this reason the system of qubits coupled with harmonic oscillator is called as Rabi model. The Hamiltonian of the coupled qubit-oscillator system [14, 15] in the presence of static bias is written as follows:

$$H = -\frac{\Delta}{2} \sigma_x - \frac{\epsilon}{2} \sigma_z + \omega a^\dagger a + \lambda \sigma_z (a^\dagger + a),$$

(2.2)

where the harmonic oscillator with a frequency $\omega$ is described by the raising and lowering operators ($a^\dagger, a|\hat{n} \equiv a^\dagger a$). The qubit-oscillator coupling strength is denoted by $\lambda$. The Fock states $\{\hat{n}|n\rangle = n|n\rangle, n = 0, 1, \ldots; a|n\rangle = \sqrt{n} |n - 1\rangle, a^\dagger |n\rangle = \sqrt{n + 1} |n + 1\rangle\}$ provide the basis for the oscillator, and the eigenstates $\sigma_z |\pm 1\rangle = \pm |\pm 1\rangle$ span the
space of the qubit.

Various approximation schemes have been introduced to solve this model which are valid only on certain regimes. For the case of a weak qubit-oscillator coupling and a small detuning frequency between the qubit and the oscillator the rotating wave approximation could be used to describe the evolution of the system. Under this approximation the Hamiltonian (2.2) reduces to that of the Jaynes-Cummings model [26] and has been studied widely for decades. The rotating wave approximation scheme incorporates terms which preserve total excitation number. It is worth noting that the model presented by (2.2) can also be reduced from the ion laser interaction Hamiltonian after a similarity transformation [27,28]. The superconducting qubits, however, lead to stronger coupling between the qubits and the oscillator and various experiments have been performed in this domain [29,30].

Moving beyond the rotating wave approximation in these strongly interacting systems one needs to incorporate terms in the qubit-oscillator interaction Hamiltonian that do not preserve the total excitation number. In the parametric regime where the oscillator frequency dominates the qubit frequency the authors of Refs. [14,15] have advanced an adiabatic approximation scheme that utilizes the disparity of the time scales of the qubit and the oscillator. The dynamical state of the fast-moving oscillator is assumed to instantaneously adjust to the slow-changing state of the qubit. This facilitates decoupling of the full Hamiltonian into sectors related to each time scale, and allows approximate evaluation [14] of eigenstates of the system. In addition, an analytical expression of the time-dependent behavior of the two level system has been obtained [14] in a relatively weak coupling limit. Studying a system of two qubits coupled to a single oscillator mode in the context of the Tavis-Cummings model the authors of Ref. [31] have analytically evaluated the collapse and revival dynamics of the qubit reduced density matrix elements. Their procedure employs the Poisson sum formula [32] that leads to expressions of the mentioned elements as infinite sums. Following the extensive experimental developments [24,25] for the superconducting qubits, a static bias term has
been included in the qubit Hamiltonian and the corresponding energy spectrum has been investigated [15,33]. A time-dependent bias acting as a control parameter has also been considered [34].

2.3 The Hamiltonian: solution via adiabatic approximation

This section gives a brief review of analyzing the Hamiltonian under an adiabatic approximation [14], [15] that hinges on the separation of the time scales governed by the high oscillator frequency and the (renormalized) low qubit frequency: $\omega \gg \Delta$. To facilitate the construction of the evolution of the states the diagonalization process [14] of the Hamiltonian (2.2) under the adiabatic approximation scheme is now described. The high-frequency oscillator is assumed to instantaneously adjust to the slow-changing state of the qubit observable $\sigma_z$ so that the construction permits, in the course of diagonalization of the oscillator mode, replacing the spin-variable $\sigma_z$ with its eigenvalue: $\pm 1$. The effective Hamiltonian of the harmonic oscillator now reads

$$H_{\mathcal{O}} = \omega a^\dagger a \pm \lambda (a^\dagger + a).$$

The number states which are shifted symmetrically by the displacement operator diagonalize the above Hamiltonian:

$$|n_\pm\rangle = D^\dagger (\pm \lambda/\omega) |n\rangle, \quad D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a),$$

(2.4)

where the degenerate eigenenergies are as follows:

$$E_n^\pm = E_n = n\omega - \frac{\lambda^2}{\omega}.$$  

(2.5)

Identically displaced states maintain orthonormality: $\langle m_\pm | n_\pm \rangle = \delta_{m,n}$, whereas the overlap between the Fock states displaced in opposite directions is given by [14]
\[ \langle m_-|n_+ \rangle = \begin{cases} 
(\frac{\lambda}{\omega})^{m-n} \exp \left( -\frac{\epsilon}{2} \right) x^{\frac{m-n}{2}} \sqrt{n! / m!} L_n^{m-n} (x) & \forall m \geq n, \\
\exp \left( -\frac{\epsilon}{2} \right) x^{\frac{n-m}{2}} \sqrt{m! / n!} L_m^{n-m} (x) & \forall m < n. 
\end{cases} \tag{2.6} \]

In (2.6) the associated Laguerre polynomial reads \( L_n^j (x) = \sum_{k=0}^{n} \binom{n+j}{n-k} \frac{x^k}{k!} \), and the parameter \( x = (2\lambda/\omega)^2 \) acts as the perturbative expansion parameter.

After diagonalizing the high frequency oscillator component of the Hamiltonian, one now turns attention to the low frequency qubit part. Tensoring the qubit states with the displaced oscillator basis states \( \{| \pm 1, n \rangle \} \) the corresponding matrix elements of the Hamiltonian may now be readily constructed. For a dominant oscillator frequency \( \Delta/\omega \ll 1 \) one may neglect \[14\] the matrix elements that mix the oscillator states with different eigenvalues of its number operator. This reduces the Hamiltonian to a block-diagonal form where each block mixes the displaced oscillator states with identical number of photons. The Hamiltonian for the \( n \)-th block assumes the form:

\[ H = \begin{pmatrix} n\omega - \lambda^2 / \omega & \delta_n \\
\delta_n & n\omega - \lambda^2 / \omega + \epsilon \end{pmatrix}, \tag{2.7} \]

where \( \delta_n = -\frac{\Delta}{2} L_n (x) \), \( \Delta = \Delta \exp \left( -\frac{\epsilon}{2} \right) \), \( \epsilon = \frac{\epsilon}{2} \). This Hamiltonian may be diagonalized in the basis of the eigenstates listed below

\[ |E^\pm_n \rangle = \sqrt{\frac{\chi_n \pm \epsilon}{2\chi_n}} |1, n_+ \rangle \pm \frac{\delta_n}{|\delta_n|} \sqrt{\frac{\chi_n \pm \epsilon}{2\chi_n}} |-1, n_- \rangle, \tag{2.8} \]

where the respective eigenvalues read

\[ E_n^\pm = n\omega - \lambda^2 / \omega \pm \chi_n, \quad \chi_n = \sqrt{\delta_n^2 + \epsilon^2}. \tag{2.9} \]

These eigenvalues and the eigenvectors are used to find the evolution of the quasi-Bell state and hybrid entangled squeezed states and hence the reduced density matrices of
the qubit and the oscillator degrees of freedom. To study the nature of the evolved state various tools have been used. This will be discussed briefly in the coming sections.

2.4 Phase space picture of quantum states

The quantum mechanical states of electromagnetic field can be described using a formulation on the phase space spanned by two conjugate variables. Unlike the case of classical mechanics the quantum state, on account of the Heisenberg uncertainty principle, can not be characterized by a probability at a single point. So there is no unique way to represent the quantum state in the phase space. But there are various quasi probability distributions available for the description of the state. Each of these distributions has its own advantages and limitations. In this work to study the characteristics of the reduced oscillator density matrices are studied using the quasi probability distributions such as the Sudarshan-Glauber diagonal $P$-representation, the Wigner $W$-distribution, and the nonnegative Husimi $Q$-function.

2.4.1 Quasi probability distributions

The overcompleteness of the coherent states allows a diagonal construction of the density matrix in the coherent state basis via the Sudarshan-Glauber $P$-representation [35,36]:

$$\rho = \int P(\beta, \beta^*) |\beta\rangle \langle \beta| \ d^2 \beta,$$

(2.10)

where the normalizability condition reads $\int P(\beta, \beta^*) d^2 \beta = 1$. For an arbitrary quantum state the relation (2.10) may be inverted and the diagonal $P$-representation is uniquely expressed [37] as the following distribution:

$$P(\beta, \beta^*) = \frac{\exp(|\beta|^2)}{\pi^2} \int \langle -\gamma|\rho|\gamma\rangle \exp(|\gamma|^2) \exp(\beta^* \gamma - \beta \gamma^*) d^2 \gamma.$$

(2.11)

The $P$-representation is not always positive semi-definite, and by analysing the $P$-representation one could explain the nonclassical nature of a quantum state. Whenever
the representation deviates from probability density properties for example if it has negative values or singularities such as derivatives of $\delta$-functions, the state is said to be to nonclassical. The $P$-representation is used to evaluate the expectation values of normally ordered functions of $a$ and $a^\dagger$. However, the complicated structure of the $P$-representation in most of the nonclassical cases makes difficult its use as a quantitative measure for nonclassicality.

Another quasi probability distribution of interest is the Wigner $W$-distribution which is related to the symmetric ordering of the canonical operators. For an arbitrary oscillator density matrix $\rho$ the Wigner quasi probability distribution is defined [38] via the displacement operator as

$$W(\beta, \beta^*) = \frac{1}{\pi^2} \int \text{Tr}[\rho D(\gamma)] \exp(\beta \gamma^* - \beta^* \gamma) \, d^2\gamma, \quad \int W(\beta, \beta^*) \, d^2\beta = 1. \quad (2.12)$$

To interpret the Wigner $W$-distribution as a probability distribution it needs to be non-negative. But it may take negative values for certain quantum states showing the nonclassical nature of a states. In such cases the negative volume of the $W$-distribution on the phase space, referred to as the negativity, can be used as a quantifier of nonclassicality [18]:

$$\delta_W = \int |W(\beta, \beta^*)| \, d^2\beta - 1. \quad (2.13)$$

The $W$-distribution can be obtained by smoothing of a singular $P$-representation via a Gaussian function of variance $1/2$:

$$W(\beta, \beta^*) = \frac{2}{\pi} \int P(\gamma, \gamma^*) \exp(-2|\beta - \gamma|^2) \, d^2\gamma. \quad (2.14)$$

The Husimi $Q$-function [16] is another quasi probability distribution which is defined as the diagonal expectation value of the oscillator density matrix in an arbitrary coherent state

$$Q(\beta, \beta^*) = \frac{1}{\pi} \langle \beta | \rho_O | \beta \rangle. \quad (2.15)$$
It is a positive semi-definite quantity that obeys the normalization condition on the phase space: \( \int Q(\beta, \beta^*) d^2\beta = 1 \). The \( Q \)-function may be regarded [39] as the convolution of the \( W \)-distribution with a Gaussian kernel possessing a variance 1/2 on the phase space:

\[
Q(\beta, \beta^*) = \frac{2}{\pi} \int W(\gamma, \gamma^*) \exp(-2|\beta - \gamma|^2) d^2\gamma,
\]

which points towards its physical interpretation as a ‘coarse-grained’ analog of the \( W \)-distribution. Moreover, the process of ‘coarse-graining’ via positive-definite Gaussian kernels of sufficiently broad variance (equal to unity) directly links [39] the \( P \)-representation with the positive semi-definite Husimi \( Q \)-function:

\[
Q(\beta, \beta^*) = \frac{1}{\pi} \int P(\gamma, \gamma^*) \exp(-|\beta - \gamma|^2) d^2\gamma.
\]

This nonnegative Husimi \( Q \)-function is used in calculating the expectation values of antinormally ordered function. The \( Q \)-function is also utilized to evaluate the Wehrl entropy [17] defined as

\[
S_W = -\int Q(\beta, \beta^*) \log Q(\beta, \beta^*) d^2\beta,
\]

which describes the delocalization of the quantum state on the phase space. These three quasi probability distributions \((P, W, Q)\) are special cases of the Cahill-Glauber \( s \)-parametrized distributions for \( s = 1, 0, -1 \), respectively [40].

### 2.5 Hilbert-Schmidt distance

Distance measures are quantitative tools that are used to find how close two states are to each other. Any two quantum states are said to be similar to each other if the distance between them is small. The Hilbert-Schmidt distance is one among the various distance measures available. The Hilbert-Schmidt distance between any two arbitrary
quantum density matrices $\rho_1$ and $\rho_2$ [19]:

$$d_{HS}(\rho_1, \rho_2) = \left( \text{Tr}(\rho_1^2) + \text{Tr}(\rho_2^2) - 2 \text{Tr}(\rho_1 \rho_2) \right)^{1/2}. \quad (2.19)$$

If the distance between them is zero then one could say these states are equivalent and if it takes the maximal value $\sqrt{2}$ then the state is said to be orthogonal. The Hilbert-Schmidt distance (2.19) between two arbitrary density matrices $\rho_1$ and $\rho_2$ may also be recast [19] via their corresponding Wigner distributions $W_1(\beta, \beta^*)$ and $W_2(\beta, \beta^*)$ as follows:

$$\left( d_{HS}(\rho_1, \rho_2) \right)^2 = \pi \int (W_1(\beta, \beta^*) - W_2(\beta, \beta^*))^2 \, d^2\beta. \quad (2.20)$$