CHAPTER-II
COMMON FIXED POINT
OF MAPPINGS SATISFYING
RATIONAL INEQUALITY
CHAPTER-II
COMMON FIXED POINT OF MAPPINGS SATISFYING RATIONAL INEQUALITY

2.1 Introduction.

In 1976 Jungck [115] generalized the Banach's classical fixed point theorem for two commutative self mappings of a complete metric space and set out a tradition of common fixed point theorems. Improving the idea of commutativity, Sessa [231] initiated a weaker notion than that of commutativity namely, weakly commutativity of maps in fixed point consideration. Further Jungck [116] introduced the notion of compatible mappings and demonstrated that weakly commuting pair of maps is compatible but the converse is not true always. After the introduction of the concept of compatibility, various type of compatibility concepts namely: Compatibility of type (A), Compatibility of type (B), Compatibility of type (C), Compatibility of type (P), weakly compatibility etc are introduced by Jungck et al [118], Pathak and Khan [190], Pathak et al[194], Pathak et al[191], Jungck [121] respectively in their papers and gave many interesting results regarding common fixed points for two to four self maps of metric spaces.

In a paper Khan [131] gave a result of fixed point of two self mappings satisfying rational inequality. Further Fisher [76] observed that the result of Khan is incorrect and it needs an extra condition for the theorem hold. Also Fisher [75] proved a similar result for two self mappings of a complete metric space satisfying a rational inequality. In this sequence Sharma and Bajaj [235] have also established a common fixed point theorem for two self mappings. Further many researchers gave results regarding common fixed point of mappings satisfying various types of rational inequalities. Using the notions of compatibility and weakly compatibility of mappings, Jeong-Rhoades [114] generalized
the results of Ahmad-Imdad [7]. Further another extension of results of Jeong-Rhoades [114], Imdad and Khan [101] gave another result by improving contraction condition and increasing the number of maps from four to six.

In this chapter, we extend the results of Imdad and Khan [101] by increasing the number of maps from six to eight. Our results generalize and improve the results of Fisher [76], Kanan [127,128], Hardy and Rogers [94], Ahmad-Imdad [7], Jeong-Rhoades [114], Imdad and Khan [101] and many others.

2.2 Preliminaries: Thought this chapter (X, d) stands for metric space.

**Definition 2.1** [231]: Two self mappings S and T on a metric space X are called weakly commuting if \( d(STx, TSx) \leq d(Sx, Tx), \forall x \in X \).

**Definition 2.2** [116]: Two self mappings S and T on a metric space X are called compatible if \( \lim_{n \to \infty} d(STx_n, TSx_n) = 0 \), whenever \( \{x_n\} \) is a sequence in X such that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \) for some \( t \) in X.

**Definition 2.3** [118]: Two self mappings S and T on a metric space X are called compatible of type (A) if \( \lim_{n \to \infty} d(STx_n, TTx_n) = 0 \) and \( \lim_{n \to \infty} d(TSx_n, SSx_n) = 0 \), whenever \( \{x_n\} \) is a sequence in X such \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \) for some \( t \) in X.

**Definition 2.4** [190]: Two self mappings S and T on a metric space X are called compatible of type (B) if

\[
\lim_{n \to \infty} d(STx_n, TTx_n) \leq \frac{1}{2} \left[ \lim_{n \to \infty} d(STx_n, St) + \lim_{n \to \infty} d(St, SSx_n) \right]
\]

and

\[
\lim_{n \to \infty} d(TSx_n, SSx_n) \leq \frac{1}{2} \left[ \lim_{n \to \infty} d(TSx_n, Tt) + \lim_{n \to \infty} d(Tt, TTx_n) \right]
\]
whenever \( \{x_n\} \) is a sequence in \( X \) such \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n \) = \( t \) for some \( t \) in \( X \).

**Definition 2.5** [194]: Two self mappings \( S \) and \( T \) on a metric space \( X \) are called compatible of type (C) if 
\[
\lim_{n \to \infty} d(STx_n, TTx_n) \leq \frac{1}{3}[\lim_{n \to \infty} d(STx_n, St) + \lim_{n \to \infty} d(St, SSx_n) + \lim_{n \to \infty} d(Tt, TTx_n)]
\]
whenever \( \{x_n\} \) is a sequence in \( X \) such \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = \) \( t \) for some \( t \) in \( X \).

**Definition 2.6** [191]: Two self mappings \( S \) and \( T \) on a metric space \( X \) are called compatible of type (P) if 
\[
\lim_{n \to \infty} d(SSx_n, TTx_n) \leq \frac{1}{3}[\lim_{n \to \infty} d(TSx_n, Ts) + \lim_{n \to \infty} d(Ts, TTx_n) + \lim_{n \to \infty} d(Tt, SSx_n)]
\]
whenever \( \{x_n\} \) is a sequence in \( X \) such \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = \) \( t \) for some \( t \) in \( X \).

**Definition 2.7** [121]: Two self mappings \( S \) and \( T \) on a metric space \( X \) are called weakly compatible if they commute at their coincidence points.

**Example 2.1**: Let \( X = [0, 3] \). Define \( S, T : [0, 3] \to [0, 3] \) by 
\[
S(x) = \begin{cases} 
  x, & \text{if } x \in [0, 1) \\
  3, & \text{if } x \in [1, 3]
\end{cases} \quad ; \quad T(x) = \begin{cases} 
  3 - x, & \text{if } x \in [0, 1) \\
  3, & \text{if } x \in [1, 3]
\end{cases}
\]
Then for any \( x \in [1, 3] \), \( x \) is a coincidence point and \( STx = TSx \), showing that \( S, T \) are weakly compatible maps on \( [0, 3] \).
**Proposition 2.1** [231]: Commutativity of two self mappings on a metric space implies weakly commutativity but converse is not true. For example let $X = [0, 1]$ and $d$ be a usual metric on $X$. For example if mappings $S, T: X \to X$ are defined by $Sx = x/(x+2)$ and $T = x/(x+2)$, for all $x \in X$. Then $S$ and $T$ are weakly commuting but not commuting.

**Proposition 2.2** [116]: Weakly commutativity of two self mappings on a metric space implies compatibility but converse is not true always. For example let $X = \mathbb{R}$, the set of real numbers. Define $S, T: X \to X$ by $Sx = x^3$ and $T = 2x^3$, for all $x \in X$. Then $S$ and $T$ are compatible but not weakly commutative.

**Proposition 2.3** [119]: Let $S, T: X \to X$ be continuous mappings. If $S$ and $T$ are compatible, then they are compatible of type (A).

**Proposition 2.4** [118]: Let $S, T: X \to X$ be compatible mappings of type (A). If one of $S$ and $T$ is continuous, then $S$ and $T$ are compatible.

**Proposition 2.5** [118]: Let $S, T: X \to X$ be continuous mappings. Then $S$ and $T$ are compatible, if and only if they are compatible of type (A).

Through the following examples, it is clear that the above propositions are not true if $S$ and $T$ are not continuous.

**Example 2.2** [118]: Let $X = \mathbb{R}$, the set of real numbers and let $S, T: X \to X$ are defined by

$S(x) = \begin{cases} 1 & \text{if } x \neq 0, \\ x & \text{if } x = 0 \end{cases}$

and

$T(x) = \begin{cases} 1 & \text{if } x \neq 0, \\ x^2 & \text{if } x = 0 \end{cases}$

with usual metric $d.$
Then $S$ and $T$ are not continuous at $x = 0$. Also $S$ and $T$ are compatible but not compatible of type (A).

**Example 2.3** [118]: Let $X = [0, 1]$, with usual metric $d(x, y) = |x - y|$.

Let $S, T: X \to X$ are defined by

$$S(x) = \begin{cases} x & \text{if } x \in [0, 1/2], \\ 1 & \text{if } x \in [1/2, 1] \end{cases} \quad \text{and} \quad T(x) = \begin{cases} 1 - x & \text{if } x \in [0, 1/2], \\ 1 & \text{if } x \in [1/2, 1] \end{cases}.$$ 

Here $S$ and $T$ are not continuous at $x = 1/2$. If the sequence $\{x_n\}$ is such that $\{x_n\} \subseteq [0, 1]$, then $S$ and $T$ are compatible of type (A) but they are not compatible.

**Proposition 2.6** [191]: Let $S, T: X \to X$ be continuous mappings. Then $S$ and $T$ are compatible if and only if they are compatible of type (P).

**Proposition 2.7** [191]: Let $S, T: X \to X$ be compatible mappings of type (A). If one of $S$ and $T$ is continuous, then $S$ and $T$ are compatible of type (P).

**Proposition 2.8** [191]: Let $S, T: X \to X$ be continuous mappings. Then

1. $S$ and $T$ are compatible if and only if they are compatible of type (P).

2. $S$ and $T$ are compatible of type (A) if and only if $S$ and $T$ are compatible of type (P).

**Proposition 2.9**: 1. Let $S, T: X \to X$ be compatible mappings and if $Sx = Tx$ then $STx = TSx$ (cf [116]).
2. Let $S, T : X \to X$ be compatible of type (A) mappings and if $Sx = Tx$, then $STx = TSx$ (cf. [117]).

3. Let $S, T : X \to X$ be compatible of type (P), mappings and if $Sx = Tx$ then $STx = TSx$ (cf. [191]).

**Proposition 2.10:**

1. Let $S, T : X \to X$ be continuous and compatible mappings then $Sx = Tx$ and $STx = TSx$ (cf. [116]).

2. Let $S, T : X \to X$ be continuous and compatible of type (A) mappings then $Sx = Tx$ and then $STx = TSx$ (cf. [117]).

3. Let $S, T : X \to X$ be continuous and compatible of type (P), mappings then $Sx = Tx$ and $STx = TSx$ (cf. [191]).

From the above propositions 2.9 and 2.10 it follows that if

1. Mappings $S, T : X \to X$ are compatible (resp. compatible of type (A), compatible of type (P)) and if $Sx = Tx$ then $STx = TSx$, i.e. $S$ and $T$ are weakly compatible.

2. Mappings $S, T : X \to X$ are continuous and compatible (resp. compatible of type (A), compatible of type (P)) then $Sx = Tx$ and $STx = TSx$, i.e. $S$ and $T$ are weakly compatible.

However, the converses of these consequences are not true always as we can see in the following examples.
Example 2.4 [121]: Let \( X = [0, 1] \), \( d \) be the absolute valued metric and the mappings \( S, T: X \to X \) are defined by

\[
S(x) = \begin{cases} 
\frac{1}{2} & \text{ if } x \in [0, \frac{1}{2}] \\
\frac{1}{2} - \frac{1}{2} & \text{ if } x \in \left(\frac{1}{2}, 1\right]
\end{cases}
\]

and

\[
T(x) = \begin{cases} 
1 - x & \text{ if } x \in [0, \frac{1}{2}] \\
0 & \text{ if } x \in \left(\frac{1}{2}, 1\right]
\end{cases}
\]

Then \( S \) and \( T \) are weakly compatible but not compatible.

Example 2.5: Let \( X = [2, 20] \), with usual metric \( d \). Let \( S, T: X \to X \) are defined by

\[
S(x) = \begin{cases} 
2 & \text{ if } x = 2 \\
12 + x & \text{ if } 2 < x \leq 5 \\
x - 3 & \text{ if } x > 5
\end{cases}
\]

\[
T(x) = \begin{cases} 
2 & \text{ if } x \in \{2\} \cup (5, 20] \\
8 & \text{ if } 2 < x \leq 5
\end{cases}
\]

Then \( S \) and \( T \) are weakly compatible but not compatible of type (A) (resp. compatible of type (P)).

2.3 Results:

By using the rational inequality, Khan [130] gave the following theorem:

Theorem 2.1. Let \( S \) and \( T \) be two self mappings of a complete metric space \( (X, d) \) satisfying:

\[
d(Sx, Ty) \leq \beta \frac{d(x, Sx)d(x, Ty) + d(y, Sx)d(y, Ty)}{d(x, Sx) + d(x, Ty)},
\]

for all \( x, y \) in \( X \) and \( 0 < \beta < 1 \). Then \( S \) and \( T \) have a unique common fixed point.
It was later shown by Fisher [76] that the theorem 2.1 was incorrect as it stood and needed an extra condition that \(d(x, Ty) + d(y, Sx) = 0\) implies that \(d(Sx, Ty) = 0\) for the theorem to hold.

Further Fisher [75] proved a following common fixed point theorem for two mappings satisfying rational inequality:

**Theorem 2.2.** Let \(S\) and \(T\) be two mappings of a complete metric space \((X, d)\) into itself such that

\[
    d(Sx, Ty) \leq \beta \left( \frac{[d(x, Sx)]^2 + [d(y, Ty)]^2}{d(x, Sx) + d(y, Ty)} \right),
\]

for all \(x, y \in X\) for which \(d(x, Sx) + d(y, Ty) \neq 0\), where \(0 < \beta < 1\). Then \(S\) and \(T\) have a unique common fixed point \(z\). Further if \(d(x, Sx) + d(y, Ty) = 0\) implies that \(d(Sx, Ty) = 0\), then \(z\) is the unique common fixed point of \(S\) and \(T\).

Using a slightly modified version of rational inequality, Sharma and Bajaj [235] proved the following theorem:

**Theorem 2.3.** Let \(S\) and \(T\) be two mappings of a complete metric space \((X, d)\) into itself such that

\[
    d(Sx, Ty) \leq \beta \frac{d(x, Sx)d(x, Ty) + d(y, Sx)d(y, Ty)}{d(x, Sx) + d(x, Ty)},
\]

for all \(x, y \in X\) for which \(d(x, Sx) + d(x, Ty) \neq 0\), where \(0 < \beta < \frac{1}{2}\), then \(S\) and \(T\) have a unique common fixed point.
Further Jeong -Rhoades [114] generalized the result of Ahmad and Imdad [7] and proved a following common fixed point theorem for four mappings satisfying rational inequality:

**Theorem 2.4.** Let $A$, $B$, $S$ and $T$ be self mappings of a complete metric space $(X, d)$ satisfying

(2.4.1) $A(X) \subseteq T(X), B(X) \subseteq S(X)$

(2.4.2) For each $x, y \in X$, either

$$d(Ax, By) \leq \alpha \left[ \frac{(d(Ax, Sx))^2 + (d(By, Ty))^2}{d(Ax, Sx) + d(By, Ty)} \right] + \beta d(Sx, Ty),$$

if $d(Ax, Sy) + d(By, Ty) \neq 0$, $\alpha, \beta > 0$, $\alpha + \beta < 1$, or

$$d(Ax, By) = 0 \text{ if } d(Ax, Sx) + d(By, Ty) = 0.$$

If either (a) the pair $(A, S)$ is compatible, $A$ or $S$ is continuous and the pair $(B, T)$ is weakly compatible or (b) the pair $(B, T)$ is compatible and $B$ or $T$ is continuous and the pair $(A, S)$ is weakly compatible, then $A$, $B$, $S$ and $T$ have a unique common fixed point $z$. Moreover, $z$ is the unique common fixed point of $A$ and $S$ and of $B$ and $T$.

**Remark:** Theorem 1 of Ahmad and Imdad [7] is a special case of theorem 2.4, since weakly commuting implies compatibility. Jeong - Rhoades [114] also mentioned that the proof of (ii) of Ahmad and Imdad [7] is incorrect. The condition $U_{2n} + U_{2n+1} = 0$ imply that $L$ and $S$ and $T$ and $J$ have coincidence points but not necessarily the same point.
Imdad and Khan [101] proved the following theorem for six mappings satisfying rational inequality:

**Theorem 2.5.**[101] Let $A, B, S, T, L$ and $J$ be self mappings of a complete metric space $(X, d)$ satisfying

(2.5.1) $AB(X) \subseteq J(X), ST(X) \subseteq L(X)$

(2.5.2) for each $x, y \in X$, either

$$d(ABx, STy) \leq \alpha_1 \left[ \frac{(d(ABx, Lx))^2 + (d(STy, Jy))^2}{d(ABx, Lx) + d(STy, Jy)} \right] + \alpha_2 d(Lx, Jy)$$

$$+ \alpha_3 [d(ABx, Jy) + d(STy, Lx)],$$

if $d(ABx, Lx) + d(STy, Jy) \neq 0$, $\alpha_i \geq 0, (i = 1, 2, 3)$ with at least one $\alpha_i$ non zero and $\alpha_1 + \alpha_2 + 2\alpha_3 < 1$.

Or, $d(ABx, STy) = 0$ if $d(ABx, Lx) + d(STy, Jy) = 0$.

(2.5.3) either the pair $(AB, L)$ is compatible and LI or $AB$ is continuous and the pair $(ST, J)$ is weakly compatible or, the pair $(ST, J)$ is compatible and $J$ or $ST$ is continuous and the pair $(AB, L)$ is weakly compatible.

Then $AB, ST, L$ and $J$ have a unique common fixed point. Furthermore if (2.5.4) the pairs $(A, B), (A, L), (B, L), (S, T), (S, J)$ and $(T, J)$ are commuting mappings then $A, B, S, T, L$ and $J$ have a unique common fixed point.

The aim of this chapter is to establish yet another extension of Theorem 2.5 by increasing the number of mappings from six to eight. In this process results due to Fisher
Kannan [127,128] Hardy-Roger [94], Ahmad-Imdad [7] and Jeong-Rhoades [114] are generalized and improved. We prove the following theorem:

**Theorem 2.6.** Let $A$, $B$, $S$, $T$, $P$, $Q$, $R$ and $L$ be self mappings of a complete metric space $(X, d)$ satisfying

\[(2.6.1)\] $AB(X) \subseteq RL(X)$, $ST(X) \subseteq PQ(X)$

\[(2.6.2)\] for each $x, y \in X$, either

\[d(ABx, STy) \leq \alpha \left[ \frac{d(ABx, PQx)^2 + d(STy, RLy)^2}{d(ABx, PQx) + d(STy, RLy)} \right] + \beta d(PQx, RLy) + \gamma[d(ABx, RLy) + d(STy, PQx)],\]

if $d(ABx, PQx) + d(STy, RLy) \neq 0$, $\alpha, \beta, \gamma \geq 0$ with at least one $\alpha, \beta, \gamma$ non zero and $\alpha + \beta + 2\gamma < 1$.

Or, $d(ABx, STy) = 0$ if $d(ABx, PQx) + d(STy, RLy) = 0$.

\[(2.6.3)\] either the pair $(AB, PQ)$ is compatible and $PQ$ or $AB$ is continuous and the pair $(ST, RL)$ is weakly compatible or, the pair $(ST, RL)$ is compatible and $ST$ or $RL$ is continuous and the pair $(AB, PQ)$ is weakly compatible. Then $AB$, $ST$, $PQ$ and $RL$ have a unique common fixed point say $z$.

\[(2.6.4)\] the pairs $(A, B)$, $(A, PQ)$, $(B, PQ)$, $(S, T)$, $(S, RL)$ and $(T, RL)$ are commute at $z$, then $A$, $B$, $S$, $T$, $PQ$ and $RL$ have a unique common fixed point.

Furthermore if

\[(2.6.5)\] the pairs $(P, Q)$, $(P, AB)$, $(Q, AB)$, $(R, AB)$, $(R, PQ)$, $(L, AB)$ and $(L, PQ)$ commute
at $z$, then $A$, $B$, $S$, $T$, $P$, $Q$, $R$ and $L$ have a unique common fixed point.

**Proof.** Let $x_0$ be an arbitrary point in $X$. Since $AB(X) \subseteq RL(X)$ we can find a point $x_1$ in $X$ such that $ABx_0 = RLx_1$. Similarly since $ST(X) \subseteq PQ(X)$, we can further choose a point $x_2$ in $X$ such that $STx_1 = PQx_2$. On continuing this argument, we can construct a sequence $\{y_n\}$ in $X$ such that

$$y_{2n} = ABx_{2n} = RLx_{2n+1} \quad \text{and} \quad y_{2n+1} = STx_{2n+1} = PQx_{2n+2} \quad \text{for} \quad n = 0, 1, 2, \ldots .$$

For the simplicity, we shall write

$$u_{2n} = d(ABx_{2n}, STx_{2n+1}) = d(y_{2n}, y_{2n+1}) \quad \text{and} \quad u_{2n+1} = d(STx_{2n+1}, ABx_{2n+2}) = d(y_{2n+1}, y_{2n+2}),$$

for $n = 0, 1, 2, \ldots .$

**Case I:** Suppose $u_{2n} + u_{2n+1} \neq 0$ for $n = 0, 1, 2, \ldots$. Then on using (2.6.2), we have

$$u_{2n+1} = d(y_{2n+1}, y_{2n+2}) = d(STx_{2n+1}, ABx_{2n+2}) = d(ABx_{2n+2}, STx_{2n+1})$$

$$\leq \alpha \left[ \frac{d(ABx_{2n+2}, PQx_{2n+2})}{d(ABx_{2n+2}, PQx_{2n+2}) + d(STx_{2n+1}, RLx_{2n+1})} \right]^2 + \frac{d(STx_{2n+1}, RLx_{2n+1})}{d(ABx_{2n+2}, PQx_{2n+2}) + d(STx_{2n+1}, RLx_{2n+1})} + \beta d(PQx_{2n+2}, RLx_{2n+1})$$

$$+ \gamma [d(ABx_{2n+2}, RLx_{2n+1}) + d(STx_{2n+1}, PQx_{2n+2})],$$

$$\leq \alpha \left[ \frac{d(y_{2n+2}, y_{2n+1})}{d(y_{2n+2}, y_{2n+1}) + d(y_{2n+1}, y_{2n})} \right]^2 + \frac{d(y_{2n+1}, y_{2n})}{d(y_{2n+2}, y_{2n+1}) + d(y_{2n+1}, y_{2n})} + \beta d(y_{2n+1}, y_{2n})$$

$$+ \gamma [d(y_{2n+1}, y_{2n}) + d(y_{2n+1}, y_{2n+2})],$$

$$\leq \alpha \left[ \frac{u_{2n+1}^2 + u_{2n}^2}{u_{2n+2}^2 + u_{2n}^2} \right] + \beta u_{2n} + \gamma [u_{2n} + u_{2n+1}].$$
\[ u_{2n+1}(u_{2n} + u_{2n-1}) \leq \alpha u_{2n}^2 + \alpha u_{2n-1}^2 + \beta u_{2n}u_{2n-1} + \gamma(u_{2n} + u_{2n-1})^2 \]

\[ u_{2n+1}^2 + u_{2n}u_{2n+1} \leq \alpha u_{2n+1}^2 + \alpha u_{2n}^2 + \beta u_{2n}u_{2n+1} + \gamma u_{2n+1}^2 + \gamma u_{2n+1}^2 + 2\gamma u_{2n}u_{2n+1} \]

\[ (1 - \alpha - \gamma)u_{2n+1}^2 + (1 - \beta - 2\gamma)u_{2n+1}u_{2n} - (\alpha + \beta + \gamma)u_{2n}^2 \leq 0. \]

\[ (1 - \alpha - \gamma)\left( \frac{u_{2n+1}}{u_{2n}} \right)^2 + (1 - \beta - 2\gamma)\left( \frac{u_{2n}}{u_{2n-1}} \right) - (\alpha + \beta + \gamma) \leq 0 \quad \cdots \quad (*) \]

\[ \frac{u_{2n+1}}{u_{2n}} \leq \frac{(2\gamma + \beta - 1) \pm \sqrt{(1 - \beta - 2\gamma)^2 + 4(1 - \alpha - \gamma)(\alpha + \beta + \gamma)}}{2(1 - \alpha - \gamma)} = \lambda < 1. \]

Since \( \lambda < 1 \) \[ \frac{\sqrt{(1 - \beta - 2\gamma)^2 + 4(1 - \alpha - \gamma)(\alpha + \beta + \gamma) - (1 - \beta - 2\gamma)}}{2(1 - \alpha - \gamma)} < 1, \]

\[ \pm \sqrt{(1 - \beta - 2\gamma)^2 + 4(1 - \alpha - \gamma)(\alpha + \beta + \gamma) - (1 - \beta - 2\gamma)} < 2(1 - \alpha - \gamma) + (1 - \beta - 2\gamma). \]

\[ (1 - \alpha - \gamma)(\alpha + \beta + \gamma) < (1 - \alpha - \gamma)^2 + (1 - \alpha - \gamma)(1 - \beta - 2\gamma), \]

\[ \alpha + \beta + \gamma < 2 - \alpha - \beta - 3\gamma \quad \Rightarrow \quad 2\alpha + 2\beta + 4\gamma < 2 \text{ or } \alpha + \beta + 2\gamma < 1 \text{ which is true as per hypothesis.} \]

It follows that \( u_{2n+1} \leq \lambda u_{2n} \). Similarly for \( u_{2n} + u_{2n-1} \neq 0, n = 0, 1, 2, 3, \ldots u_{2n} \leq \lambda u_{2n-1} \). In general, we have \( u_{n+1} \leq \lambda u_n, n = 0, 1, 2, 3, \ldots \), which implies that

\[ u_n \leq \lambda u_{n-1} \leq \lambda^2 u_{n-2} \leq \ldots \leq \lambda^n u_0 \to 0 \text{ as } n \to \infty, \text{ or } d(y_n, y_{n+1}) < \lambda^0 d(y_1, y_0). \]

Now for \( m \geq n \), we have

\[ d(y_n, y_m) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots + d(y_{m-1}, y_m) \]
\[< \lambda^n d(y_1, y_0) + \lambda^{n+1} d(y_1, y_0) + \ldots + \lambda^m d(y_1, y_0),\]

\[< (\lambda^n + \lambda^{n+1} + \ldots + \lambda^m) d(y_1, y_0) < [\lambda^n/(1 - \lambda)] d(y_1, y_0) \to 0 \text{ as } n \to \infty,\]

yields that \(\{y_n\}\) is a Cauchy sequence in \(X\). By completeness of \(X\), \(\{y_n\}\) and its subsequences also \(\{ABx_{2n}\}, \{RLx_{2n+1}\}, \{STx_{2n+1}\}\) and \(\{PQx_{2n+2}\}\) converges to the same point say \(z\) in \(X\).

Let us now assume that \(PQ\) is continuous, so that \(PQ^2x_{2n} = PQ (PQx_{2n}) \to PQz\) and \((PQ)(AB)x_{2n} \to PQz\). Also in view of compatibility of \((PQ, AB)\),

\[d((PQ)(AB)x_{2n}, (AB)(PQ)x_{2n}) \to 0 \text{ as } n \to \infty,\]

which implies that

\[d(PQz, (AB)(PQ)x_{2n}) \to 0 \text{ as } n \to \infty,\]

yields \((AB)(PQ)x_{2n} \to PQz\).

Using (2.6.2), we have

\[d((AB)(PQ)x_{2n}, STx_{2n+1}) \leq C \left[ \frac{d((AB)(PQ)x_{2n}, (PQ)^2 x_{2n})^2 + d(STx_{2n+1}, RLx_{2n+1})^2}{d((AB)(PQ)x_{2n}, (PQ)^2 x_{2n}) + d(STx_{2n+1}, RLx_{2n+1})} \right] + \beta d((PQ)^2 x_{2n}, RLx_{2n+1}) + \gamma [d((AB)(PQ)x_{2n}, RLx_{2n+1}) + d(STx_{2n+1}, (PQ)^2 x_{2n})].\]

Letting \(n \to \infty\), we have

\[d(PQz, z) \leq (\beta + \gamma) d(PQz, z),\]

yields \(PQz = z\).

By (3.2.2), we have
\[ d(ABz, STx_{2n+1}) \leq \alpha \left[ \frac{\{d(ABz, PQz)\}^2 + \{d(STx_{2n+1}, RLx_{2n+1})\}^2}{d(ABz, PQz) + d(STx_{2n+1}, RLx_{2n+1})} \right] + \beta d(PQz, RLx_{2n+1}) + \gamma \left[ d(ABz, RLx_{2n+1}) + d(STx_{2n+1}, PQz) \right]. \]

Letting \( n \to \infty \) and using \( PQz = z \), we get \( d(ABz, z) \leq (\alpha + \gamma) d(ABz, z) \), yields \( ABz = z \).

Hence \( PQz = ABz = z \), i.e. \( z \) is a common fixed point of \( PQ \) and \( AB \).

Since \( AB(X) \subset RL(X) \) there exists a point \( z' \) in \( X \) such that \( RL z' = z \), so that

\[ STz = ST(RL z'). \]

Now using (2.6.2), we have

\[ d(z, ST z') = d(ABz, ST z') \leq \alpha \left[ \frac{\{d(ABz, PQz)\}^2 + \{d(STz', RLz')\}^2}{d(ABz, PQz) + d(STz', RLz')} \right] + \beta d(PQz, RL z') + \gamma \left[ d(ABz, RL z') + d(STz', PQz) \right] = (\alpha + \gamma) d(STz', z), \]

yields that \( ST z' = z = RL z' \), i.e. \( z' \) is the coincidence point of \( ST \) and \( RL \). Now using the Weakly of \((ST, RL)\), we have \( STz = ST(RL z') = RL(ST z') = RLz \) which shows that \( z \) is also a coincidence point of the pair \((ST, RL)\).

Using (2.6.2), we have

\[ d(z, STz) = d(ABz, STz) \leq \alpha \left[ \frac{\{d(ABz, PQz)\}^2 + \{d(STz, RLz)\}^2}{d(ABz, PQz) + d(STz, RLz)} \right] + \beta d(PQz, RLz) + \gamma \left[ d(ABz, RLz) + d(STz, PQz) \right] = 0, \]
since \( d(ABz, PQz) + d(STz, RLz) = 0 \). Hence \( z = STz = RLz = ABz = PQz \), i.e. \( z \) is the common fixed point of \( AB, PQ, ST \) and \( RL \).

Further we shall prove that \( z \) is the common fixed point of \( AB, PQ, ST \) and \( RL \) if \( AB \) is continuous.

Now assume that \( AB \) is continuous so that \( AB^2x_{2n} = (AB)(ABx_{2n}) \rightarrow ABz \) and \( (AB)(PQ)x_{2n} \rightarrow ABz \). Also in view of compatibility of \( (PQ, AB) \),

\[
d((PQ)(AB)x_{2n}, (AB)(PQ)x_{2n}) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty,
\]

which implies that

\[
d(ABz, (PQ)(AB)x_{2n}) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty,
\]
yields that \( (PQ)(AB)x_{2n} \rightarrow ABz \).

Using (3.2.2), we have

\[
d((AB)(AB)x_{2n}, STx_{2n+1}) \leq \alpha \left[ \frac{d((AB)^2x_{2n}, PQ(AB)x_{2n})^2 + d(STx_{2n+1}, RLx_{2n+1})^2}{d((AB)^2x_{2n}, PQ(AB)x_{2n}) + d(STx_{2n+1}, RLx_{2n+1})} \right]
\]

\[+ \beta d((PQ)(AB)x_{2n}, RLx_{2n+1})
\]
\[+ \gamma [d((AB)^2x_{2n}, RLx_{2n+1}) + d(STx_{2n+1}, (PQ)(AB)x_{2n})].
\]

On letting \( n \rightarrow \infty \), we have

\[
d(ABz, z) \leq (b + 2\gamma)d(ABz, z),
\]
yields that \( ABz = z \) and hence \( z = ABz = RLz' \).

Then by (3.2.2), we have

\[
d((AB)^2x_{2n}, STz') \leq \alpha \left[ \frac{d((AB)^2x_{2n}, PQ(AB)x_{2n})^2 + d(STz', RLz')^2}{d((AB)^2x_{2n}, PQ(AB)x_{2n}) + d(STz', RLz')} \right]
\]
+ βd(PQ(AB) x_{2n}, RLz')

+ γ[d(AB) x_{2n}, RLz') + d(STz', PQ(AB) x_{2n})],

on letting n → ∞, we have d(z, STz') ≤ (α + γ)d(z, STz') yields that STz' = z = RLz', i.e. z' is the coincidence point of (ST, RL). Now by weakly compatibility of (ST, RL),

STz = ST(RLz') = RL(STz') = RLz implies that z is the coincidence point of (ST, RL).

Further from using (3.2.2), we have

\[ d(AB x_{2n}, STz) \leq \alpha \left[ \frac{[d(AB_{x_{2n}}, PQ_{x_{2n}})]^2 + [d(STz, RLz)]^2}{d(AB_{x_{2n}}, PQ_{x_{2n}}) + d(STz, RLz)} \right] + βd(PQ_{x_{2n}}, RLz) \]

+ γ[d(AB_{x_{2n}}, RLz) + d(STz, PQ_{x_{2n}})],

on letting n → ∞, we have d(z, STz) ≤ (β + 2γ)d(z, STz') yields that STz = z = RLz.

Also since z ∈ ST(X) ⊂ PQ(X) there exists a point z'' in X such that PQz'' = z. Then from (3.2.2), we have

\[ d(ABz'', z) = d(ABz'', STz) \leq \alpha \left[ \frac{[d(ABz'', PQz'')]^2 + [d(STz, RLz)]^2}{d(ABz'', PQz'') + d(STz, RLz)} \right] \]

+ βd(PQz'', RLz) + γ[d(ABz'', RLz) + d(STz, PQz'')] \]

\[ \leq (α + γ)d(ABz'', z), \text{ implies that } ABz'' = z. \]

Now since (AB, PQ) are compatible hence weakly compatible therefore
d(ABz, PQz) = d(AB(PQz′), PQ(AB)z′) = 0 which implies that ABz = PQz = z. Hence z is the common fixed point of AB, PQ, ST and RL.

If the pair (ST, RL) is compatible and ST or RL is continuous and the pair (AB, PQ) is weakly compatible then by similar manner it can be proved that z is the common fixed point of AB, PQ, ST and RL.

If u is another common fixed point of (ST, RL) then

\[ d(ABz, STu) + d(STu, RLu) = d(z, z) + d(u, u) = 0 \]

and so from (2.6.2), we have

\[ d(ABz, STu) = 0 \Rightarrow d(z, u) = 0 \]

so that z = u, i.e. z is the unique common fixed point of (ST RL). Similarly it can be proved that z is the unique common fixed point of (AB, PQ).

Now the question arises whether z is the unique common fixed point of AB, PQ, ST and RL. To check this suppose that v is another common fixed point of AB, PQ, ST and RL, then by (2.6.2), we have

\[ d(z, v) = d(ABz, STv) = 0, \text{ since } d(ABz, PQz) + d(STv, RV) = 0. \]

Hence z = v, i.e. z is the unique common fixed point of AB, PQ, ST and RL.

Now from (2.6.4), we have

\[ A(z) = A(ABz) = A(BAz) = (AB)Az ; \quad A(z) = A(PQz) = (PQ)Az \]

and

\[ B(z) = B(ABz) = (BA)Bz = (AB)Bz ; \quad B(z) = B(PQz) = (PQ)Bz. \]

It follows that Az and Bz are common fixed points of (AB, PQ). But since z is the unique common fixed point of (AB, PQ), we have z = Az = Bz = ABz = PQz. Similarly it
can be proved that \( z = S z = T z = S T z = R L z \). Hence \( z \) is the unique common fixed point of \( A, B, S, T, A B, P Q, S T \) and \( R L \).

Further from (2.6.5), we have

\[
Pz = P(ABz) = (AB)Pz; \quad Pz = P(PQz) = P(QPz) = (PQ)Pz.
\]

and

\[
Qz = Q(ABz) = (AB)Qz; \quad Qz = Q(PQz) = (QP)Qz = (PQ)Qz.
\]

Similarly

\[
Rz = R(ABz) = (AB)Rz; \quad Rz = R(PQz) = (PQ)Rz.
\]

and

\[
Lz = L(ABz) = (AB)Lz; \quad Lz = L(PQz) = (PQ)Lz.
\]

Hence \( Pz, Qz \) are the common fixed points of \( AB, PQ \), but by the uniqueness of \( z, z = Pz = Qz \). Similarly \( Rz, Lz \) are the common fixed points of \( AB, PQ \), implies that \( z = Rz = Lz \). Hence \( z \) is the unique common fixed point of \( A, B, R, S, T, P, Q, L \).

**Case II:** Suppose that \( u_n + u_{n+1} = 0 \) for some \( n \). Then \( y_n = y_{n+1} = y_{n-2} \). For \( n = 2k \), we have \( y_{2k+2} = ABx_{2k+2} = PQx_{2k+2} \), so there exists \( v_1, w_1 \) such that \( v_1 = ABw_1 = PQw_1 \). Similarly, there exist \( v_2, w_2 \) such that \( v_2 = STw_2 = RLw_2 \).

Since \( d(ABw_1, PQw_1) + d(STw_2, RLw_2) = 0 \), from (2.6.2), we have \( d(ABw_1, STw_2) = 0 \), i.e., \( v_1 = ABw_1 = STw_2 = v_2 \). Further \( PQv_1 = PQ(ABw_1) = AB(PQw_1) = ABv_1 \) (since \( w_1 \) is the coincidence point of \( AB, PQ \)). Similarly \( STv_2 = RLv_2 \). Define \( y_1 = ABv_1, y_2 = STv_2 \). Since \( d(ABv_1, PQv_1) + d(STv_2, RSv_2) = 0 \), it follows from (2.6.2) that \( d(ABv_1, STv_2) = 0 \), i.e. \( y_1 = ABv_1 = STv_2 = y_2 \). Thus \( ABv_1 = PQv_1 = STv_2 = RLv_2 \). But \( v_1 = v_2 \), therefore \( v_1 \) is the coincidence point of \( AB, ST, PQ \) and \( RL \).
Now we define \( w = ABv_1 \), then follows that \( w \) is also a common coincidence point of \( AB, ST, PQ \) and \( RL \). If \( ABw \neq ABv_1 = STv_1 \), then \( d(ABw, STv_1) > 0 \). But since \( d(ABw, PQw) + d(STv_1, RLv_1) = 0 \), it follows from (2.6.2) that \( d(ABw, STv_1) = 0 \) implies that \( ABw = STv_1 \), which is a contradiction. Therefore \( ABw = ABv_1 = w \) and \( w \) is a common fixed point of \( AB, ST, PQ \) and \( RL \). The rest of the proof is identical to the case I. This completes the proof.

**Corollary 2.7.** Theorem 2.6 remains true if the condition (2.6.2) is replaced by any one of the following conditions:

\[
\text{(2.7.1)} \quad d(ABx, STy) \leq \alpha \left[ \frac{(ABx, PQx)^2 + (STy, R Ly)^2}{d(ABx, PQx) + d(STy, R Ly)} \right] + \gamma (d(ABx, R Ly) + d(STy, PQx))
\]

if \( d(ABx, PQx) + d(STy, R Ly) \neq 0 \), \( \alpha, \gamma > 0 \), \( \alpha + 2\gamma < 1 \)

Or

\[ d(ABx, STy) = 0 \] if \( d(ABx, PQx) + d(STy, R Ly) = 0 \).

\[
\text{(2.7.2)} \quad d(ABx, STy) \leq \alpha \left[ \frac{(ABx, PQx)^2 + (STy, R Ly)^2}{d(ABx, PQx) + d(STy, R Ly)} \right] + \beta d(PQx, R Ly)
\]

if \( d(ABx, PQx) + d(STy, R Ly) \neq 0 \), \( \alpha, \beta > 0 \), \( \alpha + \beta < 1 \),

Or

\[ d(ABx, STy) = 0 \] if \( d(ABx, PQx) + d(STy, R Ly) = 0 \).

\[
\text{(2.7.3)} \quad d(ABx, STy) \leq \alpha \left[ \frac{(ABx, PQx)^2 + (STy, R Ly)^2}{d(ABx, PQx) + d(STy, R Ly)} \right]
\]

if \( d(ABx, PQx) + d(STy, R Ly) \neq 0 \), \( 0 < \alpha < 1 \),
Or \[ d(ABx, STy) = 0 \] if \[ d(ABx, PQx) + d(STy, RLy) = 0. \]

(2.7.4) \[ d(ABx, STy) \leq \alpha[d(ABx, PQx) + d(STy, RLy)] + \beta d(PQx, RLy) \]

\[ + \gamma[d(ABx, RLy) + d(STy, PQx)] \] if \( \alpha + \beta + 2\gamma < 1. \)

(2.7.5) \[ d(ABx, STy) \leq \alpha[d(ABx, PQx) + d(STy, RLy)] \] if \( \alpha < 1/2. \)

(2.7.6) \[ d(ABx, STy) \leq \gamma[d(ABx, RLy) + d(STy, PQx)] \] if \( \gamma < 1/2 \)

(2.7.7) \[ d(ABx, STy) \leq \beta d(PQx, RLy) \] if \( \beta < 1. \)

**Proof.** Conditions (2.7.1), (2.7.2), (2.7.3) are the special cases of condition (2.6.2) for \( \beta = 0, \gamma = 0 \) and \( \beta = \gamma = 0 \) respectively. Further since

\[
\frac{[d(ABx, PQx)]^2 + [d(STy, RLy)]^2}{d(ABx, PQx) + d(STy, RLy)} \leq \frac{[d(ABx, PQx)]^2 + [d(STy, RLy)]^2}{d(ABx, PQx) + d(STy, RLy)}
\]

\[ = d(ABx, PQx) + d(STy, RLy). \]

Then by (2.6.2) condition (2.7.4) can be obtained easily. Contraction conditions (2.7.5), (2.7.6) and (2.7.6) are the special cases to the contraction condition (2.6.4).

**Remarks**

1. If we put \( L = Q = I \), the Identity map, then we obtain the results of [101] as the special case of our results.

2. Our results are generalization and extension of many previous results such as:

(i) Condition (2.7.1) is an extension of the results of Fisher [78] and Kannan [127,128].
(ii) Condition (2.7.2) is an extension and generalizes the results of Jeong-Rhoades [114], Ahmad and Imdad [7].

(iii) Condition (2.7.3) is an extension of the results of Fisher [78].

(iv) Condition (2.7.4) is an extension of the results of Hardy-Rogers [94].

(v) Conditions (2.7.5), (2.7.6) and (2.7.7) are extensions of the results of Fisher [78] and Kannan [127,128] and many others.

Theorem 2.8. Theorem 2.6 remains true if the condition of ‘compatibility’ replaced by the condition compatibility of type (A) or compatibility of type (B) or compatibility of type (C) or compatibility of type (P) or Weakly of type (A).

Now we furnish an example to demonstrate the validity of the hypothesis and the degree of generality of our main theorem.

Example 2.6. Consider $X = [0, 1]$ with the usual metric. Define self-mappings $A$, $B$, $S$, $T$, $P$, $Q$, $R$ and $L$ as $A(x) = 2x/3$, $B(x) = 3x/4$, $S(x) = x/4$, $T(x) = 4x/5$, $P(x) = x$, $Q(x) = x/4$, $R(x) = 3x/2$ and $L(x) = x/2$.

Clearly $AB(X) = [0, 1/2] \subseteq RL(X) = [0, 1/2]$, $ST(X) = [0, 1/5] \subseteq PQ(X) = [0, 1/4]$. Also the pairs of mappings $(AB, PQ)$, $(ST, RL)$, $(A, B)$, $(A, PQ)$, $(B, PQ)$, $(S, T)$, $(S, RL)$, $(T, RL)$, $(P, AB)$, $(P, Q)$, $(Q, AB)$, $(L, PQ)$, $(R, AB)$, $(R, PQ)$ and $(L, AB)$ are commuting maps there for the pairs $(AB, PQ)$, $(ST, RL)$ are compatible and weakly compatible maps. Maps $A$, $B$, $S$, $T$, $P$, $Q$, $R$ and $L$ are continuous and so their compositions are also continuous.
Now for all \( x, y \) in \( X \) (\( x > y \)) with \( \alpha = 1/5, \beta = 1/20 \) and \( \gamma = 1/3 \); \( \alpha + \beta + 2\gamma = 3/4 < 1 \) and

\[
d(ABx, STy) = \left| \frac{x - y}{2} - \frac{y}{5} \right|.
\]

On the other hand

\[
\alpha \left[ \frac{\{d(ABx, PQx)\}^2 + \{d(STy, RLy)\}^2}{d(ABx, PQx) + d(STy, RLy)} \right] + \beta d(PQx, RLy) + \gamma [d(ABx, RLy) + d(STy, PQx)],
\]

\[
= \frac{1}{5} \left[ \frac{x - x}{2} + \frac{y - 3y}{4} \right] + \frac{1}{20} \left[ \frac{x - 3y}{4} + \frac{y - x}{2} \right] + \frac{1}{3} \left[ \frac{x - 3y}{4} + \frac{y - x}{5} \right].
\]

\[
< \frac{1}{5} \left[ \frac{x - x}{2} + \frac{y - 3y}{4} \right] + \frac{1}{20} \left[ \frac{x - 3y}{4} + \frac{y - x}{2} \right] + \frac{1}{3} \left[ \frac{x - 3y}{4} + \frac{y - x}{5} \right],
\]

[Since \( \frac{a^2 + b^2}{a + b} < \frac{a^2 + b^2 + 2ab}{a + b} = (a + b) \)]

\[
\leq \frac{1}{5} \left[ \frac{x - y}{5} + \frac{x - y}{5} \right] + \frac{1}{20} \left[ \frac{x - y}{5} + \frac{x - y}{5} \right] + \frac{1}{3} \left[ \frac{x - y}{5} + \frac{x - y}{5} \right],
\]

[since \( \frac{x}{2} > \frac{x}{4} > \frac{x}{5} > \frac{y}{5} \) and \( \frac{3y}{4} > \frac{y}{5} \) for \( x > y \)]

\[
= (2/5 + 1/20 + 2/3) \left| \frac{x - y}{2} - \frac{y}{5} \right| = (87/60) \left| \frac{x - y}{2} - \frac{y}{5} \right|,
\]

which satisfies the contraction condition (3.2.2)? Clearly 0 is the unique common fixed point of \( A, B, S, T, P, Q, R \) and \( L \).