Chapter 4

Cracks and disk in bounded isotropic medium
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4.1 Shear wave interaction by edge crack *

1. Introduction

Problems which deal with inclusions or cracks in elastic media have occupied an important place in the development of the theory of elasticity. Such solutions also have a wide range of applications in construction engineering. Cracks or inclusions are present in essentially all structural materials, either as natural defects or as a result of fabrication processes. In the presence of finite boundaries, the problems become more complicated as the interactions between the scattered waves from the crack edge and the reflected waves from the boundary of the plane start to occur. Many problems have been solved involving one or more cracks in an infinite elastic medium. Loeber and Sih (1960) and Mal (1970) have studied the problem of diffraction of elastic waves by a Griffith crack in an infinite medium. The problem of an infinite elastic strip containing an arbitrary number of unequal Griffith cracks, located parallel to its surfaces and opened by an arbitrary internal pressure has been treated by Adams (1980). Finite crack perpendicular to the surface of the infinitely long elastic strip has been studied by Chen (1978) (for an impact load) and by Srivastava et al. (2006) (for normally

incident waves).

But papers related to edge crack problem are very few. Ishida (1953) and De et al. (1989) treated the problem of edge crack in elastic strip and elastic half plane respectively. Matysiak ad pauk (1970) studied the edge crack in an elastic layer resting on Winkler foundation. An edge crack problem in a semi-infinite plane subjected to concentrated forces was studied by Chen et al. (2001). Nasima Munshi et al. (2003) solved the P-wave edge crack problem in an infinitely long elastic strip.

In our paper, we have treated the diffraction of SH-wave by an edge crack in an infinitely long elastic strip. Applying the Fourier transform, the mixed boundary value problem is converted to the solution of dual integral equations. The dual integral equations have been finally reduced to a Fredholm integral equation of second kind by applying Abel’s transform. Expression for the stress intensity factor (SIF) and crack opening displacement have been plotted. Also stress (scattered field) outside the crack has been calculated and shown by three dimensional graph.

2. Formulation of the problem

![Figure 4.1: Geometry of the crack.](image)

We consider the problem of diffraction of SH-wave by a finite edge crack in an infinitely long elastic strip of width \( h_1 \). The crack is located in the region \( 0 \leq x_1 \leq a, \) \( -\infty < z_1 < \infty, y_1 = 0 \). Normalizing all the lengths with respect to ‘\( a \)’ and putting
$x_1 = x$, $y_1 = y$, $z_1 = z$, $h_1 = h$, it is found that the location of the crack is $0 \leq x \leq 1$, $-\infty < z < \infty$, $y = 0$, (figure 4.1) referred to cartesian coordinate system $(x, y, z)$.

Let a normally incident time-harmonic SH-wave travels in the direction of the positive $y$–axis. The only non-vanishing $z$–component of displacement which is independent of $z$ is $w(x, y, t) = W(x, y)e^{-i\omega t}$.

Now our problem reduces to the solution of the equation

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + k_2^2 W = 0 \quad (4.1.1)$$

where $k_2 = \frac{a\omega}{c^2}$,

subject to the boundary conditions

$$\tau_{yz}(x, 0) = \tau_0 e^{-i\omega t}, \quad 0 \leq x \leq 1, \quad (4.1.2)$$

$$W(x, 0) = 0, \quad 1 \leq x \leq h, \quad (4.1.3)$$

$$\tau_{xz}(0, y) = 0, \quad |y| < \infty, \quad (4.1.4)$$

$$\tau_{xz}(h, y) = 0, \quad |y| < \infty. \quad (4.1.5)$$

Henceforth, the time factor $e^{-i\omega t}$ which is common to all field variables would be omitted in the sequel.

The non vanishing stresses are

$$\tau_{yz} = \mu \frac{\partial W}{\partial y}, \quad (4.1.6)$$

$$\tau_{xz} = \mu \frac{\partial W}{\partial x}. \quad (4.1.7)$$

The solution of equation (4.1.1) can be taken as

$$W(x, y) = \int_{-\infty}^{\infty} A(\xi)e^{-\alpha \xi}e^{i\xi x}d\xi + \int_{0}^{\infty} [B(\zeta)e^{\beta \zeta} + C(\zeta)e^{-\beta \zeta}] \sin(\zeta y)d\zeta, \quad y > 0, \quad (4.1.8)$$

where $\alpha = \sqrt{\xi^2 - k_2^2}$, $\xi > k_2$,

$$\alpha = -i\sqrt{k_2^2 - \xi^2}, \quad \xi < k_2,$$

and $\beta = \sqrt{\zeta^2 - k_2^2}$, $\zeta > k_2$,

$$\beta = -i\sqrt{k_2^2 - \zeta^2}, \quad \zeta < k_2.$$

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Therefore the expression of stresses are
\[ a\tau_{yx}(x, y) = -\mu \int_{-\infty}^{\infty} \alpha A(\xi)e^{-\alpha y} e^{i\xi x} d\xi + \mu \int_{0}^{\infty} \zeta [B(\zeta)e^{\beta x} + C(\zeta)e^{-\beta x}] \cos(\zeta y) d\zeta, \quad (4.1.9) \]
\[ a\tau_{xz}(x, y) = i\mu \int_{-\infty}^{\infty} \xi A(\xi)e^{-\alpha y} e^{i\xi x} d\xi + \mu \int_{0}^{\infty} \beta [B(\zeta)e^{\beta x} - C(\zeta)e^{-\beta x}] \sin(\zeta y) d\zeta. \quad (4.1.10) \]

From the boundary conditions (4.1.4) and (4.1.5), B(\zeta) and C(\zeta) can be found to be
\[ B(\zeta) = \frac{2i\zeta}{\pi \beta (e^{2\beta \zeta} - 1)} \left[ \int_{-\infty}^{\infty} \frac{\xi A(\xi)}{\alpha^2 + \zeta^2} d\xi - e^{h\beta} \int_{-\infty}^{\infty} \frac{\xi A(\xi)e^{ih\xi}}{\alpha^2 + \zeta^2} d\xi \right] \quad (4.1.11) \]
\[ C(\zeta) = \frac{2i\zeta}{\pi \beta (1 - e^{-2h\beta})} \left[ \int_{-\infty}^{\infty} \frac{\xi A(\xi)}{\alpha^2 + \zeta^2} d\xi - e^{-h\beta} \int_{-\infty}^{\infty} \frac{\xi A(\xi)e^{ih\xi}}{\alpha^2 + \zeta^2} d\xi \right] \quad (4.1.12) \]

Now, from boundary conditions (4.1.2) and (4.1.3), we obtain the following dual integral equations for the determination of the unknown function A(\xi):
\[ \int_{-\infty}^{\infty} \alpha A(\xi)e^{i\xi x} d\xi = p(x), \quad 0 \leq x \leq 1, \quad (4.1.13) \]
\[ \int_{-\infty}^{\infty} A(\xi)e^{i\xi x} d\xi = 0, \quad 1 \leq x \leq h, \quad (4.1.14) \]
where \[ p(x) = \frac{\tau_0 a}{\mu} + \int_{0}^{\infty} \zeta [B(\zeta)e^{\beta x} + C(\zeta)e^{-\beta x}] d\zeta. \quad (4.1.15) \]

3. Method of Solution

In order to reduce the dual integral equations (4.1.13) and (4.1.14) to a single Fredholm integral equation, we assume that
\[ A(\xi) = \frac{\tau_0 a}{2\mu} \int_{0}^{1} t g(t) J_0(\xi t) dt \quad (4.1.16) \]
so that equation (4.1.14) is automatically satisfied.

Equation (4.1.13) can be written as
\[ \int_{0}^{1} t g(t) \int_{0}^{\infty} \xi [1 + H(\xi)] J_0(\xi t) \cos(\xi x) d\xi dt = \frac{\mu p(x)}{\tau_0 a}, \quad 0 \leq x \leq 1 \quad (4.1.17) \]
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where

\[ H(\xi) = \left( \frac{\alpha}{\xi} - 1 \right) \]  \hspace{1cm} (4.1.18)

Now,

\[ \alpha = (\xi^2 - k_2^2)^{\frac{1}{2}} = \xi \left( 1 - \frac{k_2^2}{\xi^2} \right)^{\frac{1}{2}} \rightarrow \xi \rightarrow \infty \]

\[ \Rightarrow \frac{\alpha}{\xi} \rightarrow 1 \text{ as } \xi \rightarrow \infty \]

\[ \Rightarrow H(\xi) = \left( \frac{\alpha}{\xi} - 1 \right) \rightarrow 0 \text{ as } \xi \rightarrow \infty \]

Using Abel’s transform in equation (4.1.17) we obtain the following Fredholm integral equation of second kind:

\[ g(t) + \int_0^1 u g(u) L(u, t) du = 1 \]  \hspace{1cm} (4.1.19)

where

\[ L(u, t) = L_1(u, t) - L_2(u, t) - L_3(u, t), \]  \hspace{1cm} (4.1.20)

\[ L_1(u, t) = \int_0^\infty \xi H(\xi) J_0(\xi u) J_0(\xi t) d\xi, \]  \hspace{1cm} (4.1.21)

\[ L_2(u, t) = \int_0^\infty \frac{\zeta^2 I_0(\beta u)}{\beta (e^{2\beta t} - 1)} [I_0(\beta t) + L_0(\beta t)] d\zeta, \]  \hspace{1cm} (4.1.22)

\[ L_3(u, t) = \int_0^\infty \frac{\zeta^2 I_0(\beta u) e^{-2\beta t}}{\beta (1 - e^{-2\beta t})} [I_0(\beta t) - L_0(\beta t)] d\zeta, \]  \hspace{1cm} (4.1.23)

\( I_0() \) is the modified Bessel function of imaginary argument of order zero and \( L_0() \) is the modified struve function of order zero.

In order to make the numerical analysis easier, the semi-infinite integral has therefore been converted to finite integrals by using simple contour integration technique. The integral (4.1.21) has no poles, it has only branch point at \( \xi = k_2 \).

Let

\[ H(\xi) = M(\alpha, \xi) = \left( \frac{\alpha}{\xi} - 1 \right) \]
Let us consider the following two integrals.

\[ I_1 = \int_{C_1} \xi M(\alpha, \xi) J_0(\xi t) H_0^{(1)}(\xi u) d\xi, \quad u > t \]

and

\[ I_2 = \int_{C_2} \xi M(\alpha, \xi) J_0(\xi t) H_0^{(2)}(\xi u) d\xi, \quad u > t \]

and \( C_1, C_2 \) are the contours shown in the following figure:

![Figure 4.2: Contour of the integral \( L_1(u, t) \)](image)

where

\[
\begin{align*}
\alpha &= \sqrt{\xi^2 - k_2^2} \\
\alpha' &= \sqrt{k_2^2 - \xi^2} \\
s &= \sqrt{\eta^2 + k_2^2}
\end{align*}
\]

For contour \( C_1 \)

\[
\begin{align*}
\int_0^{k_2} \xi M(i\alpha', \xi) J_0(\xi t) H_0^{(1)}(\xi u) d\xi + \int_{k_2}^{\infty} \xi M(\alpha, \xi) J_0(\xi t) H_0^{(1)}(\xi u) d\xi \\
+ \int_{-\infty}^0 i\eta M(is, i\eta) J_0(i\eta t) H_0^{(1)}(i\eta u) i \, d\eta &= 0 \quad (4.1.24)
\end{align*}
\]
For contour $C_2$

$$
\int_0^{k_2} \xi M(-i\alpha', \xi) J_0(\xi t) H_0^{(2)}(\xi u) \, d\xi + \int_{k_2}^\infty \xi M(\alpha, \xi) J_0(\xi t) H_0^{(2)}(\xi u) \, d\xi \\
+ \int_0^0 i\eta M(-i\eta, -i\eta) J_0(-i\eta t) H_0^{(2)}(-i\eta u) \, i \, d\eta = 0
$$

$$
\Rightarrow \quad 2L_1(u, t) - \int_0^{k_2} \xi M(-i\alpha', \xi) J_0(\xi t) H_0^{(1)}(\xi u) \, d\xi - \int_{k_2}^\infty \xi M(\alpha, \xi) J_0(\xi t) H_0^{(1)}(\xi u) \, d\xi \\
+ \int_0^0 i\eta M(-i\eta, -i\eta) J_0(-i\eta t) H_0^{(2)}(-i\eta u) \, i \, d\eta = 0 \quad (4.1.25)
$$

Now,

$$
H_0^{(2)}(-i\eta u) = -H_0^{(1)}(i\eta u)
$$

$$
J_0(-i\eta u) = J_0(i\eta u)
$$

$$
M(is, i\eta) = M(-is, -i\eta) = \frac{s}{\eta} - 1
$$

$$
M(i\alpha', \xi) - M(-i\alpha', \xi) = \frac{i\alpha'}{\xi} - 1 - \left( -\frac{i\alpha'}{\xi} - 1 \right) = \frac{2i\alpha'}{\xi}
$$

Adding (4.1.24) and (4.1.25) and using the above relations, we get

$$
L_1(u, t) = -i \int_0^{k_2} (k_2^2 - \xi^2)^{1/2} J_0(\xi t) H_0^{(1)}(\xi u) \, d\xi
$$

Putting $\xi = k_2\eta$, the integral $L_1(u, t)$ can be converted to the following finite integral

$$
L_1(u, t) = -ik_2^2 \int_0^1 (1 - \eta^2)^{1/2} J_0(k_2\eta t) H_0^{(1)}(k_2\eta u) \, d\eta, \quad u > t,
$$

$$
= -ik_2^2 \int_0^1 (1 - \eta^2)^{1/2} J_0(k_2\eta u) H_0^{(1)}(k_2\eta t) \, d\eta, \quad u < t. \quad (4.1.26)
$$
4. Quantities of Physical Interest

The shear stress $\tau_{yz}(x,y)$ in the plane $z=0$ in the neighbourhood of the crack can be found from equation (4.1.9) and is given by

$$a\tau_{yz}(x,0) = -\mu \int_{-\infty}^{\infty} \alpha A(\xi)e^{ix}\xi d\xi + \mu \int_{0}^{\infty} \zeta[B(\zeta)e^{\beta x} + C(\zeta)e^{-\beta x}]d\zeta$$  \hspace{1cm} (4.1.27)

Substituting the values of $B(\zeta)$ and $C(\zeta)$ from equations (4.1.11) and (4.1.12), the expression for the stress can be presented as

$$\tau_{yz}(x,0) = -\tau_0 \int_{0}^{1} t g(t) \int_{0}^{\infty} \sqrt{\xi^2 - k^2} J_0(\xi t) \cos(\xi x) d\xi + O(1)$$
$$= -\tau_0 \int_{0}^{1} t g(t) \int_{0}^{\infty} \xi J_0(\xi t) \cos(\xi x) d\xi + O(1) \text{ as } \xi \to \infty, x > 1$$
$$= -\tau_0 \int_{0}^{1} t g(t) \int_{0}^{\infty} J_0(\xi t) \sin(\xi x) d\xi + O(1)$$
$$= -\tau_0 \frac{d}{dx} \int_{0}^{1} \frac{tg(t)}{\sqrt{x^2 - t^2}} dt + O(1)$$
$$= \frac{\tau_0 x}{\sqrt{x^2 - 1}} g(1) + O(1), \quad x > 1$$  \hspace{1cm} (4.1.28)

Defining the stress intensity factor $K$ by

$$K = \lim_{x \to 1+} \left| \frac{\sqrt{x-1} \tau_{yz}(x,0)}{\tau_0} \right|$$  \hspace{1cm} (4.1.29)

We obtain

$$K = \frac{|g(1)|}{\sqrt{2}}$$  \hspace{1cm} (4.1.30)

Now, crack opening displacement

$$\Delta W(x,0) = W(x,0+) - W(x,0-)$$  \hspace{1cm} (4.1.31)

can be obtained from (4.1.8) as

$$\Delta W(x,0) = 2 \int_{-\infty}^{\infty} A(\xi)e^{ix}d\xi, \quad 0 \leq x \leq 1$$  \hspace{1cm} (4.1.32)
which on substitution of the value of $A(\xi)$ from (4.1.16) takes the form

$$
\Delta W(x, 0) = \frac{2\tau_0 a}{\mu} \int_0^1 tg(t) dt \int_0^\infty J_0(\xi t) \cos(\xi x) d\xi, \quad 0 \leq x \leq 1
$$

$$
= \frac{2\tau_0 a}{\mu} \int_x^1 \frac{tg(t)}{\sqrt{t^2 - x^2}} dt, \quad 0 \leq x \leq 1. \quad (4.1.33)
$$

Scattered field $\tau_{yz}(x, y)$ for $x > 1$, $y > 0$ is calculated from equations (4.1.9), (4.1.11), (4.1.12) and (4.1.16) and is represented by the expression

$$
\tau_{yz}(x, y) = -\tau_0 \int_0^\infty \int_0^1 \alpha tg(t) J_0(\xi t) e^{-\alpha y} \cos(\xi x) d\xi dt
$$

$$
+ \tau_0 \int_0^\infty \int_0^1 \frac{\zeta^2 e^{-2h\beta}(e^{\beta y} + e^{-\beta y}) \cos(\zeta y)}{\beta(1 - e^{-2h\beta})} I_0(t\beta) tg(t) d\zeta dt \quad (4.1.34)
$$

5. Numerical results and discussion

![Figure 4.3: Dynamic SIF versus dimensionless frequency $k_2$.](image)

The integral equation (4.1.19) has been solved by the method of Fox and Goodwin (1953), for different values of dimensionless frequency $k_2$ and strip width $h$. The integral in (4.1.19) has been represented by a quadrature formula involving values of
the desired function $g(t)$ at pivotal points inside the specified range of integration, and then converted to a set of simultaneous linear algebraic equations. The solution of the set of linear algebraic equations gives the first approximation to the required pivotal values of $g(t)$ which have been improved by the difference-correction technique.

![Graph](image1)

**Figure 4.4:** Crack opening displacement versus dimensionless distance; $h = 3$.

![Graph](image2)

**Figure 4.5:** Crack opening displacement versus dimensionless distance; $h = 4$.

After solving the integral equation, the stress intensity factor (SIF) $K$ at the edge of the crack tip has been calculated numerically and plotted against $k_2$ (figure 4.3) for
different values of strip width $h(= 2, 2.5, 3)$. It is observed that the SIF is increasing with the increase in the values of $k_2$. For lower values of $k_2$, stress intensity factors for different values of $h$ are close and differ much for higher values of $k_2$. The crack opening displacement (COD) has been plotted against $x$ ($0 \leq x \leq 1$) for different values of $k_2$ ($= 0.5, 1.0, 1.5$) and $h$ ($= 3, 4$) (figures 4.4, 4.5). In both figures it is clear that COD increases as $k_2$ increases and as $x$ increases, it decreases and becomes zero at $x = 1$. The maximum values of COD are at $x = 0$. To calculate dimensionless scattered field $|\tau_{yz}(x,y)/\tau_0|$ outside the crack, double integrals in the expression (4.1.34) are evaluated for $k_2 = 0.6$ and different values of $x$ and $y$ (figures 4.6, 4.7). From figures 4.6 and 4.7, it has been observed that scattered field is wave like nature and decreases with increase in the values of $x$ and $y$.

Figure 4.6: Scattered field outside the crack ($h= 3.5$ and $k_2 = 0.6$).
Figure 4.7: Scattered field outside the crack ($h=4.5$ and $k_2=0.6$).
4.2 P-wave interaction with a circular disk in an infinite cylinder†

1. Introduction

The dynamic interaction of elastic waves with cracks or inclusions are of considerable importance in view of their extensive applications in mechanical engineering and also in seismology and geophysics. Cracks usually weaken and soften the mechanical properties of materials and rigid flat inclusions result in concentration of stresses. The problems involving rigid inclusions in elastic medium have been studied by many authors. Bycroft (1956) and Gladwell (1968) solve the problems of vibration rigid disc in semi-infinite elastic solid. Vibrations of rigid disc on a transversely isotropic elastic half space was treated by Kirkner (1982). Li and Fan (2001) and Selvadurai (2002) considered the problem of rigid circular inclusion at the interface of two bounded elastic half-space. The problem of torsional oscillation of rigid disk in an infinite cylinder was studied by Manna et al. (2003). Senjuntichai et al. (2003, 2006) analyzed the problems of vertical vibration of rigid circular disc. Wu (2006) investigated the problem of nano adhesion between a rigid circular disk and an infinite elastic surface. The vertical vibration of a flexible plate with rigid core resting on a semi-infinite saturated soil had been investigated by Chen et al. (2007) where the special case of rigid plate was considered. Very recently Morteza et al. (2010) solved the problem of forced vertical vibration of rigid circular disc on a transversely isotropic half-space.

In this paper P-wave interaction with a circular disc in an infinite elastic cylinder has been analyzed. Hankel and Fourier transforms are used to reduce the elastodynamic problem to a pair of dual integral equations which is further reduced to Fredholm integral equation of second kind and solved numerically. The stress intensity factor

at the edge of the disk is calculated and represented by means of graphs. Also the
displacement outside the disk has been calculated numerically and plotted for different
parameters.

2. Formulation of the problem

We consider the problem of interaction of P-wave incidence with a circular disk
of radius $a$ in a cylinder of radius $b$. Normalizing the lengths by ‘$a$’ and putting
\[ \frac{b}{a} = b_1, \quad \frac{z}{a} = z, \quad \frac{r}{a} = r \]
where $(r_1, \theta, z_1)$ is cylindrical co-ordinate system attached to the center of the disk and $z_1$-axis is along the axis of the cylinder. In terms of the
co-ordinate system $(r, \theta, z)$ the location of the disk is $0 \leq r \leq 1$, $z = 0$ (figure 4.8).
The displacement components $u_r(r, z, t)$ and $u_z(r, z, t)$ can be expressed in terms of
two wave potentials $\phi(r, z, t)$ and $\psi(r, z, t)$ as
\[ u_r(r, z, t) = \frac{\partial \phi}{\partial r} - \frac{\partial \psi}{\partial z} \quad \text{and} \quad u_z(r, z, t) = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial r} + \frac{\psi}{r}. \]
$u_\theta$ vanishes everywhere due to symmetry of the problem.

Non vanishing stress components are
\[ \tau_{rr}(r, z, t) = 2\mu \left( \frac{\partial \phi}{\partial r} - \frac{\partial \psi}{\partial z} \right) + \lambda \nabla^2 \phi \]
\[ \tau_{zz}(r, z, t) = 2\mu \left( \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial r} + \frac{\psi}{r} \right) + \lambda \nabla^2 \phi \]
\[ \tau_{rz}(r, z, t) = \mu \left[ \phi \left( 2 \frac{\partial \phi}{\partial r} - \frac{\partial \psi}{\partial z} \right) + \frac{\partial}{\partial r} \left( \frac{\partial \psi}{\partial r} + \frac{\psi}{r} \right) \right] \]
where $\lambda, \mu$ are Lamé’s constants and
\[ \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \]

Putting $\phi(r, z, t) = \phi(r, z)e^{-i\omega t}$ and $\psi(r, z, t) = \psi(r, z)e^{-i\omega t}$, the displacement equa-
tions of motion can be written in terms of $\phi$ and $\psi$ as
\[ \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} + k_1^2 \phi = 0 \]
and
\[ \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} - \frac{\psi}{r^2} + \frac{\partial^2 \psi}{\partial z^2} + k_1^2 \psi = 0 \]
Figure 4.8: Geometry of a penny shaped disk in a cylinder of finite radius.

where \( k_1^2 = \frac{a^2 \omega^2}{c_1^2} \), \( k_2^2 = \frac{a^2 \omega^2}{c_2^2} \), \( c_1^2 = \frac{\lambda + 2\mu}{\rho} \), \( c_2^2 = \frac{\mu}{\rho} \).

Henceforth, the time factor \( e^{-i\omega t} \) which is common to all field variables would be omitted in the sequel.

The boundary conditions of the problem are

\[
\begin{align*}
    u_r(r,0) &= 0, \quad 0 < r < b & (4.2.4) \\
    u_z(r,0) &= -v_0, \quad 0 \leq r \leq 1 & (4.2.5) \\
    \tau_{zz}(r,0) &= 0, \quad 1 < r < b & (4.2.6) \\
    \tau_{rr}(b,z) &= 0, \quad -\infty < z < \infty & (4.2.7) \\
    \tau_{rz}(b,z) &= 0, \quad -\infty < z < \infty & (4.2.8)
\end{align*}
\]

where it is assumed that a plane time harmonic elastic wave originating at \( z = -\infty \) be incident normally on the disk and is defined by \( v_0 e^{i(kz-\omega t)} \) and \( v_0 \) is a constant.

By the use of integral transform technique, solutions of the equations (4.2.2) and (4.2.3) can be obtained as

\[
\begin{align*}
    \phi(r,z) &= \int_0^\infty A_1(\xi)J_0(\xi r)e^{-\alpha_1 z}d\xi + \int_0^\infty B_1(\zeta)I_0(\beta_1 r)sin(\zeta z)d\zeta & (4.2.9) \\
    \psi(r,z) &= \int_0^\infty A_2(\xi)J_1(\xi r)e^{-\alpha_2 z}d\xi + \int_0^\infty B_2(\zeta)I_1(\beta_2 r)cos(\zeta z)d\zeta & (4.2.10)
\end{align*}
\]
where \( \alpha_i = \sqrt{\xi^2 - k_i^2} \), \( k_i < \xi \)
\[= -i\sqrt{k_i^2 - \xi^2} \quad , \quad k_i > \xi \]
and \( \beta_i = \sqrt{\zeta^2 - k_i^2} \), \( k_i < \zeta \)
\[= -i\sqrt{k_i^2 - \zeta^2} \quad , \quad k_i > \zeta, \]
and \( A_i(\xi), B_i(\zeta), \ i = 1, 2 \) are unknown constants to be determined. \( J_0 \) and \( J_1 \) are respectively the zero and first order Bessel function of the first kind while \( I_0 \) and \( I_1 \) are zero and first order modified Bessel function of the first kind respectively.

From boundary condition (4.2.4), we find that

\[A_2(\xi) = \frac{\xi}{\alpha_2} A_1(\xi).\]

Using the above relation expressions for displacements \( u_r, u_z \) and stresses \( \tau_{zz}, \tau_{rr}, \tau_{rz} \) can be expressed as

\[u_r(r, z) = -\int_0^\infty [e^{-\alpha_1 z} - e^{-\alpha_2 z}] \xi A_1(\xi) J_1(\xi r) d\xi + \int_0^\infty [\beta_1 B_1(\zeta) I_1(\beta_1 r) + \beta_2 B_2(\zeta) I_1(\beta_2 r)] \sin(\zeta z) d\zeta \quad (4.2.11)\]

\[u_z(r, z) = -\int_0^\infty \frac{[\alpha_1 e^{-\alpha_1 z} - \frac{\xi^2}{\alpha_2} e^{-\alpha_2 z}]}{[\alpha_1 e^{-\alpha_1 z} - \frac{\xi^2}{\alpha_2} e^{-\alpha_2 z}]} A_1(\xi) J_0(\xi r) d\xi + \int_0^\infty [\zeta B_1(\zeta) I_0(\beta_1 r) + \beta_2 B_2(\zeta) I_0(\beta_2 r)] \cos(\zeta z) d\zeta \quad (4.2.12)\]

\[\tau_{zz}(r, z)/\mu = \int_0^\infty [(2\xi^2 - k_2^2) e^{-\alpha_1 z} - 2\xi^2 e^{-\alpha_2 z}] A_1(\xi) J_0(\xi r) d\xi - \int_0^\infty [(2\beta_1^2 + k_2^2) B_1(\zeta) I_0(\beta_1 r) + 2\beta_2 B_2(\zeta) I_0(\beta_2 r)] \sin(\zeta z) d\zeta \quad (4.2.13)\]

\[\tau_{rr}(r, z)/\mu = -\int_0^\infty [(2\alpha_1^2 + k_2^2) e^{-\alpha_1 z} - 2\xi^2 e^{-\alpha_2 z}] J_0(\xi r) A_1(\xi) d\xi + \frac{2}{r} \int_0^\infty [e^{-\alpha_1 z} - e^{-\alpha_2 z}] \xi A_1(\xi) J_1(\xi r) d\xi + \int_0^\infty \left[[(2\zeta^2 - k_2^2) I_0(\beta_1 r) - \frac{2}{r} \beta_1 I_1(\beta_1 r)] B_1(\zeta) \right]
+ \frac{2}{r} \beta_2 I_0(\beta_2 r) - \frac{1}{r} I_1(\beta_2 r) \right] \beta_2 B_2(\zeta) \sin(\zeta z) d\zeta \quad (4.2.14)\]
and

\[
\tau_{xz}(r,z)/\mu = \int_0^{\infty} \left[ 2\alpha_1 e^{-\alpha_1 z} - \frac{(\alpha_2^2 + \xi^2)}{\alpha_2} e^{-\alpha_2 \xi} \right] \xi J_1(\xi r) A_1(\xi) \, d\xi \\
+ \int_0^{\infty} \left[ 2\xi \beta_1 B_1(\xi) I_1(\beta_1 r) + (\xi^2 + \beta_2^2) B_2(\xi) I_1(\beta_2 r) \right] \cos(\xi z) \, d\xi \quad (4.2.15)
\]

From boundary conditions (4.2.7) and (4.2.8), the unknown functions \(B_1(\xi)\) and \(B_2(\xi)\) can be found to be

\[
B_1(\xi) = \left[ (\xi^2 + \beta_2^2) I_1(\beta_2 b) S_1(\xi) - 2\xi \{ \beta_2 I_0(\beta_2 b) - \frac{1}{b} I_1(\beta_2 b) \} \right] S_2(\xi) / M(\xi) \quad (4.2.16)
\]

\[
B_2(\xi) = \left[ 2\xi \beta_1 I_1(\beta_1 b) S_1(\xi) - \{ (\xi^2 - k_2^2) I_0(\beta_1 b) - \frac{2}{b} \beta_1 I_1(\beta_1 b) \} \right] S_2(\xi) / M(\xi) \quad (4.2.17)
\]

where

\[
M(\xi) = \left[ (2\xi^2 - k_2^2) I_0(\beta_1 b) - \frac{2}{b} \beta_1 I_1(\beta_1 b) \right] (\xi^2 + \beta_2^2) I_1(\beta_2 b) \\
- 4\xi^2 \beta_1 I_1(\beta_1 b) [ \beta_2 I_0(\beta_2 b) - \frac{1}{b} I_1(\beta_2 b) ] \quad (4.2.18)
\]

and

\[
S_1(\xi) = \frac{2\xi}{\pi} \int_0^{\infty} \left[ \frac{k_2^2 - 2\xi^2}{\xi^2 + \beta_1^2} + \frac{2\beta_2^2}{\xi^2 + \beta_2^2} \right] J_0(\xi b) A_1(\xi) \, d\xi \\
- \frac{4\xi}{\pi b} \int_0^{\infty} \left[ \frac{1}{\xi^2 + \beta_1^2} - \frac{1}{\xi^2 + \beta_2^2} \right] \xi J_1(\xi b) A_1(\xi) \, d\xi \quad (4.2.19)
\]

\[
S_2(\xi) = -\frac{2}{\pi} \int_0^{\infty} \left[ \frac{k_1^2}{\xi^2 + \beta_1^2} - \frac{k_2^2}{\xi^2 + \beta_2^2} \right] \xi J_1(\xi b) A_1(\xi) \, d\xi \quad (4.2.20)
\]

Now, boundary conditions (4.2.5) and (4.2.6) lead to the following dual integral equations from which the unknown function \(A_1(\xi)\) is to be determined.

\[
\int_0^{\infty} \frac{1}{\xi} \left[ 1 + H(\xi) \right] J_0(\xi r) A_1(\xi) \, d\xi = q(r) \quad , \quad 0 \leq r \leq 1 \quad (4.2.21)
\]

and

\[
\int_0^{\infty} A_1(\xi) J_0(\xi r) \, d\xi = 0 \quad , \quad 1 < r < b \quad (4.2.22)
\]

where

\[
q(r) = -\frac{2\nu_0}{(k_1^2 + k_2^2)} - \frac{2}{(k_1^2 + k_2^2)} \int_0^{\infty} \left[ \xi B_1(\xi) I_0(\beta_1 r) + \beta_2 B_2(\xi) I_0(\beta_2 r) \right] \, d\xi \quad (4.2.23)
\]

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and \[ H(\xi) = \frac{2(\xi^2 - \alpha_1 \alpha_2)}{\alpha_2 (k_1^2 + k_2^2)} - 1 \rightarrow 0 \text{ as } \xi \rightarrow \infty \] (4.2.24)

Since \[ \frac{\xi(\alpha_1 \alpha_2 - \xi^2)}{\alpha_2} = \frac{\xi(\sqrt{\xi^2 - k_1^2} \sqrt{\xi^2 - k_2^2} - \xi^2)}{\sqrt{\xi^2 - k_2^2}} \]
\[ = \frac{\xi^2 (1 - \frac{1}{2} k_1^2) (1 - \frac{1}{2} k_2^2)}{(1 - \frac{1}{2} \xi^2)} - \xi^2 \]
\[ = \frac{\xi^2 [1 - \frac{1}{2} (k_1^2 + k_2^2)] - \xi^2}{(1 - \frac{1}{2} \xi^2)} \]
\[ = -\frac{1}{2} (k_1^2 + k_2^2) \text{ as } \xi \rightarrow \infty \]

3. Method of solution

In order to solve the dual integral equations (4.2.21) and (4.2.22), \( A_1(\xi) \) is taken in the form
\[ A_1(\xi) = \frac{2v_0 \xi}{\pi (k_1^2 + k_2^2)} \int_0^\infty g(t) \cos(\xi t) dt \] (4.2.25)

so that equation (4.2.22) is automatically satisfied.

Substitution of the value of \( A_1(\xi) \) from equation (4.2.25) in equation (4.2.21), yields a Fredholm integral equation of second kind
\[ g(t) + \int_0^1 g(u) L(u, t) du = -2 \] (4.2.26)

where \[ L(u, t) = L_1(u, t) + L_2(u, t) + L_3(u, t) \] (4.2.27)

\[ L_1(u, t) = \int_0^\infty H(\xi) \cos(\xi u) \cos(\xi t) d\xi \] (4.2.28)

\[ L_2(u, t) = \frac{8}{\pi^2 (k_1^2 + k_2^2)} \int_0^\infty \left\{ \left( k_2^2 - 2 \xi^2 \right) K_0(\beta_1 b) - \frac{2}{b} \beta_1 K_1(\beta_1 b) \right\} \cosh(\beta_1 u) \]
\[ + 2 \left\{ \frac{\beta_2}{b} K_0(\beta_2 b) + \frac{1}{b} \beta_2 K_1(\beta_2 b) \right\} \cosh(\beta_2 u) \right\} M_1(\zeta, t) d\zeta \] (4.2.29)
\[ M_1(\zeta, t) = [\zeta(\zeta^2 + \beta_2^2)I_1(\beta_2 b)\cosh(\beta_1 t) + 2\zeta\beta_1\beta_2 I_1(\beta_1 b)\cosh(\beta_2 t)] \frac{\zeta}{M(\zeta)} \quad (4.2.30) \]

\[ L_3(u, t) = \frac{8}{\pi^2(k_1^2 + k_2^2)} \int_0^\infty \left[ k_1^2 \beta_1 k_1(\beta_2 b)\cosh(\beta_1 u) - k_2^2 \beta_2 k_1(\beta_2 b)\cosh(\beta_2 u) \right] M_2(\zeta, t) d\zeta \quad (4.2.31) \]

\[ M_2(\zeta, t) = [2\zeta^2 \{\beta_2 I_0(\beta_2 b) - \frac{1}{b} I_1(\beta_2 b)\} \cosh(\beta_1 t) + \beta_2 \{(2\zeta^2 - k_2^2) I_0(\beta_1 b) - \frac{2}{b} \beta_1 I_1(\beta_1 b)\} \cosh(\beta_2 t)] / M(\zeta) \quad (4.2.32) \]

\[ I_0() \] and \( I_1() \) are the modified Bessel function of imaginary argument of order zero and one respectively.

Now, with the help of equation (4.2.25) equations (4.2.19) and (4.2.20) can be written as

\[ S_1(\zeta) = \frac{4\nu_0 \zeta}{\pi^2(k_1^2 + k_2^2)} \left[ \int_0^1 g(t) \{(k_1^2 - 2\zeta^2)K_0(\beta_1 b)\cosh(\beta_1 t) + 2\beta_2^2 K_0(\beta_2 b)\cosh(\beta_2 t)\} dt - \frac{2}{b} \int_0^1 g(t) \{(\beta_1 K_1(\beta_1 b)\cosh(\beta_1 t) - \beta_2 K_1(\beta_2 b)\cosh(\beta_2 t)\} dt \right] \quad (4.2.33) \]

\[ S_2(\zeta) = -\frac{4\nu_0}{\pi^2(k_1^2 + k_2^2)} \int_0^1 g(t) \left[ k_1^2 \beta_1 K_1(\beta_1 b)\cosh(\beta_1 t) - k_2^2 \beta_2 K_1(\beta_2 b)\cosh(\beta_2 t) \right] dt \quad (4.2.34) \]

In order to make the numerical analysis easier, the semi-infinite integral has therefore been converted to finite integrals by using simple contour integration technique.

The integral (4.2.28) has no poles, it has only branch points at \( \xi = k_1 \) and \( \xi = k_2 \). The infinite integral in equation (4.2.28) can be converted into integrals with finite limits.

Let us consider the following two integrals.

\[ I_1 = \frac{2}{\pi t} \int_{C_1} M(\xi, \alpha_1, \alpha_2)e^{i\xi u}\cos(\xi t)d\xi, \quad u > t \]
and \[ I_2 = \frac{2}{\pi t} \int_{C_2} M(\xi, \alpha_1, \alpha_2)e^{-i\xi u}\cos(\xi t)d\xi, \quad u > t \]

where \( M(\xi, \alpha_1, \alpha_2) = H(\xi) = \frac{2(\xi^2 - \alpha_1\alpha_2)}{\alpha_2(k_1^2 + k_2^2)} - 1 \)
and $C_1, C_2$ are the contours shown in the following figure:

![Figure 4.9: Contour of the integral $L_1(u, t)$](image)

where as

$$
\alpha_1 = \sqrt{\xi^2 - k_1^2}, \quad \xi > k_1
$$

$$
\alpha_2 = \sqrt{\xi^2 - k_2^2}, \quad \xi > k_2
$$

$$
\alpha'_1 = \sqrt{k_1^2 - \xi^2}, \quad k_1 > \xi
$$

$$
\alpha'_2 = \sqrt{k_2^2 - \xi^2}, \quad k_2 > \xi
$$

Integrating $I_1$ along the contour $C_1$

$$
\int_0^{k_2} M(\xi, \imath \alpha'_1, \imath \alpha'_2) e^{\imath \xi u \cos(\xi t)} d\xi + \int_{k_2}^{k_1} M(\xi, \imath \alpha'_1, \alpha_2) e^{\imath \xi u \cos(\xi t)} d\xi \\
+ \int_{k_1}^{\infty} M(\xi, \alpha_1, \alpha_2) e^{\imath \xi u \cos(\xi t)} d\xi = 0 \quad (4.2.35)
$$
Integrating $I_2$ along the contour $C_2$

\[
\int_0^{k_2} M(\xi, -i\alpha_1', -i\alpha_2')e^{-i\xi u \cos(\xi t)}d\xi + \int_{k_2}^{k_1} M(\xi, -i\alpha_1', \alpha_2)e^{-i\xi u \cos(\xi t)}d\xi \\
+ \int_{k_1}^{\infty} M(\xi, \alpha_1, \alpha_2)e^{-i\xi u \cos(\xi t)}d\xi = 0 \tag{4.2.36}
\]

Using the above relations and adding (4.2.33) and (4.2.34),

\[
2L_1(u, t) + \int_0^{k_2} \left[ M(\xi, i\alpha_1', i\alpha_2') - M(\xi, -i\alpha_1', -i\alpha_2') \right]e^{i\xi u \cos(\xi t)}d\xi \\
+ \int_{k_2}^{k_1} \left[ M(\xi, i\alpha_1', \alpha_2) - M(\xi, -i\alpha_1', \alpha_2) \right]e^{i\xi u \cos(\xi t)}d\xi \\
+ \int_{k_1}^{\infty} \left[ M(\xi, \alpha_1, \alpha_2) - M(\xi, \alpha_1, \alpha_2) \right]e^{i\xi u \cos(\xi t)}d\xi = 0 \tag{4.2.37}
\]

Now

\[
M(\xi, i\alpha_1', i\alpha_2') - M(\xi, -i\alpha_1', -i\alpha_2') = 2\xi \left( \frac{(\xi^2 + \alpha_1' \alpha_2')}{i\alpha_2'} + \frac{(\xi^2 + \alpha_1' \alpha_2')}{i\alpha_2'} \right) \\
= \frac{-4i\xi}{(k_1^2 + k_2^2)} \frac{\alpha_2'}{(k_1^2 + k_2^2)} \\
= \frac{-4i\xi \{\xi^2 + \sqrt{k_1^2 - \xi^2} \sqrt{k_2^2 - \xi^2} \}}{(k_1^2 + k_2^2) \sqrt{k_2^2 - \xi^2}} \\
= \frac{-4i\xi \{\xi^2 + \sqrt{1 - \xi^2} \sqrt{\gamma^2 - \xi^2} \}}{(1 + \gamma^2) \sqrt{\gamma^2 - \xi^2}}
\]

and

\[
M(\xi, i\alpha_1', \alpha_2) - M(\xi, -i\alpha_1', \alpha_2) = 2\xi \left( \frac{\xi^2 - i\alpha_1' \alpha_2'}{\alpha_2} - \frac{\xi^2 + i\alpha_1' \alpha_2'}{\alpha_2} \right) \\
= \frac{-4i\xi \sqrt{k_1^2 - \xi^2}}{(k_1^2 + k_2^2)} \\
= \frac{-4i\xi \sqrt{1 - \xi^2}}{(1 + \gamma^2)}
\]

where $\gamma = \frac{k_2}{k_1} < 1$ and $\xi$ is replaced by $k_1 \xi$. 91
Finally the integral (4.2.28), $L_1(u, t)$ can be converted to the following finite integral

\[
L_1(u, t) = 2ik_1 \int_0^\gamma \frac{\xi [\xi^2 + (1 - \xi^2)^\frac{1}{2} (\gamma^2 - \xi^2)^\frac{1}{2}]}{(1 + \gamma^2)(\gamma^2 - \xi^2)^\frac{1}{2}} e^{ixu} \cos(\xi t) d\xi \\
+ 2ik_1 \int_\gamma^1 \frac{\xi (1 - \xi^2)^\frac{1}{2}}{(1 + \gamma^2)} e^{ixu} \cos(\xi t) d\xi , \quad u > t
\]

(4.2.38)

**4. Stress intensity factor and displacement outside the disk**

The stress distribution on the disk on the plane $z = 0$ can be found from (4.2.13) and is given by

\[
\tau_{zz}(r, 0) = -\mu k_2^2 \int_0^\infty A_1(\xi) J_0(\xi r) d\xi
\]

(4.2.39)

Substituting the values of $A_1(\xi)$ from equation (4.2.25), the expression for the stress can be presented as

\[
\tau_{zz}(r, 0) = -\frac{2v_0 \mu k_2^2}{(k_1^2 + k_2^2)} \int_0^1 g(t) dt \int_0^\infty \xi J_0(\xi r) \cos(\xi t) d\xi \\
= -\frac{2v_0 \mu k_2^2}{(k_1^2 + k_2^2)} \int_0^1 g(t) dt \left( \frac{d}{dt} \int_0^\infty J_0(\xi r) \sin(\xi t) d\xi \right) \\
= -\frac{2v_0 \mu k_2^2}{(k_1^2 + k_2^2) \sqrt{1 - r^2}} g(1) + O(1) , \quad |r| \leq 1
\]

(4.2.40)

Defining the stress intensity factor $K$ by

\[
K = \lim_{r \to 1^+} \left| \frac{\sqrt{1 - r}}{\mu v_0} \tau_{zz}(r, 0) \right|
\]

(4.2.41)

We obtain,

\[
K = \frac{\sqrt{2} \gamma^2}{(1 + \gamma^2)} |g(1)|
\]

(4.2.42)
Now from equation (4.2.12) after substituting the values of $A_1(\xi), B_1(\zeta)$ and $B_2(\zeta)$, we get the vertical displacement outside the disk as

\[
 u_z(r, z) = -\frac{2v_0}{\pi(k_1^2 + k_2^2)} \int_0^1 g(t) \int_0^\infty \left[ \alpha_1 e^{-\alpha_1 z} - \frac{\xi^2}{\alpha_2} e^{-\alpha_2 z} \right] J_0(\xi r) \cos(\xi t) d\xi \, dt 
+ \int_0^\infty \left[ (\zeta^2 + \beta_2^2) I_0(\beta_1 r) I_1(\beta_2 b) + 2\zeta \beta_1 \beta_2 I_0(\beta_2 r) I_1(\beta_1 b) \right] \cos(\zeta z) \frac{S_1(\zeta)}{M(\zeta)} d\zeta 
- \int_0^\infty \left[ 2\zeta^2 I_0(\beta_1 r) \{ \beta_2 I_0(\beta_2 b) - \frac{1}{b} I_1(\beta_2 b) \} \right] \cos(\zeta z) \frac{S_1(\zeta)}{M(\zeta)} d\zeta 
+ \beta_2 I_0(\beta_2 r) \{ (2\zeta^2 - k_2^2) I_0(\beta_1 b) - \frac{2}{b} \beta_1 I_1(\beta_1 b) \} \cos(\zeta z) \frac{S_2(\zeta)}{M(\zeta)} d\zeta \quad (4.2.43)
\]

5. Numerical results and discussion

To calculate stress intensity factor (SIF), $K$ it is necessary to obtain the value of $g(1)$ and for which integral (4.2.26) is solved numerically. Value of SIF($K$) has been plotted against dimensionless frequency $k_1$ (fig. 4.10, fig. 4.11) for different radius of the cylinder ($b = 2.5, 3, 3.5$) and for two different values of $k_2$ (= 0.2, 0.3) subject to the condition $\gamma = \frac{k_2}{k_1} < 1$.

![Figure 4.10: Dynamic SIF versus dimensionless frequency $k_1$.](image-url)
Figure 4.11: Dynamic SIF versus dimensionless frequency $k_1$.

$|u_z(r,z)/v_0|$, $b=1.5$, $k_2=0.2$, $k_1=0.3$

Figure 4.12: Displacement versus dimensionless distance.

Figure 4.10 and figure 4.11 show that SIF is increasing initially and then decreasing and showing wave like nature with increasing values of $k_1$. Also it can be seen that SIF is increasing as radius of the cylinder increases and for higher values of the radius SIF
Figure 4.13: Displacement versus dimensionless distance.

is sharply decreasing with increase in $k_1$ which reveals the fact that in infinite medium SIF decreases as $k_1$ increases.

The dimensionless vertical displacement defined by $u_z/v_0$ away from the disk has been calculated from (4.2.43) for two different radius of the cylinder ($b = 1.5, 2$) and for fixed values of $k_1$ and $k_2$ (fig. 4.12, fig. 4.13). In all cases it is found that displacement is decreasing as $r$ increases i.e., for infinite medium displacement satisfies radiation condition.