Chapter-V

Rigid strips in orthotropic medium
Chapter 5

Rigid strips in orthotropic medium

5.1 P-wave interaction with a pair of rigid strips embedded in an orthotropic strip

1. Introduction

The dynamic interaction of rigid strips with an elastic isotropic or orthotropic medium is a subject of considerable interest in mechanics. A static analysis of this kind is useful for understanding the behavior of foundations under external loads in the topic of soil-structure interaction. The dynamic analysis of this kind is of importance to earth-quake engineering, machine, vibrations and Seismology. The performance of engineered systems is affected by inhomogeneities such as cracks and inclusions present in the material. Cracks and rigid inclusions in an elastic material have become the subject of investigations. Presently use of anisotropic materials are increasing due to their strength. Increasing use of anisotropic media demands that the study should be extensive. A detailed reference of work done on the determination of the dynamic stress field around a crack or inclusion in elastic solid has been given by Sih (1977), Sih and Chen (1981). However in the presence of finite boundaries, the problem becomes complicated since they involve additional geometric parameters, describing
the dimension of the solids. Forced vertical vibration of a single strip was treated by Wickham (1977). Mandal et al. (1992, 1997) solved the problem of forced vibration of two and four rigid strips on semi-infinite elastic medium. Mandal et al. (1998) also treated the diffraction problem by four rigid strips in orthotropic medium. Shear wave interaction with a pair of rigid strips in elastic strip was analyzed by Pramanick et al. (1999). Wu Da-zhi et al. (2006) considered the torsional vibration problem of rigid circular plate on transversely isotropic saturated soil. Very recently Morteza et al. (2010a, 2010b) considered the vibration problem of rigid circular disc on transversely isotropic media.

In this paper we analyzed the diffraction of elastic P-wave by two rigid strips embedded in an infinite orthotropic strip. Using Hilbert transform technique, the mixed boundary value problem has been reduced to the Fredholm integral equation of second kind which has been solved numerically. Stress intensity factors at both the edges of the strips have been calculated and shown graphically for different parameters and materials. Finally, vertical displacement has been calculated outside strips and shown by 3D-graphs.

2. Formulation of the problem

Let us consider an infinitely long orthotropic elastic strip of width $2h_1$ containing two coplanar rigid strips embedded in it. The location of the strips are $b \leq |X| \leq a$, $Y = 0$, $|Z| < \infty$, with reference to the cartesian co-ordinate axes $(X,Y,Z)$. Normalizing all the lengths with respect to “$a$” and putting $\frac{X}{a} = x$, $\frac{Y}{a} = y$, $\frac{Z}{a} = z$, $\frac{b}{a} = c$, $\frac{h_1}{a} = h$, the location of the rigid strips are defined by $c \leq |x| \leq 1$, $y = 0$, $|z| < \infty$ (figure 5.1).

Let a time harmonic wave given by $u = 0$ and $v = v_0 e^{i(ky-\omega t)}$, where $k = \frac{\omega}{c_s \sqrt{\epsilon_0}}$, $c_s = \left(\frac{\mu_0}{\rho}\right)^{\frac{1}{2}}$ with $\rho$ being the density of the material, $\omega$ is the circular frequency and $v_0$ is a constant, travelling in the direction of the positive y-axis and be incident normally on the strips.
The non-zero stress components $\tau_{yy}$, $\tau_{xy}$ and $\tau_{xx}$ are given by

$$\frac{\tau_{yy}}{\mu_{12}} = c_{12} \frac{\partial u}{\partial x} + c_{22} \frac{\partial v}{\partial y}$$  \hspace{1cm} (5.1.1)$$

$$\frac{\tau_{xy}}{\mu_{12}} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$  \hspace{1cm} (5.1.2)$$

$$\frac{\tau_{xx}}{\mu_{12}} = c_{11} \frac{\partial u}{\partial x} + c_{12} \frac{\partial v}{\partial y}$$  \hspace{1cm} (5.1.3)$$

where $u$ and $v$ are displacement components and $c_{ij} \ (i, j = 1, 2)$ are non-dimensional parameters related to the engineering elastic constants $E_i$, $\mu_{ij}$ and $\nu_{ij} \ (i, j = 1, 2, 3)$ by the relations

$$c_{11} = \frac{E_1}{\mu_{12}}(1 - \nu_{12}^2 E_2/E_1)$$

$$c_{22} = \frac{E_2}{\mu_{12}}(1 - \nu_{12}^2 E_2/E_1) = c_{11} E_2/E_1$$  \hspace{1cm} (5.1.4)$$

$$c_{12} = \nu_{12} E_2/\mu_{12}(1 - \nu_{12}^2 E_2/E_1) = \nu_{12} c_{22} = \nu_{21} c_{11}$$

for generalized plane stress and

$$c_{11} = \frac{(E_1/\Delta\mu_{12})(1 - \nu_{23} \nu_{32})}{\Delta \mu_{12}}$$

$$c_{22} = \frac{(E_2/\Delta\mu_{12})(1 - \nu_{13} \nu_{31})}{\Delta \mu_{12}}$$  \hspace{1cm} (5.1.5)$$

$$c_{12} = \frac{E_1(\nu_{21} + \nu_{13} \nu_{32} E_2/E_1)/\Delta \mu_{12}}{E_2(\nu_{12} + \nu_{23} \nu_{31} E_1/E_2)/\Delta \mu_{12}}$$

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\[ \Delta = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - \nu_{12}\nu_{23}\nu_{31} - \nu_{13}\nu_{21}\nu_{32} \]

for plane strain. The constants \( E_i \) and \( \nu_{ij} \) satisfy the Maxwell’s relation

\[ \nu_{ij}/E_i = \nu_{ji}/E_j \]  

(5.1.6)

The equation of motion for orthotropic material in terms of displacements are

\[ c_{11} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + (1 + c_{12}) \frac{\partial^2 v}{\partial x \partial y} = \frac{a^2}{c_1^2} \frac{\partial^2 u}{\partial t^2} \]

\[ c_{22} \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} + (1 + c_{12}) \frac{\partial^2 u}{\partial x \partial y} = \frac{a^2}{c_2^2} \frac{\partial^2 v}{\partial t^2} \]

Therefore, substituting \( u(x,y,t) = u(x,y)e^{-i\omega t} \) and \( v(x,y,t) = v(x,y)e^{-i\omega t} \), our problem reduces to the solution of the equations

\[ c_{11} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + (1 + c_{12}) \frac{\partial^2 v}{\partial x \partial y} + k_s^2 u = 0 \]  

(5.1.7)

\[ c_{22} \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} + (1 + c_{12}) \frac{\partial^2 u}{\partial x \partial y} + k_s^2 v = 0 \]  

(5.1.8)

where \( k_s^2 = \frac{a^2\omega^2}{c_1^2} \).

Thus the problem is to find the stress distribution near the edges of strips subject to the following boundary conditions:

\[ v(x, 0+) = v(x, 0-) = -v_0, \quad c \leq |x| \leq 1 \]  

(5.1.9)

\[ \tau_{yy}(x,0) = 0, \quad |x| < c, \quad 1 < |x| < h \]  

(5.1.10)

\[ u(x,0) = 0, \quad |x| < h \]  

(5.1.11)

\[ \tau_{xx}(\pm h,y) = 0 \]  

(5.1.12)

\[ \tau_{xy}(\pm h,y) = 0 \]  

(5.1.13)

Henceforth, the time factor \( e^{-i\omega t} \) which is common to all field variables will be omitted in the sequel.

The solution of equations (5.1.7) and (5.1.8) are found to be

\[ u(x, y) = \frac{2}{\pi} \int_0^\infty \left[ A_1(\xi)e^{-\nu_1|y|} + A_2(\xi)e^{-\nu_2|y|} \right] \sin(\xi x) d\xi \]

\[ + \frac{2}{\pi} \int_0^\infty \left[ A_3(\zeta) \sinh(\nu_3 x) + A_4(\zeta) \sinh(\nu_4 x) \right] \sin(\zeta y) d\zeta \]  

(5.1.14)
\[
v(x, y) = \frac{2}{\pi} \int_{0}^{\infty} \frac{1}{\xi} \left[ \alpha_1 A_1(\xi) e^{-\nu_1 y} + \alpha_2 A_2(\xi) e^{-\nu_2 y} \right] \cos(\xi x) d\xi \\
+ \frac{2}{\pi} \int_{0}^{\infty} \frac{1}{\zeta} \left[ \alpha_3 A_3(\zeta) \cosh(\nu_3 x) + \alpha_4 A_4(\zeta) \cosh(\nu_4 x) \right] \cos(\zeta y) d\zeta \quad (5.1.15)
\]

where \( A_i(\xi) (i = 1, 2, 3, 4) \) are the unknown functions to be determined, \( \nu_1^2 \) and \( \nu_2^2 \) are the roots of the equation

\[
c_{22} \nu_4^4 + \{(c_{12}^2 + 2c_{12} - c_{11}c_{22})\xi^2 + (1 + c_{22})k_s^2\} \nu^2 + (c_{11}\xi^2 - k_s^2)(\xi^2 - k_s^2) = 0 \quad (5.1.16)
\]

and \( \nu_3^2, \nu_4^2 \) are the roots of the equation

\[
c_{11} \nu_4^4 + \{(c_{12}^2 + 2c_{12} - c_{11}c_{22})\xi^2 + (1 + c_{11})k_s^2\} \nu^2 + (c_{22}\xi^2 - k_s^2)(\xi^2 - k_s^2) = 0 \quad (5.1.17)
\]

where \( \alpha_i = \frac{c_{11}\xi^2 - k_s^2 - \nu_i^2}{(1 + c_{12})\nu_i} \) (i = 1, 2) \( (5.1.18) \)

and \( \alpha_i = \frac{\xi^2 - k_s^2 - c_{11}\nu_i^2}{(1 + c_{12})\nu_i} \) (i = 3, 4) \( (5.1.19) \)

From the boundary condition (5.1.11) it is found that

\[
A_2(\xi) = -A_1(\xi) \quad (5.1.20)
\]

Therefore expressions for displacements \( u, v \) and stresses \( \tau_{yy}, \tau_{xy}, \tau_{xx} \) can be finally written as

\[
u(x, y) = \frac{2}{\pi} \int_{0}^{\infty} \frac{1}{\xi} \left[ \alpha_1 A_1(\xi) e^{-\nu_1 y} - \alpha_2 e^{-\nu_2 y} \right] A_1(\xi) \sin(\xi x) d\xi \\
+ \frac{2}{\pi} \int_{0}^{\infty} \frac{1}{\zeta} \left[ A_3(\zeta) \sinh(\nu_3 x) + A_4(\zeta) \sinh(\nu_4 x) \right] \sin(\zeta y) d\zeta \\
\quad (5.1.21)
\]

\[
u(x, y) = \frac{2}{\pi} \int_{0}^{\infty} \frac{1}{\xi} \left[ \alpha_1 e^{-\nu_1 y} - \alpha_2 e^{-\nu_2 y} \right] A_1(\xi) \cos(\xi x) d\xi \\
+ \frac{2}{\pi} \int_{0}^{\infty} \frac{1}{\zeta} \left[ \alpha_3 A_3(\zeta) \cosh(\nu_3 x) + \alpha_4 A_4(\zeta) \cosh(\nu_4 x) \right] \cos(\zeta y) d\zeta \quad (5.1.22)
\]
\[
\frac{\tau_{yy}}{\mu_{12}} = \frac{2}{\pi} \int_0^\infty \left[ (c_{12}\xi - sgn(y)\frac{c_{22}\alpha_1\nu_1}{\xi})e^{-\nu_1|y|} - (c_{12}\xi - sgn(y)\frac{c_{22}\alpha_2\nu_2}{\xi})e^{-\nu_2|y|} \right] \\
\times A_1(\xi)\cos(\xi x)\,d\xi + \int_0^\infty \left[ (c_{12}\nu_3 - c_{22}\alpha_3)A_3(\zeta)\cosh(\nu_3 x) + (c_{12}\nu_4 - c_{22}\alpha_4)A_4(\zeta)\cosh(\nu_4 x) \right] \sin(\zeta y)\,d\zeta
\]

\[
\frac{\tau_{xy}}{\mu_{12}} = -\frac{2}{\pi} \int_0^\infty \left[ (\nu_1 + \alpha_1)e^{-\nu_1 y} - (\nu_2 + \alpha_2)e^{-\nu_2 y} \right] A_1(\xi)\sin(\xi x)\,d\xi \\
+ \int_0^\infty \left[ (\zeta + \frac{\nu_3\alpha_3}{\xi})A_3(\zeta)\sinh(\nu_3 x) + (\zeta + \frac{\nu_4\alpha_4}{\xi})A_4(\zeta)\sinh(\nu_4 x) \right] \cos(\zeta y)\,d\zeta, \quad y > 0
\]

\[
\frac{\tau_{xx}}{\mu_{12}} = \frac{2}{\pi} \int_0^\infty \left[ (c_{11}\xi - \frac{c_{12}\alpha_1\nu_1}{\xi})e^{-\nu_1|y|} - (c_{11}\xi - \frac{c_{12}\alpha_2\nu_2}{\xi})e^{-\nu_2|y|} \right] A_1(\xi)\cos(\xi x)\,d\xi \\
+ \int_0^\infty \left[ (c_{11}\nu_3 - c_{12}\alpha_3)A_3(\zeta)\cosh(\nu_3 x) + (c_{11}\nu_4 - c_{12}\alpha_4)A_4(\zeta)\cosh(\nu_4 x) \right] \sin(\zeta y)\,d\zeta, \quad y > 0
\]

The boundary conditions (5.1.9) and (5.1.10) yield the following pair of dual integral equations

\[
\int_0^\infty \frac{1}{\xi} \left[ 1 + H(\xi) \right] A(\xi)\cos(\xi x)\,d\xi = p(x) \quad c \leq |x| \leq 1
\]

\[
\int_0^\infty A(\xi)\cos(\xi x)\,d\xi = 0 \quad |x| < c \quad 1 < |x| < h
\]

where
\[
A(\xi) = \frac{\alpha_1\nu_1 - \alpha_2\nu_2}{\xi} A_1(\xi)
\]

\[
H(\xi) = \left( \frac{\alpha_1 - \alpha_2}{\alpha_1\nu_1 - \alpha_2\nu_2} \right) \frac{\xi}{d} - 1 \quad \text{as} \quad \xi \to \infty
\]

\[
p(x) = -\frac{\pi}{2c}\nu_0 - \frac{1}{c} \int_0^\infty \frac{1}{\zeta} \left[ \alpha_3 A_3(\zeta)\cosh(\nu_3 x) + \alpha_4 A_4(\zeta)\cosh(\nu_4 x) \right] \,d\zeta
\]

\[
d = \frac{c_11 + N_1N_2}{N_1N_2(N_1 + N_2)}
\]
and
\[
N_1^2 = \frac{1}{2c_{22}} \left[ -(c_{12}^2 + 2c_{12} - c_{11}c_{22}) + \sqrt{(c_{12}^2 + 2c_{12} - c_{11}c_{22})^2 - 4c_{11}c_{22}} \right] \quad (5.1.32)
\]
\[
N_2^2 = \frac{1}{2c_{22}} \left[ -(c_{12}^2 + 2c_{12} - c_{11}c_{22}) - \sqrt{(c_{12}^2 + 2c_{12} - c_{11}c_{22})^2 - 4c_{11}c_{22}} \right] \quad (5.1.33)
\]

Since,
\[
\frac{\alpha_1 - \alpha_2}{\alpha_1 \nu_1 - \alpha_2 \nu_2} = \frac{c_{11} \xi^2 - k_2^2 - \nu_1^2}{(1+c_{12}) \nu_1} - \frac{c_{11} \xi^2 - k_2^2 - \nu_1^2}{(1+c_{12}) \nu_2}
\]
\[
= - \frac{\nu_2 (c_{11} \xi^2 - k_2^2 - \nu_1^2)}{(1+c_{12})} - \frac{\nu_1 (c_{11} \xi^2 - k_2^2 - \nu_2^2)}{(1+c_{12})}
\]
\[
= - \frac{\xi^2 c_{11} (N_2 - N_1) + k_2^2 (N_1 - N_2) + \xi^2 N_1 N_2 (N_2 - N_1)}{\xi^3 N_1 N_2 (N_1^2 - N_2^2)}
\]
\[
= \frac{c_{11} (N_1 - N_2) + N_1 N_2 (N_1 - N_2)}{\xi N_1 N_2 (N_1^2 - N_2^2)}
\]
\[
= \frac{1}{\xi} \left[ \frac{c_{11} + N_1 N_2}{N_1 N_2 (N_1 + N_2)} \right] = \frac{d}{\xi} \quad \text{as} \quad \xi \to \infty
\]

Using the boundary conditions (5.1.12) and (5.1.13), \(A_3(\zeta)\) and \(A_4(\zeta)\) are expressed in terms of the function \(A(\xi)\) as:
\[
M(\zeta) A_3(\zeta) = \left( \zeta + \frac{\alpha_4 \nu_4}{\zeta} \right) i_1(\zeta) \sinh(\nu_4 h) - (c_{11} \nu_4 - c_{12} \alpha_4) i_2(\zeta) \cosh(\nu_4 h) \quad (5.1.34)
\]
\[
M(\zeta) A_4(\zeta) = - \left( \zeta + \frac{\alpha_3 \nu_3}{\zeta} \right) i_1(\zeta) \sinh(\nu_3 h) + (c_{11} \nu_3 - c_{12} \alpha_3) i_2(\zeta) \cosh(\nu_3 h) \quad (5.1.35)
\]

where
\[
M(\zeta) = \left( \zeta + \frac{\alpha_4 \nu_4}{\zeta} \right) (c_{11} \nu_4 - c_{12} \alpha_4) \cosh(\nu_4 h) \sinh(\nu_4 h)
\]
\[
- \left( \zeta + \frac{\alpha_3 \nu_3}{\zeta} \right) (c_{11} \nu_4 - c_{12} \alpha_4) \sinh(\nu_4 h) \cosh(\nu_4 h) \quad (5.1.36)
\]

\[
i_1(\zeta) = \frac{2}{\pi} \int_0^\infty \left[ \frac{c_{11} \xi^2 + c_{12} (k_2^2 + \nu_2^2)}{\nu_1^2 + \xi^2} - \frac{c_{11} \xi^2 + c_{12} (k_2^2 + \nu_1^2)}{\nu_2^2 + \xi^2} \right] \frac{A(\xi) \cos(\xi h)}{\nu_1^2 - \nu_2^2} d\xi \quad (5.1.37)
\]

\[
i_2(\zeta) = - \frac{2}{\pi} \int_0^\infty \left[ \frac{c_{12} \nu_2^2 + c_{11} \xi^2 - k_2^2}{\nu_1^2 + \xi^2} - \frac{c_{12} \nu_1^2 + c_{11} \xi^2 - k_2^2}{\nu_2^2 + \xi^2} \right] \frac{\xi A(\xi) \sin(\xi h)}{\nu_1^2 - \nu_2^2} d\xi \quad (5.1.38)
\]

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3. Method of solution

In order to reduce the dual integral equations (5.1.26) and (5.1.27) to a single Fredholm integral equation, we assume that

\[ A(\xi) = \int_{c}^{1} \frac{h(t^2)}{t} \{ 1 - \cos(\xi t) \} dt \]  

(5.1.39)

where the unknown function \( h(t^2) \) is to be determined.

Substituting \( A(\xi) \) from (5.1.39) in equations (5.1.27), we note that

\[
\int_{0}^{c} A(\xi) \cos(\xi x) d\xi = \pi \int_{c}^{1} \frac{h(t^2)}{t} \left[ \delta(x) - \frac{1}{2} \delta(x + t) - \frac{1}{2} \delta(|x - t|) \right] dt
\]

so that equation (5.1.27) is automatically satisfied.

Again, the substitution of the value of \( A(\xi) \) from (5.1.39) in equation (5.1.26) yields

\[
\frac{1}{2} \int_{c}^{1} \frac{h(t^2)}{t} \log \left| \frac{x^2 - t^2}{x^2} \right| dt = p(x)
\]

\[
- \int_{c}^{1} \frac{h(t^2)}{t} dt \int_{0}^{\infty} \xi^{-1} H(\xi) \cos(\xi x) \{ 1 - \cos(\xi t) \} d\xi
\]

(5.1.40)

where the result

\[
\int_{0}^{\infty} \frac{\cos(\xi x) \{ 1 - \cos(\xi t) \}}{\xi} d\xi = \frac{1}{2} \log \left| \frac{x^2 - t^2}{x^2} \right|
\]

has been used.

Differentiating both sides of equation (5.1.40) with respect to \( x \) and subsequently multiplying by \( -\frac{2\xi}{x} \), we obtain

\[
\frac{2}{\pi} \int_{c}^{1} \frac{t h(t^2)}{t^2 - x^2} dt = \frac{2x}{\pi} \int_{c}^{1} \frac{h(t^2)}{t} dt \int_{0}^{\infty} \frac{1}{\xi} \left[ \alpha_{3} \nu_{3} A_{5}(\xi) \sinh(\nu_{3} x) + \alpha_{4} \nu_{4} A_{6}(\xi) \sinh(\nu_{4} x) \right] d\xi - \int_{0}^{\infty} H(\xi) \sin(\xi x) \{ 1 - \cos(\xi t) \} d\xi, \quad c \leq |x| \leq 1
\]

(5.1.41)

It is known that using Hilbert transform technique, the solution of the integral equation of the form

\[
\frac{2}{\pi} \int_{a}^{b} \frac{th(t^2)}{(t^2 - y^2)} dt = R(y), \quad a < y < b
\]

can be obtained in the form

\[
h(t^2) = \frac{2}{\pi} \left( \frac{t^2 - a^2}{b^2 - t^2} \right)^{\frac{1}{2}} \int_{a}^{b} \left( \frac{b^2 - y^2}{y^2 - a^2} \right)^{\frac{1}{2}} \frac{yR(y)}{(y^2 - t^2)} dy + \frac{D}{(t^2 - a^2)^{\frac{1}{2}} (b^2 - t^2)^{\frac{1}{2}}}
\]

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with condition that $R$ must be an even function of $y$ so as to make integral convergent and $D$ is an arbitrary constant.

Using Hilbert transform technique, the solution of the integral equation (5.1.41) is given by

$$h(u^2) + \int_c^1 \frac{h(t^2)}{t}[k_1(u^2, t^2) + k_2(u^2, t^2)] dt = \frac{D}{(u^2 - c^2)^{\frac{1}{2}}(1 - u^2)^{\frac{1}{2}}}$$

(5.1.42)

where

$$k_1(u^2, t^2) = 4 \frac{\pi}{\pi^2} \left( \frac{u^2 - c^2}{1 - u^2} \right)^{\frac{1}{2}} \int_c^1 \left( \frac{1 - x^2}{x^2 - c^2} \right)^{\frac{1}{2}} \frac{x^2}{x^2 - u^2} dx \left[ \int_0^\infty \frac{1}{\zeta} [\alpha_3 \nu_3 A_5(\zeta) \sinh(\nu_3 x) + \alpha_4 \nu_4 A_6(\zeta) \sinh(\nu_4 x)] d\zeta \right]$$

(5.1.43)

$$k_2(u^2, t^2) = -4 \frac{\pi}{\pi^2} \left( \frac{u^2 - c^2}{1 - u^2} \right)^{\frac{1}{2}} \int_c^1 \left( \frac{1 - x^2}{x^2 - c^2} \right)^{\frac{1}{2}} \frac{x^2}{x^2 - u^2} dx \times \int_0^\infty H(\xi) \sin(\xi x) \{1 - \cos(\xi t)\} d\xi$$

(5.1.44)

$$A_5(\zeta) = \left[ \left( \zeta + \frac{\alpha_4 \nu_4}{\zeta} \right) i_3(\zeta) \sinh(\nu_4 h) - (c_{11} \nu_4 - c_{12} \alpha_4) i_4(\zeta) \cosh(\nu_4 h) \right] / M(\zeta)$$

(5.1.45)

$$A_6(\zeta) = -\left[ \left( \zeta + \frac{\alpha_3 \nu_3}{\zeta} \right) i_3(\zeta) \sinh(\nu_3 h) + (c_{11} \nu_3 - c_{12} \alpha_3) i_4(\zeta) \cosh(\nu_3 h) \right] / M(\zeta)$$

(5.1.46)

and

$$i_3(\zeta) = \frac{2}{\pi} \int_0^\infty \left[ \frac{\zeta \{c_{11} \xi^2 + c_{12} (k_1^2 + \nu_1^2)\}}{\nu_1^2 + \zeta^2} - \frac{\zeta \{c_{11} \xi^2 + c_{12} (k_1^2 + \nu_1^2)\}}{\nu_2^2 + \zeta^2} \right] \frac{\xi \{1 - \cos(\xi t)\} \cos(\xi h)}{\nu_1^2 - \nu_2^2} d\xi$$

(5.1.47)

$$i_4(\zeta) = \frac{2}{\pi} \int_0^\infty \left[ \frac{c_{12} \nu_2^2 + c_{11} \xi^2 - k_2^2}{\nu_1^2 + \zeta^2} - \frac{c_{12} \nu_2^2 + c_{11} \xi^2 - k_2^2}{\nu_2^2 + \zeta^2} \right] \frac{\xi \{1 - \cos(\xi t)\}}{(\nu_1^2 - \nu_2^2)} \sin(\xi h) d\xi$$

(5.1.48)

In order to determine the arbitrary constant $D$, multiplying equation (5.1.40) by

$$\frac{x}{(x^2 - c^2)^{\frac{1}{2}}(1 - x^2)^{\frac{1}{2}}}$$

and integrating from $c$ to 1 with respect to $x$, we obtain

$$\int_c^1 \frac{h(u^2)}{u} du = -\frac{\pi \nu_0}{c \log \left( \frac{1 + c}{1 - c} \right)} - \frac{4}{\pi \log \left( \frac{1 + c}{1 - c} \right)} \left[ \int_c^1 \frac{x B_1(x, t^2)}{(x^2 - c^2)^{\frac{1}{2}}(1 - x^2)^{\frac{1}{2}}} dx \right]$$

$$+ \int_c^1 \frac{h(t^2)}{t} dt \int_c^1 \frac{x B_2(x, t^2)}{(x^2 - c^2)^{\frac{1}{2}}(1 - x^2)^{\frac{1}{2}}} dx$$

(5.1.49)

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where

\[ B_1(x, t^2) = \frac{1}{d} \int_0^\infty \frac{1}{\zeta} \left[ \alpha_3 A_5(\zeta) \cosh(\nu_3 x) + \alpha_4 A_6(\zeta) \cosh(\nu_4 x) \right] d\zeta \quad (5.1.50) \]

\[ B_2(x, t^2) = \int_0^\infty \xi^{-1} H(\xi) \cos(\xi x) \{1 - \cos(\xi t)\} d\xi \quad (5.1.51) \]

Again, substituting \( h(u^2) \) from equation (5.1.43) in the equation (5.1.49) and simplifying, we obtain

\[ D = -\frac{2\nu_0 c}{d \log \left| \frac{1-c}{1+c} \right|} - \frac{8c}{\pi^2 \log \left| \frac{1-c}{1+c} \right|} \int_c^1 \frac{h(t^2)}{t} \left[ \int_c^1 \frac{x \{ B_1(x, t^2) + B_2(x, t^2) \}}{(x^2 - c^2)^{\frac{1}{2}}(1 - x^2)^{\frac{1}{2}}} \, dx \right] \]

\[ \quad + \frac{2c}{\pi} \int_c^1 \frac{h(t^2)}{t} \, dt \int_c^1 \frac{1}{u} \left\{ k_1(u^2, t^2) + k_2(u^2, t^2) \right\} du \quad (5.1.52) \]

Eliminating \( D \) from equations (5.1.42) and (5.1.52) and simplifying we obtain

\[ [(u^2 - c^2)(1 - u^2)]^{\frac{1}{2}} h(u^2) + \int_c^1 \frac{h(t^2)}{t} \left[ k_a(u^2, t^2) + k_b(u^2, t^2) + k_c(u^2, t^2) \right] dt = -\frac{2\nu_0 c}{d \log \left| \frac{1-c}{1+c} \right|} \quad (5.1.53) \]

where

\[ k_a(u^2, t^2) = \frac{4}{\pi^2} (u^2 - c^2) \int_c^1 \left( \frac{1 - x^2}{x^2 - c^2} \right)^{\frac{1}{2}} \frac{x^2}{x^2 - u^2} \left[ \frac{\partial}{\partial x} \{ B_1(x, t^2) + B_2(x, t^2) \} \right] \, dx \quad (5.1.54) \]

\[ k_b(u^2, t^2) = \frac{8c}{\pi^2 \log \left| \frac{1-c}{1+c} \right|} \int_c^1 \frac{x \{ B_1(x, t^2) + B_2(x, t^2) \}}{(x^2 - c^2)^{\frac{1}{2}}(1 - x^2)^{\frac{1}{2}}} \, dx \quad (5.1.55) \]

\[ k_c(u^2, t^2) = -\frac{8c}{\pi^2 u} \left( \frac{u^2 - c^2}{1 - u^2} \right)^{\frac{1}{2}} \int_c^1 \left( \frac{1 - x^2}{x^2 - c^2} \right)^{\frac{1}{2}} \frac{x^2}{x^2 - u^2} \]

\[ \quad \times \left[ \frac{\partial}{\partial x} \{ B_1(x, t^2) + B_2(x, t^2) \} \right] \, dx \quad (5.1.56) \]

Next for further simplification we put

\[ [(u^2 - c^2)(1 - u^2)]^{\frac{1}{2}} h(u^2) = H(u^2) \]

and make the substitution

\[ u^2 = c^2 \cos^2 \phi + \sin^2 \phi \quad \text{and} \quad t^2 = c^2 \cos^2 \theta + \sin^2 \theta \]
in equation (5.1.53) which then reduces to the form
\[ G(\phi) + \int_0^\frac{\pi}{2} \frac{G(\theta)}{c^2 \cos^2 \theta + \sin^2 \theta} [k'_a(\phi, \theta) + k'_b(\phi, \theta) + k'_c(\phi, \theta)] d\theta = -\frac{2v_0 c}{d \log \left| \frac{1-c}{1+c} \right|} \]  
(5.1.57)

where
\[ G(\phi) = H(c^2 \cos^2 \phi + \sin^2 \phi) \]  
(5.1.58)
\[ G(\theta) = H(c^2 \cos^2 \theta + \sin^2 \theta) \]  
(5.1.59)
\[ k'_a(\phi, \theta) = k_a(c^2 \cos^2 \phi + \sin^2 \phi, c^2 \cos^2 \theta + \sin^2 \theta) \]  
(5.1.60)
\[ k'_b(\phi, \theta) = k_b(c^2 \cos^2 \phi + \sin^2 \phi, c^2 \cos^2 \theta + \sin^2 \theta) \]  
(5.1.61)
\[ k'_c(\phi, \theta) = k_c(c^2 \cos^2 \phi + \sin^2 \phi, c^2 \cos^2 \theta + \sin^2 \theta) \]  
(5.1.62)

When \( h \) tends to infinity (\( h \to \infty \)), the medium become infinite. In this case, expression for \( p(x) \) given by equation (5.1.30) becomes \( p(x) = -\frac{\pi}{2c_0} v_0 \), since \( A_3(\zeta) \) and \( A_4(\zeta) \) given by equations (5.1.34) - (5.1.38) become zero.

\( A_3(\zeta) \) can be written as
\[ A_3(\zeta) = \frac{1}{2M(\zeta)} \left[ \left( \zeta + \frac{\alpha_4 v_4}{\zeta} \right) i_1(\zeta)(e^{\nu_4 h} - e^{-\nu_4 h}) - (c_{11} \nu_4 - c_{12} \alpha_4) i_2(\zeta)(e^{\nu_4 h} + e^{-\nu_4 h}) \right] \]

where
\[ M(\zeta) = \frac{1}{4} \left[ \left( \zeta + \frac{\alpha_4 v_4}{\zeta} \right)(c_{11} \nu_3 - c_{12} \alpha_3)(e^{\nu_3 h} + e^{-\nu_3 h})(e^{\nu_3 h} - e^{-\nu_3 h}) \right. \]
\[- \left. \left( \zeta + \frac{\alpha_3 v_3}{\zeta} \right)(c_{11} \nu_4 - c_{12} \alpha_4)(e^{\nu_4 h} - e^{-\nu_4 h})(e^{\nu_4 h} + e^{-\nu_4 h}) \right] \]

Therefore,
\[ A_3(\zeta) = \frac{1}{M_1(\zeta)} \left[ \left( \zeta + \frac{\alpha_4 v_4}{\zeta} \right) i_1(\zeta)(1 - e^{-2\nu_4 h}) - (c_{11} \nu_4 - c_{12} \alpha_4) i_2(\zeta)(1 + e^{-2\nu_4 h}) \right] \]

and
\[ M_1(\zeta) = \frac{e^{\nu_3 h}}{2} \left[ \left( \zeta + \frac{\alpha_4 v_4}{\zeta} \right)(c_{11} \nu_3 - c_{12} \alpha_3)(1 + e^{-2\nu_3 h})(1 - e^{-2\nu_3 h}) \right. \]
\[- \left. \left( \zeta + \frac{\alpha_3 v_3}{\zeta} \right)(c_{11} \nu_4 - c_{12} \alpha_4)(1 - e^{-2\nu_4 h})(1 + e^{-2\nu_4 h}) \right] \]

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As $h \to \infty$, $M_1(\zeta) \to \infty$ and therefore $A_3(\zeta) \to 0$.

Similarly, $A_4(\zeta) \to 0$.

So in this case the dual integral equations (5.1.26) and (5.1.27) become

$$\int_0^\infty \frac{1}{\xi} [1 + H(\xi)] A(\xi) \cos(\xi x) d\xi = -\frac{\pi}{2c} v_0, \quad c \leq |x| \leq 1$$

$$\int_0^\infty A(\xi) \cos(\xi x) d\xi = 0, \quad |x| < c, \quad |x| > 1$$

This problem has been analyzed in details by Sarkar et al. (1995).

4. Quantities of physical interest

The stress $\tau_{yy}(x, y)$ for $y \to 0$ in the neighbourhood of the strip can be found from equation (5.1.23) and is given by

$$\tau_{yy}(x, 0^\pm) = \mp \frac{2\mu_{12} c_{22}}{\pi} \int_0^\infty A(\xi) \cos(\xi x) d\xi, \quad c \leq |x| \leq 1 \quad (5.1.63)$$

Now $\Delta \tau_{yy}(x, 0) = \tau_{yy}(x, 0^+) - \tau_{yy}(x, 0^-)$

then $\Delta \tau_{yy}(x, 0) = -\frac{4}{\pi} \mu_{12} c_{22} \int_0^\infty A(\xi) \cos(\xi x) d\xi \quad (5.1.64)$

Substituting the value of $A(\xi)$ from equation (5.1.39) in the equation (5.1.64), we get

$$\Delta \tau_{yy}(x, 0) = 2 \mu_{12} c_{22} \frac{h(x^2)}{x} \quad (5.1.65)$$

Since $h(x^2) = [(x^2 - c^2)(1 - x^2)]^{-\frac{1}{2}} H(x^2)$ and $x^2 = c^2 \cos^2 \phi + \sin^2 \phi$,

equation (5.1.65) becomes

$$\Delta \tau_{yy}(x, 0) = \frac{2 \mu_{12} c_{22} G(\phi)}{x[(x^2 - c^2)(1 - x^2)]^{\frac{1}{2}}} \quad (5.1.66)$$

So the stress intensity factors $N_c$ and $N_1$ at the two tips of the strip can be expressed as

$$N_c = \lim_{x \to c^+} \left[ \frac{\Delta \tau_{yy}(x, 0)}{\pi c_{22} \mu_{12}} (x - c)^{\frac{1}{2}} \right]$$

$$= \frac{2}{\pi} \frac{G(0)}{c \sqrt{2c(1 - c^2)}} \quad (5.1.67)$$
and
\[ N_1 = \lim_{x \to 1^-} \left[ \frac{\Delta \tau_{yy}(x,0)}{\pi c_{22} \mu_{12}} (1 - x)^{\frac{1}{2}} \right] \]
\[ = \frac{2}{\pi} \frac{G(\frac{\pi}{2})}{\sqrt{2(1 - c^2)}} \]  
(5.1.68)

Making \( c \) tend to zero, the two strips merge into one and in that case
\[ N_1 = \frac{\sqrt{2}}{\pi} G(\pi/2) \]

Now from equation (5.1.22) after substituting the value of \( A_1(\xi) \) and using the equations (5.1.28) and (5.1.39), we get the vertical displacement outside the strip as
\[
v(x, y) = \frac{2}{\pi} \int_{0}^{\infty} \frac{1}{\xi} \left( \alpha_1 e^{-\nu_1 y} - \alpha_2 e^{-\nu_2 y} \right) \frac{\xi A(\xi) \cos(\xi x)}{(\alpha_1 \nu_1 - \alpha_2 \nu_2)} d\xi \\
+ \frac{2}{\pi} \int_{0}^{\infty} \frac{1}{\zeta} \left[ \alpha_3 A_3(\zeta) \cosh(\nu_3 x) + \alpha_4 A_4(\zeta) \cosh(\nu_4 x) \right] \cos(\zeta y) d\zeta \\
= \frac{2}{\pi} \int_{c}^{1} \frac{h(t^2)}{t} dt \left[ \int_{0}^{\infty} \left( \alpha_1 e^{-\nu_1 y} - \alpha_2 e^{-\nu_2 y} \right) \frac{1 - \cos(\xi t)}{(\alpha_1 \nu_1 - \alpha_2 \nu_2)} \right] d\xi \\
+ \int_{0}^{\infty} \frac{1}{\zeta} \left[ \alpha_3 A_5(\zeta) \cosh(\nu_3 x) + \alpha_4 A_6(\zeta) \cosh(\nu_4 x) \right] \cos(\zeta y) d\zeta \]  
(5.1.69)

Table 1. Engineering elastic constants.

<table>
<thead>
<tr>
<th></th>
<th>( E_1 ) (Pa)</th>
<th>( E_2 ) (Pa)</th>
<th>( \mu_{12} ) (Pa)</th>
<th>( \nu_{12} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type I</td>
<td>E-type glass-epoxy composite</td>
<td>9.79 \times 10^9</td>
<td>42.3 \times 10^9</td>
<td>3.66 \times 10^9</td>
</tr>
<tr>
<td>Type II</td>
<td>Stainless steel-aluminium composite</td>
<td>79.76 \times 10^9</td>
<td>85.91 \times 10^9</td>
<td>30.02 \times 10^9</td>
</tr>
</tbody>
</table>

5. Numerical calculations and discussions

The method of Fox and Goodwin (1953) has been used to solve the integral equation (5.1.57) numerically for different values of dimensionless frequency \( k_s \), material strip width \( 2h \) and separating distance of the strips \( 2c \). The integral in (5.1.57) has been represented by a quadrature formula involving values of the desired function \( G \) at pivotal points in the range of integration which leads to a set of algebraic linear simultaneous equations. The solution of the set of linear algebraic equations gives a
first approximation to the required pivotal values of $G$ which has been improved by the use of difference correction technique. After solving the integral equation (5.1.57) for different values of engineering elastic constants of several orthotropic materials listed in table-1, the stress intensity factors (SIF), $k_c$ and $k_1$ at both the ends of the strip given by equations (5.1.67) and (5.1.68) has been plotted against $k_s$ for different values of $h$ and $c$ and for different materials.

Figure 5.2: Stress intensity factor($N_c$) verses frequency($k_s$).

Figure 5.3: Stress intensity factor($N_c$) verses frequency($k_s$).

Figure 5.4: Stress intensity factor($N_c$) verses frequency($k_s$).

Figure 5.5: Stress intensity factor($N_c$) verses frequency($k_s$).
In figure 5.2 and figure 5.6, \( N_c \) (SIF, at the inner edge of the strip) and \( N_1 \) (SIF, at the outer edge of the strip) have been plotted against \( k_s \) for \( h=2.0 \) and \( h=2.5 \) and for different strip lengths (\( c=0.2, 0.4, 0.6 \)) for material type-I. In figure 5.4 and figure 5.8, \( N_c \) and \( N_1 \) have been plotted against \( k_s \) for \( c=0.4 \) and \( c=0.6 \) and for different material strip widths (\( h=2.0, 2.5, 3.0 \)) for material type-I. Same set of parameters
stated above the graphs for \( N_c \) and \( N_1 \) have been plotted in figures 5.3, 5.7, 5.5, 5.9 for material type-II.

For a particular value of material strip width \( h \) (= 2.0, 2.5) the value of \( N_c \) decreases initially and after increasing again it decreases with the increase in \( k_s \) for material type-I (figure 5.2) where as for material type-II, it is slowly decreasing with the increase in \( k_s \) (figure 5.3) for different values of strip length \( c \) (= 0.2, 0.4, 0.6). It is also observed that with the increase in \( c \) the value of \( N_c \) increases. When strip length \( c \) is fixed, the value of \( N_c \) is higher for higher values of \( h \) (= 2.0, 2.5, 3.0) (figure 5.4 and figure 5.5) for both type of materials. Fig. 5.6 - fig. 5.9 show that \( N_1 \) has initial decreasing tendency and then increases with the increase in \( k_s \) for both the materials. For fixed \( c \), \( N_c \) is higher when material strip width \( h \) is higher. In all the cases it is seen that as the length of the strip increases the value of \( N_1 \) decreases.

![Figure 5.10: Displacement (|v(x,y)|) verses distances (x,y)](image)

Finally in fig. 5.10 - fig. 5.13, the vertical displacement \( v(x,y) \) has been plotted outside the strips \((0 < x < c, \ 1 < x < h)\) for fixed values of \( h \) (= 2.5), \( k_s \) (= 0.4) and \( c \) (= 0.6) for both the materials. In fig. 5.10 and fig. 5.11, \( v(x,y) \) has been plotted the inner side of the strip \((0 < x < c)\) and in figure 5.12 and figure 5.13 for outer side of the strip \((1 < x < h)\). In all cases wave like nature of the displacement has been observed.
Figure 5.11: Displacement (\(|v(x, y)|\)) verses distances (x, y)

Figure 5.12: Displacement (\(|v(x, y)|\)) verses distances (x, y)

Figure 5.13: Displacement (\(|v(x, y)|\)) verses distances (x, y)