Total Domination Polynomials of Cartesian Products of some graphs

5.1 Introduction

In this chapter, we determine the total domination polynomials of Cartesian products of certain classes of graphs with $K_2$. We establish an interesting relation between domination polynomials and total domination polynomials of some graph classes. Moreover, we determine the total domination polynomials of the Cartesian product of some graphs with the cycle $C_4$. Further, the total domination number of some graphs are also determined. We need the following theorems for our further discussions.

Theorem 5.1.1. (see [1]) If a graph $G$ consists of $m$ components $G_1, G_2, \ldots, G_m$, then $D(G, x) = D(G_1, x) \ldots D(G_m, x)$.

\footnote{A part of this chapter has been published in Journal of Pure and Applied Mathematics: Advances and Applications, Volume 16, Number 2, 2016, Pages 97-108.}
Theorem 5.1.2. (see [1]) For every $n \geq 4$,

$$D(P_n, x) = x [D(P_{n-1}, x) + D(P_{n-2}, x) + D(P_{n-3}, x)] ,$$

with initial values $D(P_1, x) = x$, $D(P_2, x) = x^2 + 2x$, $D(P_3, x) = x^3 + 3x^2 + x$.

Theorem 5.1.3. (see [1]) For every $n \geq 4$,

$$D(C_n, x) = x [D(C_{n-1}, x) + D(C_{n-2}, x) + D(C_{n-3}, x)] ,$$

with initial values $D(C_1, x) = x$, $D(C_2, x) = x^2 + 2x$, $D(C_3, x) = x^3 + 3x^2 + 3x$.

Theorem 5.1.4. (see [1]) $D(K_{m,n}, x) = ((1 + x)^m - 1)((1 + x)^n - 1) + x^m + x^n$.

Theorem 5.1.5. (see [1]) If there exists $1 \leq j \leq k$, such that $s_j \geq 3$, then

$$D(\theta_{s_1, s_2, \ldots, s_j, \ldots, s_k}, x) = x[D(\theta_{s_1, s_2, \ldots, s_j-1, \ldots, s_k}, x) + D(\theta_{s_1, s_2, \ldots, s_{j-2}, \ldots, s_k}, x) + D(\theta_{s_1, s_2, \ldots, s_{j-3}, \ldots, s_k}, x)].$$

Theorem 5.1.6. (see [30]) If $L_n$ is the graph $P_n \square K_2$, then domination polynomial of $L_n$ satisfies the recurrence $D(L_n, x) = x(x + 2)D(L_{n-1}, x) + x(x + 1)D(L_{n-2}, x) + x^2(x + 1)D(L_{n-3}, x) - x^3D(L_{n-5}, x)$ with initial values,

<table>
<thead>
<tr>
<th>$n$</th>
<th>$P_n \square K_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$x^2 + 2x$</td>
</tr>
<tr>
<td>2</td>
<td>$x^4 + 4x^3 + 6x^2$</td>
</tr>
<tr>
<td>3</td>
<td>$x^6 + 6x^5 + 15x^4 + 16x^3 + 3x^2$</td>
</tr>
<tr>
<td>4</td>
<td>$x^8 + 8x^7 + 28x^6 + 52x^5 + 48x^4 + 47x^3 + 2x^3$</td>
</tr>
<tr>
<td>5</td>
<td>$x^{10} + 10x^9 + 45x^8 + 116x^7 + 178x^6 + 148x^5 + 47x^4 + 2x^3$</td>
</tr>
<tr>
<td>6</td>
<td>$x^{12} + 12x^{11} + 66x^{10} + 216x^9 + 453x^8 + 604x^7 + 470x^6 + 168x^5 + 17x^4$</td>
</tr>
</tbody>
</table>
5.2 On Cartesian products

In this section, we establish a relation between domination polynomials and total domination polynomials of some graphs. We prove that for any graph $G$, there exists a graph $H$ such that the set of all open neighborhoods of vertices of $K_2 \Box G$ is exactly the same as the set of all closed neighborhoods of vertices of $H$.

Theorem 5.2.1. For a bipartite graph $G$, $D_t(K_2 \Box G, x) = [D(G, x)]^2$.

Proof. Let $X = \{x_1, x_2, x_3, \ldots, x_m\}$, $Y = \{y_1, y_2, y_3, \ldots, y_n\}$ be the bipartition of $G$. If $V(K_2) = \{a, b\}$, then the bipartition of $K_2 \Box G$ is $\{(a, x_1), (a, x_2), \ldots, (a, x_m), (b, y_1), (b, y_2), \ldots, (b, y_n)\} \cup \{(b, x_1), (b, x_2), \ldots, (b, x_m), (a, y_1), (a, y_2), \ldots, (a, y_n)\}$. Then, for $i = 1, 2, \ldots, m$ and for $j = 1, 2, \ldots, n$, we have,

\[
N((a, x_i)) = \{(a, y)/y \sim x_i \text{ in } G\} \cup \{(b, x_i)\}
\]
\[
N((b, x_i)) = \{(b, y)/y \sim x_i \text{ in } G\} \cup \{(a, x_i)\}
\]
\[
N((a, y_j)) = \{(a, x)/x \sim y_j \text{ in } G\} \cup \{(b, y_j)\}
\]
\[
N((b, y_j)) = \{(b, x)/x \sim y_j \text{ in } G\} \cup \{(a, y_j)\}.
\]

Let $H_1$ be a bipartite graph with partite sets $\{(b, x_1), (b, x_2), \ldots, (b, x_m)\}$ and $\{(a, y_1), (a, y_2), \ldots, (a, y_n)\}$, such that $(b, x_i) \sim (a, y_j)$ if and only if $(a, x_i) \sim (a, y_j)$ in $K_2 \Box G$. Similarly, we construct another bipartite graph $H_2$ with partite sets $\{(a, x_1), (a, x_2), \ldots, (a, x_m)\}$ and $\{(b, y_1), (b, y_2), \ldots, (b, y_n)\}$, such that $(a, x_i) \sim (b, y_j)$ if and only if $(b, x_i) \sim (b, y_j)$ in $K_2 \Box G$. It can be observed that both $H_1$ and $H_2$ are isomorphic to $G$ and a set $N$ is an open neighborhood of a vertex in $K_2 \Box G$ if and only if $N$ is a closed neighborhood of a vertex in $H_1$ or $H_2$. 

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Therefore, a set \( S \subseteq V(G) \) is a total dominating set of \( K_2 \Box G \) if and only if it is a dominating set of \( H_1 \cup H_2 \). So, from Theorem 5.1.1 we have,

\[
D_t(K_2 \Box G, x) = D(G_1 \cup G_2, x) = D(G_1, x)D(G_2, x) = [D(G, x)]^2.
\]

This completes the proof.

**Corollary 5.2.2.** For a bipartite graph \( H \), \( \gamma_t(K_2 \Box H) = 2 \gamma(H) \).

*Proof.* By definition, \( \gamma(H) \) is the smallest power of \( x \) in the domination polynomial and \( \gamma_t(H) \) is the smallest power of \( x \) in the total domination polynomial of \( H \). From Theorem 5.2.1 we have \( D_t(K_2 \Box H, x) = [D(H, x)]^2 \). So, the least power of \( x \) in \( D_t(K_2 \Box H, x) \) is twice that of \( D(H, x) \), which proves our result.

**Corollary 5.2.3.** From Theorem 5.2.1, we obtain the following results.

1. \( D_t(K_2 \Box P_n, x) = [D(P_n, x)]^2 \).
2. \( \gamma_t(K_2 \Box P_n) = 2 \gamma(P_n) = 2 \lceil \frac{n}{3} \rceil \)
3. \( D_t(K_2 \Box C_{2n}, x) = [D(C_{2n}, x)]^2 \).
4. \( \gamma_t(K_2 \Box C_{2n}) = 2 \gamma(C_{2n}) = 2 \lceil \frac{2n}{3} \rceil \)
5. \( D_t(K_2 \Box K_{m,n}, x) = [D(K_{m,n}, x)]^2 \).
6. \( \gamma_t(K_2 \Box K_{m,n}) = 4 \).
7. \( D_t(K_2 \Box B_{m,n}, x) = [D(B_{m,n}, x)]^2 \), where \( B_{m,n} \) is the bi-star graph.
8. If $T$ is a tree, then $D_t(K_2 \Box T, x) = [D(T, x)]^2$.

Proof. The proof is straightforward.

**Theorem 5.2.4.** If $G$ is a non bipartite graph with $n$ vertices, then there exists a bipartite graph $H$ with $2n$ vertices such that $D_t(K_2 \Box G, x) = D(H, x)$.

Proof. Let $V(G) = \{1, 2, 3, \ldots, n\}$ and $V(K_2) = \{a, b\}$. Let $H$ be a bipartite graph with vertex set $\{(a, 1), (a, 2), \ldots, (a, n)\} \cup \{(b, 1), (b, 2), \ldots, (b, n)\}$ such that $(a, i)$ is adjacent to $(b, j)$ if and only if $i$ is adjacent to $j$ in $G$. Then for $1 \leq i, j \leq n$, $N_{K_2 \Box G}((a, i)) = N_H[(b, i)]$ and $N_{K_2 \Box G}((b, j)) = N_H[(a, j)]$. Therefore, a TD-set of $K_2 \Box G$ dominates $H$ and if $S$ dominates $H$, then it is a TD-set of $K_2 \Box G$. This completes the proof.

**Lemma 5.2.5.** A set $S$ is a dominating set of a cycle $C_n$ if and only if it has non empty intersection with the set of vertices of each and every path of length two in $C_n$.

Proof. Let $S$ be a dominating set and $P = uvw$ be a path of length two in $C_n$. If $S \cap \{u, v, w\} = \phi$, then the vertex $v$ is not adjacent to any vertex in $S$ and so $S$ cannot be a dominating set of $C_n$. The converse is obvious.

**Theorem 5.2.6.** If $n$ is odd, then $D_t(K_2 \Box C_n, x) = D(C_{2n}, x)$.

Proof. Let the vertices of $K_2 \Box C_n$ be labeled as shown in figure 5.1. Let $H_G$ be the open neighborhood hypergraph of $K_2 \Box C_n$. Let $C_{2n}$ be the cycle with vertex set $\{(r, 1), (r, 2), \ldots, (r, n), (t, 1), (t, 2), \ldots, (t, n)\}$ such that $(r, i)$ is adjacent to $(t, j)$ if and only if $i$ is adjacent to $j$ in $C_n$. It can be observed that if $v \in V(K_2 \Box C_n)$,
then the vertices of $N(v)$ form a path of length two in $C_{2n}$. Also the set of vertices of any path of length two in $C_{2n}$ is an edge in $H_G$. Therefore, the proof follows from Theorem 2.1.14 and Lemma 5.2.5.

Figure 5.1: The graph $K_2 \square C_n$

5.3 TD-Polynomials of Cayley graphs

In this section, we find the total domination polynomials of some cubic cayley graphs.

**Theorem 5.3.1.** Let $G = \text{Cay}(\mathbb{Z}_n, S)$, where $S = \{a, b, b^{-1}\}$ is a generating set of $\mathbb{Z}_n$ such that $a^{-1} = a$ and $a \notin \langle b \rangle$. Then,

$$D_t(G, x) = \begin{cases} [D(C_{\frac{n}{2}}, x)]^2, & \text{if } n/2 \text{ is even} \\ D(C_n, x), & \text{otherwise.} \end{cases}$$

**Proof.** Here $G \cong K_2 \square C_2$. So the proof follows from Corollary 5.2.3 and Theorem 5.2.6.
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**Theorem 5.3.2.** Let \( G = \text{Cay}(\mathbb{Z}_n, S) \), where \( S = \{a, b, b^{-1}\} \) such that \( a^{-1} = a \) and \( b \) is a generator of \( \mathbb{Z}_n \). Then,

\[
D_t(G, x) = \begin{cases} 
D(C_n, x), & \text{if } n \text{ is a multiple of 4} \\
[D(C_{\frac{n}{2}}, x)]^2, & \text{otherwise}
\end{cases}
\]

**Proof.** Since \( b \) is a generator of \( \mathbb{Z}_n \), the graph \( G = \text{Cay}(\mathbb{Z}_n, S) \) can be labeled as shown in figure 5.2.

![Figure 5.2: The graph \( G = \text{Cay}(\mathbb{Z}_n, S) \)](image)

**Case 1:** When \( n \) is a multiple of 4.

Let \( C_n \) be the cycle \( (b^n, b^{\frac{n}{2}+1}, b^2, b^{\frac{n}{2}+3}, b^4, \ldots, b^{n-2}, b^{\frac{n}{2}-1}, b^n) \). It can be observed that if \( v \in V(G) \), then the vertices of \( N(v) \) form a path of length two in \( C_n \) and any path of length two in \( C_n \) is an open neighborhood of a vertex in \( G \). Therefore, by Lemma 5.2.5, \( D_t(G, x) = D(C_n, x) \).

**Case 2:** When \( n \) is not a multiple of 4.

In this case the graph \( G = \text{Cay}(\mathbb{Z}_n, S) \) is bipartite with bipartition \( X = \{b^k: k \text{ is odd}\} \) and \( Y = \{b^s: s \text{ is even}\} \). Let the components of the open neighborhood hypergraph of \( G \) are \( H_X = (Y, \{N(x): x \in X\}) \) and \( H_Y = (X, \{N(x): x \in Y\}) \). Since \( H_X \) is isomorphic to \( H_Y \), by Theorem 2.1.14.
5.4. TD-Polynomials of regular graphs

\[ D_t(G, x) = [C(H_X, x)]^2 \]. Also \( E(H_X) = \{ N(b), N(b^{2+2}), N(b^3), \ldots, N(b^n) \} \)

\[ = \{ b^n, b^{n+1}, b^2 \}, \{ b^{n+1}, b^2, b^{n+3}, 2 \}, \ldots, \{ b^{n+1}, b^n, b^{n+1} \} \}. \]

Let \( C_n \) be the cycle, \( (b^n, b^{n+1}, b^2, b^{n+3}, b^4, \ldots, b^{n+1}, b^n) \), with vertex set \( Y \). It is obvious that if \( v \in Y \), then the vertices of \( N_G(v) \) form a path of length two in \( C_n \). Also any path of length two in \( C_n \) is an open neighborhood of a vertex in \( Y \). Therefore, by Lemma 5.2.5, \( C(H_X, x) = D(C_n, x) \).

This completes the proof. \( \square \)

5.4 TD-Polynomials of regular graphs

In this section, we investigate how the domination polynomials of some regular graphs and total domination polynomials of their Cartesian product with \( K_2 \) are related.

**Lemma 5.4.1.** If \( G \) is an \((m-1)\)-regular bipartite graph, then

\[ D(G, x) = mx^2 \left[ 1 + \binom{2m-2}{1} x + \ldots + \binom{2m-2}{m-3} x^{m-1} \right] + \left[ \frac{m}{m-2} + 2 \right] x^m + \left( \frac{2m}{m+1} \right) x^{m+1} + \left( \frac{2m}{m+2} \right) x^{m+2} + \ldots + \left( \frac{2m}{2m} \right) x^{2m}. \]

**Proof.** Let \( X = \{a_1, a_2, \ldots, a_m\} \) and \( Y = \{b_1, b_2, \ldots, b_m\} \) be the bipartition of \( G \). Assume that \( a_i \) is not adjacent to \( b_i \) for all \( i \). Let \( S \) be a set of vertices of \( G \). If \( \{a_r, b_r\} \subseteq S \) for some \( r \), then \( S \) is a dominating set of \( G \). Since there are \( m \) pairs, one pair can be selected in \( m \) ways. So when \( k < m \), the coefficient of \( x^k \) in \( D(G, x) \) is \( m \binom{2m-2}{k-2} \). When \( k = m \), since \( X \) and \( Y \) are also dominating
sets of $G$, the coefficient of $x^m$ is $m\binom{2m-2}{m-2} + 2$. When $k > m$, any subset of the vertices of $G$ contains a pair $a_i, b_i$. So the coefficient of $x^k$ in $D(G, x)$ is $\binom{2m}{k}$.

This completes the proof.

**Theorem 5.4.2.** $D_t(K_2\square K_n, x) = D(G, x)$, where $G$ is an $(n-1)$-regular bipartite graph on $2n$ vertices.

**Proof.** Let the vertex sets of $K_2$ and $K_n$ be $\{a, b\}$ and $\{1, 2, 3, \ldots, n\}$ respectively. If $A = \{(a, 1), (a, 2), \ldots, (a, n)\}$ and $B = \{(b, 1), (b, 2), \ldots, (b, n)\}$, then $V(K_2\square K_n) = A \cup B$. Also $N((a, i)) = \{(b, i)\} \cup A \setminus \{(a, i)\}$ and $N((b, i)) = \{(a, i)\} \cup B \setminus \{(b, i)\}$. We construct an $(n-1)$-regular bipartite graph $G$ in which the open neighborhoods $N((a, i))$ and $N((b, i))$ are represented as star graphs with root vertices $(b, i)$ and $(a, i)$ respectively.

![Figure 5.3: The graph $G$](image)

Let $S$ be a dominating set of $G$. Then for all $i$, either $(a, i) \in S$ or $(a, i)$ is adjacent to some vertex in $S$. If $(a, i) \in S$ for all $i$, then $S \cap N((b, i)) \neq \emptyset$ for all $i$. If $(a, r) \notin S$ for some $r$, then there exists $(b, s) \in S$ for some $s \neq r$. Therefore, $S \cap N_{K_2\square K_n}((b, i)) \neq \emptyset$ for all $i$. Similarly we can prove that $S \cap$
5.5. TD-Polynomials of friendship graphs

$N_{K_2□K_n}((a,i)) \neq \phi$ for all $i$. Therefore, $S$ is a vertex covering set of the open neighborhood hypergraph $H_{K_2□K_n}$. Conversely, let $S$ is a vertex covering set of $H_{K_2□K_n}$. We prove that $S$ is a dominating set of $G$. Consider a vertex $(a,i)$. If $(a,i) \in S$, then there is nothing to prove. If $(a,i) \notin S$, then $(b,j) \in S$ for some $j \neq i$. So $(a,i)$ is adjacent to $(b,j)$ in $G$. Similarly we can prove the case of $(b,j)$ also. Therefore, $S$ is a dominating set of $G$. Thus the result follows from Theorem 2.1.14.

5.5 TD-Polynomials of friendship graphs

In this section, we determine the total domination polynomials of Cartesian product of friendship graphs with $K_2$.

Here we need the following.

Definition 5.5.1. (see [1]) The friendship graph $F_n$ with $2n+1$ vertices and $3n$ edges, is the graph formed by the join of $K_1$ with $n$ copies of $K_2$.

Theorem 5.5.2. $D_t(K_2□F_n,x) = D(\theta_{3,3,\ldots,3},x)_{(2n \text{ times})}$.

Proof. Let $V(F_n) = \{u,1,2,3,\ldots,n\}$ such that $N(u) = \{1,2,3,\ldots,n\}$. For an even vertex $x$, $N(x) = \{u,x-1\}$ and for an odd vertex $x$, $N(x) = \{u,x+1\}$. Let $V(K_2) = \{a,b\}$. Then in $K_2□F_n$,

$N((a,u)) = \{(b,u),(a,1),(a,2),\ldots,(a,n)\}$

$N((b,u)) = \{(a,u),(b,1),(b,2),\ldots,(b,n)\}$

For even $x$, $N((a,x)) = \{(a,u),(b,x),(a,x-1)\}$
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\[
N((b, x)) = \{(b, u), (a, x), (b, x - 1)\}
\]

For odd $x$, \(N((a, x)) = \{(a, u), (b, x), (a, x + 1)\}\)

\[
N((b, x)) = \{(b, u), (a, x), (b, x + 1)\}
\]

Since the Friendship graph $F_n$ is not bipartite, by Theorem 5.2.4 there exists a bipartite graph $H$ such that $D_t(K_2 \square F_n, x) = D(H, x)$. In $H$, we have,

\[
N((a, u)) = \{(b, 1), (b, 2), \ldots, (b, n)\}
\]

\[
N((b, u)) = \{(a, 1), (a, 2), \ldots, (a, n)\}
\]

If $x$ is even, then $N((a, x)) = \{(b, u), (b, x - 1)\}$ and

If $x$ is odd, then $N((a, x)) = \{(b, u), (b, x + 1)\}$

Note that $H$ is the theta graph $\theta_{\underbrace{3, 3, \ldots, 3}_{2n \text{ times}}}$. It can be observed that for each vertex $s$ of $K_2 \square F_n$ there exists a vertex $t$ in $H$ such that $N_{K_2 \square F_n}(s) = N_H[t]$.

Therefore, $D_t(K_2 \square F_n, x) = D(H, x)$. This completes the proof. \hfill \Box

The construction of $H$ in the case of $F_2$ is shown in figure 5.4.

5.6 Total domination polynomials of $C_4 \square G$

In this section, we determine the total domination polynomials of Cartesian product of certain classes of graphs with the cycle $C_4$.

**Theorem 5.6.1.** For a bipartite graph $G$,

\[
D_t(C_4 \square G, x) = [D(K_2 \square G, x)]^2.
\]
5.6. Total domination polynomials of $C_4 \Box G$

Proof. Since the graph $K_2 \Box K_2 \Box G$ is isomorphic to $C_4 \Box G$, the result follows from Theorem 2.1.14 and 5.2.1. □

![Graphs](image)

Figure 5.4: The graphs $F_2, K_2 \Box F_2$ and $H$

**Corollary 5.6.2.** From Theorem 5.6.1 we obtain the following results.

1. $D_t(C_4 \Box P_n, x) = [D(L_n, x)]^2$,
2. $\gamma_t(C_4 \Box P_n) = 2 \left\lceil \frac{n + 1}{2} \right\rceil$,
3. $D_t(C_4 \Box C_{2n}, x) = [D(K_2 \Box C_{2n}, x)]^2$,
4. $\gamma_t(C_4 \Box C_{2n}) = \begin{cases} 2n, & \text{if } n \text{ is even} \\ 2n + 2, & \text{otherwise.} \end{cases}$
5. $D_t(C_4 \Box K_{1,n}, x) = [D(K_2 \Box K_{1,n}, x)]^2$,
6. $\gamma_t(C_4 \Box K_{1,n}) = 4$.  

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TD-Polynomials of Splitting Graphs

6.1 Introduction

In this chapter, we are concerned with total domination polynomials of splitting graphs of order $k$ of a graph $G$. Moreover, we introduce the terminology of iterated splitting graph $S^r(G)$ of a graph $G$ and determine its total domination polynomial.

We need the following results.

**Theorem 6.1.1.** (see [18]) For the path graph $P_n$, where $n > 1$, we have

$$C(P_n, x) = \sum_{i=0}^{n} \binom{i+1}{n-i} x^i.$$

**Theorem 6.1.2.** (see [18]) For the cycle graph $C_n$, where $n \geq 3$, we have

$$C(C_n, x) = \sum_{i=1}^{n} \binom{n}{i} \binom{i}{n-i} x^i.$$

**Definition 6.1.3.** (see [43]) The splitting graph of a graph $G$ is defined as, for

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1 A part of this chapter has been published in *Advances and Applications in Discrete Mathematics*, Volume 18, Number 3, 2017, Pages 331-343.
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Each vertex \( v \) of \( G \), take a new vertex \( v' \) and join \( v' \) to all vertices of \( G \) adjacent to \( v \). The graph \( \text{spl}(G) \) thus obtained is called the splitting graph of \( G \).

**Definition 6.1.4.** The splitting graph of order \( k \) of a graph \( G \), denoted by \( \text{spl}^k(G) \) is defined as for each vertex \( v \) of \( G \), take \( k \) new vertices \( v_1, v_2, \ldots, v_k \) and join each of these vertices to all vertices of \( G \) adjacent to \( v \).

**Lemma 6.1.5.** If \( G \) is bipartite, then \( \text{spl}^k(G) \) is bipartite.

**Proof.** Let \((X, Y)\) be the bipartition of \( G \) and \( X', Y' \) be collection of new vertices of \( \text{spl}^k(G) \) corresponding to the vertices of \( X \) and \( Y \) respectively. Then \( X \cup X' \) and \( Y \cup Y' \) are the partite sets of \( \text{spl}^k(G) \). This proves the result. \( \square \)

### 6.2 On splitting graphs

It is noted that for a graph \( G \), the TD-polynomials of splitting graph of a graph \( G \) is closely related to the total domination polynomial of \( G \). In this section, the total domination polynomial of splitting graph of \( G \) is determined.

**Theorem 6.2.1.** For a connected graph \( G \) with \( n \) vertices,

\[
D_t\left(\text{spl}^k(G), x\right) = D_t(G, x)(1 + x)^{nk}.
\]

**Proof.** For a vertex \( v \) in \( G \), let \( v^1, v^2, \ldots, v^k \) be the new vertices in \( \text{spl}^k(G) \). Then for \( i = 1, 2, \ldots, k \), \( N_{\text{spl}^k(G)}(v^i) = N_G(v) \). Also \( N_{\text{spl}^k(G)}(v) \supseteq N_G(v) \). Therefore, if \( S \) is a total dominating set of \( G \), then \( S \) is a total dominating set of \( \text{spl}^k(G) \). From the construction of \( \text{spl}^k(G) \), it can be observed that if \( K \) is a total dominating set of \( \text{spl}^k(G) \), then \( K \cap V(G) \) is a total dominating set of \( G \). Since \( G \) has \( n \) vertices,
6.2. On splitting graphs

$spl^k(G)$ has $n(k+1)$ vertices. So the $nk$ new vertices need not be present in a total dominating set of $spl^k(G)$. Note that from $nk$ vertices $r$ vertices can be selected in $\binom{nk}{r}$ ways. Therefore, $D_t(spl^k(G), x) = D_t(G, x) \left[ 1 + \binom{nk}{1} + \binom{nk}{2} + \ldots + \binom{nk}{nk} \right]$. This proves the result. 

\[ \square \]

Corollary 6.2.2. For the path $P_{2n}$, where $n \geq 1$, we have

\[ D_t(spl^k(P_{2n}), x) = \left[ \sum_{i=1}^{n} \left( \frac{i+1}{n-i} \right) x^i \right]^2 (1+x)^{2nk}. \]

Proof. From Theorem 2.2.5 we have, $\left[ \sum_{i=1}^{n} \left( \frac{i+1}{n-i} \right) x^i \right]^2$. Since $spl^k(P_{2n})$ has $2nk$ new vertices, the proof follows from Theorem 6.2.1. 

\[ \square \]

Corollary 6.2.3. For the path $P_{2n+1}$, where $n \geq 1$, we have

\[ D_t(spl^k(P_{2n+1}), x) = x^2 \left[ \sum_{i=0}^{n+1} \left( \frac{i+1}{n+1-i} \right) x^i \right] \left[ \sum_{i=0}^{n-2} \left( \frac{i+1}{n-(2+i)} \right) x^i \right] (1+x)^{(2n+1)k}. \]

Proof. From Theorem 2.2.7 we have, $D_t(P_{2n+1}, x) = x^2 \left[ \sum_{i=0}^{n+1} \left( \frac{i+1}{n+1-i} \right) x^i \right] \left[ \sum_{i=0}^{n-2} \left( \frac{i+1}{n-(2+i)} \right) x^i \right]$. Thus the proof follows from Theorem 6.2.1. 

\[ \square \]

Corollary 6.2.4. For the cycle graph $C_{2n}$, where $n \geq 2$, we have

\[ D_t(spl^k(C_{2n}), x) = \left[ \sum_{i=1}^{n} \frac{n}{i} \left( \frac{i}{n-i} \right) x^i \right]^2 (1+x)^{2nk}. \]

Proof. From Theorem 2.2.8 we have, $D_t(C_{2n}, x) = \left[ \sum_{i=1}^{n} \frac{n}{i} \left( \frac{i}{n-i} \right) x^i \right]^2$. Therefore, the proof follows from Theorem 6.2.1. 

\[ \square \]
Corollary 6.2.5. If \( n \) is odd, then
\[
D_t\left(\text{spl}^k(C_n), x\right) = \left[ \sum_{i=1}^{n} \binom{n}{i} \left( \frac{i}{n-i} \right) x^i \right] (1 + x)^{nk}.
\]

Proof. If \( n \) is odd, from Theorem 2.2.10, we have, \( D_t(C_n, x) = \sum_{i=1}^{n} \binom{n}{i} \left( \frac{i}{n-i} \right) x^i \). Then the proof follows from Theorem 6.2.1. \( \square \)

Corollary 6.2.6. Using Theorem 6.2.1 we can infer the following results.

(i) \( D_t(\text{spl}^k(K_{m,n}), x) = [(1 + x)^{m+n} - (1 + x)^m - (1 + x)^n + 1] [1 + x]^{(m+n)k} \),

(ii) \( D_t(\text{spl}^k(K_{1,n}), x) = x [(1 + x)^n - 1] [1 + x]^{(1+n)k} \),

(iii) \( D_t(\text{spl}^k(K_n), x) = [(1 + x)^n - 1 - x] [1 + x]^{nk} \).

Proof. The proof is straightforward. \( \square \)

6.3 On iterated splitting graph

In this section, we introduce the terminology of iterated splitting graph of a graph \( G \) and derive its total domination polynomial.

Definition 6.3.1. The iterated splitting graph \( S^i(G) \) of a graph \( G \) is defined as \( S^i(G) = S^i(S^{i-1}(G)) \), where \( i = 2, 3, \ldots, k \) and \( S^1(G) \) denotes the splitting graph \( \text{spl}(G) \) of \( G \).

Lemma 6.3.2. If \( G \) is bipartite, then \( S^k(G) \) is bipartite.

Proof. The proof is similar to that of Lemma 6.1.5. \( \square \)
Theorem 6.3.3. For a connected graph $G$ with $n$ vertices, the total domination polynomial of the iterated splitting graph of $G$ is

$$D_t(S^k(G)) = D_t(G, x)(1 + x)^{n2^{k-1}}.$$  

Proof. Let $G$ be a connected graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. Let $v_1^k, v_2^k, \ldots, v_n^k$ be the new vertices in the iterated splitting graph $S^k(G)$ such that $N_{S^k(G)}(v_i^k) = N_G(v_i)$. From the construction of $S^k(G)$, for $1 \leq j \leq k - 1$, we have, $N_{S^k(G)}(v_j^k) \supseteq N_{S^k(G)}(v_i^k) = N_G(v_i)$. Therefore, a vertex covering set of $G$ is a vertex covering set of $S^k(G)$. Also if $S$ is a vertex covering set of $S^k(G)$, then $S \cap V(G)$ is a vertex covering set of $G$. Since the iterated splitting graph $S^k(G)$ has $n2^k$ vertices, the total domination polynomial of $S^k(G)$ is $D_t(G, x)(1 + x)^{n2^{k-1}}$. This completes the proof.

Corollary 6.3.4. For the path $P_{2n}$, where $n \geq 1$, we have

$$D_t(S^k(P_{2n}), x) = \left[\sum_{i=1}^{n} \left(\frac{i + 1}{n - (i + 1)}\right)x^i\right]^2(1 + x)^{n2^k}.$$  

Proof. From Theorem 2.2.5 we have, $\left[\sum_{i=1}^{n} \left(\frac{i + 1}{n - (i + 1)}\right)x^i\right]^2$. Since the iterated splitting graph $S^k(P_{2n})$ has $n2^{k+1}$ vertices, from Theorem 6.3.3 we have,

$$D_t(S^k(P_{2n}), x) = D_t(P_{2n}, x)(1 + x)^{2n2^{k-1}} = \left[\sum_{i=1}^{n} \left(\frac{i + 1}{n - (i + 1)}\right)x^i\right]^2(1 + x)^{n2^k}.$$  

This proves the result.
6.3. On iterated splitting graph

**Corollary 6.3.5.** For the path $P_{2n+1}$, where $n \geq 1$, we have

$$D_t(S^k(P_{2n+1}), x) = x^2 \left[ \sum_{i=0}^{n+1} \frac{i+1}{n+1} x^i \right] \left[ \sum_{i=0}^{n-2} \frac{i+1}{n-(2+i)} x^i \right] (1+x)^{(2n+1)2^{k-1}}.$$  

**Proof.** From Theorem 2.2.7 we have,

$$D_t(P_{2n+1}, x) = x^2 \left[ \sum_{i=0}^{n+1} \frac{i+1}{n+1} x^i \right] \left[ \sum_{i=0}^{n-2} \frac{i+1}{n-(2+i)} x^i \right].$$

Thus the proof follows from Theorem 6.3.3. 

---

**Corollary 6.3.6.** For the cycle graph $C_{2n}$, where $n \geq 2$, we have

$$D_t(S^k(C_{2n}), x) = \left[ \sum_{i=1}^{n} \frac{n}{i} \left( \frac{i}{n-i} \right) x^i \right]^2 (1+x)^{n2^k}.$$  

**Proof.** From Theorem 2.2.8 we have, $D_t(C_{2n}, x) = \left[ \sum_{i=1}^{n} \frac{n}{i} \left( \frac{i}{n-i} \right) x^i \right]^2$. Therefore, the proof follows from Theorem 6.3.3. 

---

**Corollary 6.3.7.** If $n$ is odd, then

$$D_t(S^k(C_n), x) = \left[ \sum_{i=1}^{n} \frac{n}{i} \left( \frac{i}{n-i} \right) x^i \right] (1+x)^{n2^{k-1}}.$$  

**Proof.** If $n$ is odd, from Theorem 2.2.10, we have, $D_t(C_n, x) = \sum_{i=1}^{n} \frac{n}{i} \left( \frac{i}{n-i} \right) x^i$. Then the proof follows from Theorem 6.3.3. 

---

**Corollary 6.3.8.** Using Theorem 6.3.3 we can infer the following results.

(i) $D_t(S^k(K_{m,n}), x) = \left[ (1+x)^{m+n} - (1+x)^m - (1+x)^n + 1 \right] [1+x]^{(m+n)2^{k-1}},$

(ii) $D_t(S^k(K_{1,n}), x) = x \left[ (1+x)^n - 1 \right] [1+x]^{(1+n)2^{k-1}},$

(iii) $D_t(S^k(K_n), x) = \left[ (1+x)^n - 1 - x \right] [1+x]^{n2^{k-1}}$. 


6.3. On iterated splitting graph

Proof. (i) We have, \( D_t(K_{m,n}, x) = [(1 + x)^m - 1] [(1 + x)^n - 1] \). Since, \( S^k(K_{m,n}) \) has \((m + n)2^k\) vertices, the proof follows from Theorem 6.3.3.

(ii) The proof follows immediately by substituting \( m = 1 \) in (i).

(iii) The TD-Polynomial of \( K_n \) is \([ (1 + x)^n - 1 - x ] \). Since \(| V(K_n) | = n\), \( S^k(K_n) \) has \( n2^k \) vertices. Thus the proof follows from Theorem 6.3.3.

\[ \square \]
Chapter 7

Global Bipartite domination in Graphs

7.1 Introduction

In this chapter the concepts of the global bipartite domination number, $\gamma_{gb}(G)$ and global bipartite total domination number, $\gamma_{gbt}(G)$ of a connected bipartite graph $G$ are introduced. We study some of the general properties of $\gamma_{gb}$ and $\gamma_{gbt}$. Moreover, we determine the global bipartite domination number and global bipartite total domination number of certain classes of graphs. Connected spanning subgraphs of $K_{m,n}$ with global bipartite domination number and global bipartite total domination number $m + n$ or $m + n - 1$ are characterized.

\footnote{A part of this chapter has been published in Malaya Journal of Mathematik. Volume 4, Number 3, 2016, Pages 438-442.}
7.2 Global bipartite domination

Definition 7.2.1. Let $G$ be a connected bipartite graph with bipartition $(X, Y)$, with $|X| = m$ and $|Y| = n$. The relative complement of $G$ in $K_{m,n}$, denoted by $\hat{G}$, is the graph obtained by deleting all edges of $G$ from $K_{m,n}$ (i.e., $K_{m,n} - E(G)$). A global bipartite dominating set (GBDS) of $G$ is a set $S$ of vertices of $G$ such that it dominates $G$ and its relative complement $\hat{G}$. The global bipartite domination number, $\gamma_{gb}(G)$, is the minimum cardinality of a global bipartite dominating set of $G$.

Example 7.2.2. For the graph given in Figure 7.1, $S = \{1, 2, 3\}$ is the minimum global bipartite dominating set of $G$. So $\gamma_{gb}(G) = 3$.

It can be observed that global bipartite domination is defined for connected bipartite graphs only.

Theorem 7.2.3. For any connected spanning subgraph $G$ of $K_{m,n}$, $\gamma(G) \leq \gamma_{gb}(G) \leq m + n$.

Proof. A global bipartite dominating set of $G$ is a dominating set of $G$ and so
7.2. Global bipartite domination

\[ \gamma(G) \leq \gamma_{gb}(G) \]. The set of all vertices of \( G \) is clearly a GBDS of \( G \) so, \( \gamma_{gb}(G) \leq m + n \). Therefore, \( \gamma(G) \leq \gamma_{gb}(G) \leq m + n \).

Remark 7.2.4. The bounds in Theorem 7.2.3 are sharp. For the complete bipartite graph \( K_{m,n} \), \( \gamma_{gb}(K_{m,n}) = m + n \). For the path graph \( P_4 \), \( \gamma(P_4) = \gamma_{gb}(P_4) = 2 \). So \( K_{m,n} \) has the largest possible GBD number. Also the bounds in Theorem 7.2.3 are strict. For the graph \( K_{2,3} - e \), \( \gamma(K_{2,3} - e) = 2 \) and \( \gamma_{gb}(K_{2,3} - e) = 4 \).

Theorem 7.2.5. For a spanning subgraph \( G \) of \( K_{m,n} \), if \( G \) and \( \hat{G} \) do not contain any isolated vertices, then \( \gamma_{gb}(G) \leq \min\{m,n\} \).

Proof. Let \((X,Y)\) be the bipartition of \( G \) with \(|X| = m \leq |Y| = n \). Since \( G \) and \( \hat{G} \) does not contain isolated vertices, \( X \) is a G.B.D.S. of \( G \). Therefore, \( \gamma_{gb}(G) \leq m \).

Theorem 7.2.6. For any two positive integers \( m \) and \( n \), \( \gamma_{gb}(K_{m,n}) = m + n \).

Proof. Let \( G \) be a complete bipartite graph with partitions \( X \) and \( Y \). Then \( uv \in E(G) \) for every \( u \in X \) and \( v \in Y \). Let \( \hat{G} \) denotes the relative complement of \( G \) in \( K_{m,n} \). Then \( \hat{G} \) contains \( m + n \) isolated vertices. Hence every global bipartite dominating set of \( G \) must contain all vertices of \( \hat{G} \) and so \( \gamma_{gb}(G) \geq m + n \). Now \( V(G) \) is a global bipartite dominating set of \( G \). Hence \( \gamma_{gb}(G) = m + n \).

Theorem 7.2.7. For a spanning subgraph \( G \) of \( K_{m,n} \), a vertex \( v \) is in every global bipartite dominating set of \( G \) if and only if \( v \) is an isolated vertex in \( \hat{G} \).

Proof. If \(|V(G)| \leq 3\), the proof is trivial. So let \(|V(G)| > 3 \). If \( v \) is an isolated vertex in \( \hat{G} \), then \( v \) is in every global bipartite dominating set of \( G \). Conversely if \( v \) is not an isolated vertex in \( \hat{G} \), then there exist at least two vertices \( u \) and \( w \).
such that $u$ is adjacent to $v$ in $G$ and $w$ is adjacent to $v$ in $\hat{G}$. So $V(G) \setminus \{v\}$ is a global bipartite dominating set of $G$. This completes the proof. \qed

**Theorem 7.2.8.** Let $G$ be a connected bipartite graph with partite sets $X$ and $Y$. Let $S = V_1 \cup V_2$ be a GBDS of $G$, where $V_1 \subseteq X$ and $V_2 \subseteq Y$. Then if $V_1 = \emptyset$, then $V_2 = Y$ and if $V_2 = \emptyset$, then $V_1 = X$.

**Proof.** Let $S = V_1 \cup V_2$ be a global bipartite dominating set of $G$ such that $V_1 \subseteq X$ and $V_2 \subseteq Y$. If $V_1 = \emptyset$, then $S \subseteq Y$. Since $G$ is bipartite, the vertices in $Y$ are not adjacent. Therefore, $S$ is a GBDS of $G$ only if $S \supseteq Y$. Therefore, $S = V_2 = Y$. Similarly, we can prove that if $V_2 = \emptyset$ then $V_1 = X$. \qed

**Theorem 7.2.9.** Let $(X,Y)$ be the bipartition of a connected graph $G$. Then $X$ is a GBDS of $G$ if and only if $|N(y)| < |X|$, $\forall y \in Y$.

**Proof.** Let $X$ be a GBDS of $G$. If possible assume that there exists a vertex $y \in Y$ such that $|N(y)| = |X|$. Then $y$ is an isolated vertex in $\hat{G}$, contradicting the fact that $X$ is a GBDS of $G$. Conversely, since $G$ is connected, $X$ is dominating set of $G$. So it is sufficient to show that $X$ dominates $\hat{G}$ also. Let $y \in Y$, then $N(y)$ is a proper subset of $X$. So $y$ is adjacent to at least one vertex of $X$ in $\hat{G}$. This completes the proof. \qed

**Theorem 7.2.10.** Let $G$ be a connected sub graph of $K_{m,n}$. Then $\gamma_{gb}(G) = m + n - 1$ if and only if $G \cong K_{m,n} - e$.

**Proof.** Let $G \cong K_{m,n} - e$. where $e = uv \in E(K_{m,n})$. So $uv \notin E(G)$ and hence $uv \in E(\hat{G})$. Since $\hat{G}$ contains $m + n - 2$ isolated vertices, every global bipartite dominating set of $G$ contains either $u$ or $v$ and all vertices of $V(G) \setminus \{u,v\}$. 96
Thus, \( \gamma_{gb}(G) \geq m + n - 1 \). Since \( V(G) - \{u\} \) is a GBDS of \( G \), it follows that \( \gamma_{gb}(G) \leq m + n - 1 \). Therefore, we obtain \( \gamma_{gb}(G) = m + n - 1 \). Conversely assume that \( \gamma_{gb}(G) = m + n - 1 \). To prove \( G \cong K_{m,n} - e \). We observe that \( \gamma_{gb}(K_{m,n}) = m + n \) and \( \gamma_{gb}(K_{m,n} - e) = m + n - 1 \). Let \( G \) be a proper subgraph of \( K_{m,n} - e \) containing \( m + n \) vertices. Then \( \widehat{G} \) contains at most \( m + n - 3 \) isolated vertices. In that case \( \hat{G} \) contains a path \( uvw \). Then \( V(G) - \{u, w\} \) is a GBDS of \( G \). So \( \gamma_{gb}(G) \leq m + n - 2 \). This completes the proof.

**Theorem 7.2.11.** Let \( G \) be a graph with bipartition \( (X, Y) \). If \( G \) has a \( \gamma \)-set \( S = V_1 \cup V_2 \), where \( V_1 \subseteq X \) and \( V_2 \subseteq Y \) then \( S \) is a \( \gamma_{gb} \)-set of \( G \) if and only if

\[
\bigcap_{x \in V_1} N(x) \subseteq V_2 \quad \text{and} \quad \bigcap_{y \in V_2} N(y) \subseteq V_1.
\]

**Proof.** Let \( \bigcap_{x \in V_1} N(x) \subseteq V_2 \) and \( \bigcap_{y \in V_2} N(y) \subseteq V_1 \). Since \( S \) is a \( \gamma \)-set of \( G \), it suffices to show that \( S \) dominates the relative compliment of \( G \). Let \( u \in X \). If \( u \in \bigcap_{y \in V_2} N(y) \), then \( u \in V_1 \). If \( u \notin \bigcap_{y \in V_2} N(y) \) then \( u \) is adjacent to at least one vertex of \( V_2 \) in \( \widehat{G} \). Similarly, we can prove that if \( v \in Y \) then \( v \in V_2 \) or \( v \) is adjacent to at least one vertex of \( V_1 \) in \( \widehat{G} \). Conversely, let \( S \) dominates \( \widehat{G} \). Let \( x \) be an arbitrary vertex in \( X \). If \( x \in \bigcap_{y \in V_2} N(y) \), then in \( \widehat{G} \), \( x \) is not adjacent to any vertex of \( V_2 \). Since \( S \) dominates \( \widehat{G} \), we can deduce that \( x \in V_1 \). If \( x \notin \bigcap_{y \in V_2} N(y) \), then \( x \) is adjacent to at least one element of \( V_2 \) in \( \widehat{G} \). Hence the proof.

**Corollary 7.2.12.** Let \( G \) be a connected bipartite graph with \( n \) vertices, \( n \geq 4 \). Then \( \gamma_{gb}(G \circ K_1) = n \), where \( G \circ K_1 \) denotes the corona of the graphs \( G \) and \( K_1 \).

**Proof.** If \( G \cong K_{1,n} \), the proof is trivial. Otherwise, let \( (X, Y) \) be the bipartition of \( G \circ K_1 \). Let \( S = V_1 \cup V_2 \), where \( V_1 \subseteq X \) and \( V_2 \subseteq Y \), be the set of all
pendant vertices of $G \circ K_1$. Clearly $S$ is $\gamma$-set of $G \circ K_1$. Also $\bigcap_{x \in V_1} N(x) = \phi$ and $\bigcap_{y \in V_2} N(y) = \phi$. Therefore, the proof follows immediately from Theorem 7.2.11.

**Corollary 7.2.13.** For $n \geq 10$, $\gamma_{gb}(P_n) = \gamma(P_n) = \lceil \frac{n}{3} \rceil$.

**Proof.** Let $(1, 2, 3, \ldots, n)$ be the path $P_n$. Then $X = \{x: x \text{ is even}, x \leq n\}, Y = \{y: y \text{ is odd}, y \leq n\}$ is the bipartition of $P_n$. Let $S_1 = \{i: i \equiv 1 (\text{mod } 3), i \leq n\}$ and $S_2 = \{i: i + 1 \equiv 0 (\text{mod } 3), i \leq n\}$. Then either $S_1$ or $S_2$ is a $\gamma$-set of $P_n$. Also for $i = 1, 2, \bigcap_{x \in S_i \cap X} N(x) = \phi$ and $\bigcap_{y \in S_i \cap Y} N(y) = \phi$. Thus the proof follows from Theorem 7.2.11.

The global bipartite domination number, $\gamma_{gb}(P_n)$ for $1 < n < 10$ is given in Table 7.1.

<table>
<thead>
<tr>
<th>$P_n$</th>
<th>$P_2$</th>
<th>$P_3$</th>
<th>$P_4$</th>
<th>$P_5$</th>
<th>$P_6$</th>
<th>$P_7$</th>
<th>$P_8$</th>
<th>$P_9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_{gb}(P_n)$</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

**Corollary 7.2.14.** For an even integer $n \geq 10$, $\gamma_{gb}(C_n) = \gamma(C_n) = \lceil \frac{n}{3} \rceil$.

**Proof.** The proof is exactly similar to Corollary 7.2.13.

**Theorem 7.2.15.** For any two positive integers $a$ and $b$ with $a < b$, there exists a graph $G$ such that $\gamma(G) = a$ and $\gamma_{gb}(G) = b$.

**Proof.** Consider the graph $K_{b-a,a}$, with partite sets $W = \{w_1, w_2, \ldots, w_{b-a}\}$ and $U = \{u_1, u_2, \ldots, u_a\}$. Let $G$ be the graph obtained from $K_{b-a,a}$ by adding
new vertices $v_1, v_2, \ldots, v_a$ and join $v_i$ with $u_i$ for $i = 1, 2, \ldots, a$. Let $S$ be a dominating set of $G$. Since for each $i$, $v_i$ is adjacent to $u_i$ only, $|S| \geq a$. Now $U$ is a dominating set of $G$. So $|S| \leq a$. Hence $\gamma(G) = a$. In $\hat{G}$, the vertices $w_1, w_2, \ldots, w_{b-a}$ are isolated. So $W$ is a subset of every $\gamma_{gb}$-set of $G$. Therefore, the set $\{u_1, u_2, \ldots, u_a, w_1, w_2, \ldots, w_{b-a}\}$ is a $\gamma_{gb}$-set of $G$. Hence $\gamma_{gb}(G) = b$. This completes the proof. 

A graph $G$ with $\gamma(G) = 2$ and $\gamma_{gb}(G) = 6$ is given in figure 7.2

![Graph with $\gamma(G) = 2$ and $\gamma_{gb}(G) = 6$](image)

Figure 7.2: Graph $G$ with $\gamma = 2$ and $\gamma_{gb} = 6$

**Lemma 7.2.16.** If $G$ is an $r$-regular connected bipartite graph with bipartition $(X, Y)$ then $|X| = |Y|$.

**Proof.** In an $r$-regular connected bipartite graph $G$ with bipartition $(X, Y)$, each edge contributes exactly one to the degree sums $r|X|$ and $r|Y|$. Therefore, $r|X| = r|Y| = |E|$ and so $|X| = |Y|$. This completes the proof.

**Theorem 7.2.17.** If $G$ is an $(n - 1)$-regular bipartite graph with $2n$ vertices, then $\gamma_{gb}(G) = n$.

**Proof.** Let $(X, Y)$ be the bipartition of $G$. Since $G$ has $2n$ vertices, from Lemma 7.2.16 we have $|X| = |Y| = n$. Since $G$ is $(n - 1)$ regular, $\hat{G}$ has $n$ components
and all of them are paths with two vertices. So $\gamma(\hat{G}) = n$. Then by Theorem 7.2.11, we can find a $\gamma$-set of $\hat{G}$ such that it dominates $G$ also. Therefore, $\gamma_{gb}(G) = n$.

**Theorem 7.2.18.** Let $G$ be a healthy spider with $2n + 1$ vertices, then $\gamma_{gb}(G) = n + 1$.

*Proof.* Let $S$ be a $\gamma$-set of $G$, then $|S| = n$ and $u \notin S$ (see figure 7.3). So $S$ dominates all vertices except $u$ in $\hat{G}$. So $S \cup \{u\}$ is a $\gamma_{gb}$-set of $G$. This completes the proof. ☐

---

**Figure 7.3:** Healthy Spider

**Figure 7.4:** Wounded Spider
Theorem 7.2.19. If $G$ is a wounded spider with $n+k+1$ vertices, then $\gamma_{gb}(G) = k + 1$.

Proof. Observe that $\gamma(G) = k + 1$. Also, the set $S = \{1, 2, 3, \ldots, k, u\}$ is a $\gamma_{gb}$-set of $G$ (see figure 7.4).

Thus the proof follows. \hfill \Box

Theorem 7.2.20. $\gamma_{gb}(B_n) = 4$, where $B_n$ is the book graph on $2n + 2$ vertices.

Proof. Let the vertices of $B_n$ be labeled as shown in figure 7.4. Then $X = \{v, u_1, u_2, \ldots, u_n\}$, $Y = \{u, v_1, v_2, \ldots, v_n\}$ is the bipartition of $B_n$. Clearly the set $\{u, v\}$ is the $\gamma$-set of $B_n$. Also $\{u, v, u_1, v_1\}$ is a $\gamma$-set of $\hat{B}_n$. Therefore, $\gamma_{gb}(B_n) = 4$. \hfill \Box

Figure 7.5: Book Graph

Theorem 7.2.21. For $n \geq 3$, $\gamma_{gb}(S(K_n)) = n$, where $S(K_n)$ is the subdivision of the complete graph $K_n$. 
7.3. Global bipartite total domination

Proof. Let $X$ be the set of all old vertices and $Y$ be the set of all new vertices of $S(K_n)$. Then $(X, Y)$ is a bipartition of $S(K_n)$. In $S(K_n)$, the degree of each vertex in $X$ is $n - 1$ and the degree of each vertex in $Y$ is 2. We construct a $\gamma$-set of $S(K_n)$ as follows. Let $S \subseteq X$ such that $|S| = n - 2$. Then $S$ dominates all but one vertex $u$ in $Y$. Also $N(u) = \{x, y\}$ and $X - S = \{x, y\}$. So $S \cup \{u\}$ is a $\gamma$-set of $S(K_n)$. Note that any $\gamma$-set of $S(K_n)$ contains exactly $n - 2$ vertices from $X$ and one vertex from $Y$. Since $S \cup \{u\}$ does not dominate $x$ and $y$ in $\hat{G}$, this set is not a $\gamma_{gb}$-set. So a $\gamma_{gb}$-set of $S(K_n)$ contains at least $n$ vertices. Clearly the set $X$ is a global bipartite dominating set of $S(K_n)$. Therefore, $\gamma_{gb}(S(K_n)) = n$. \qed

Remark 7.2.22. $\gamma_{gb}(S(K_2)) = 3$.

Proof. Since $S(K_2) = P_3$, the proof follows. \qed

7.3 Global bipartite total domination

In this section, we introduce the concept of global bipartite total domination in graphs. We study some of its general properties and determine the global bipartite total domination number of certain classes of graphs.

Definition 7.3.1. Let $G$ be a connected bipartite graph with bipartition $(X, Y)$, with $|X| = m$ and $|Y| = n$. Let $\hat{G}$ denotes the relative complement of $G$ in $K_{m,n}$. Let $S$ be a total dominating set of $G$. If $S$ dominates $\hat{G}$, then $S$ is called a global bipartite total dominating set (GBTDS) of $G$. The global bipartite total domination number, $\gamma_{gbt}(G)$ is the minimum cardinality of a global bipartite total dominating set of $G$. 

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Example 7.3.2. For the graph $G$ given in figure 7.6, $S_1 = \{1, 3, 4, 5\}$, $S_2 = \{1, 3, 4, 6\}$ and $S_3 = \{2, 3, 4, 6\}$ are the minimum global bipartite total dominating sets of $G$, so that $\gamma_{gbt}(G) = 4$.

![Figure 7.6: A graph $G$ and $\hat{G}$](image)

Theorem 7.3.3. For a connected spanning subgraph $G$ of $K_{m,n}$,

$$\gamma_t(G) \leq \gamma_{gbt}(G) \leq m + n.$$  

Proof. Since every $\gamma_{gbt}$-set is a total dominating set we have, $\gamma_{gbt}(G) \geq \gamma_t(G)$. Also $V(G)$ is a GBTDS of $G$. Thus we have the result. \qed

Remark 7.3.4. The bounds of Theorem 7.3.3 are sharp. For the graph $K_2$, $\gamma_t(K_2) = \gamma_{gbt}(K_2) = 2$ and for $K_{m,n}$, $\gamma_{gbt}(K_{m,n}) = m + n$. Also the bounds of Theorem 7.3.3 are strict. For the graph $G$ given in figure 7.6, $\gamma_{gbt}(G) = 4$.

Theorem 7.3.5. Let $G$ be a connected bipartite graph with partite sets $X,Y$. Let $S = V_1 \cup V_2$, where $V_1 \subseteq X$ and $V_2 \subseteq Y$, be a total dominating set of $G$. Then $S$ is a GBTDS of $G$ if and only if $\bigcap_{x \in V_1} N(x) \subseteq V_2$ and $\bigcap_{y \in V_2} N(y) \subseteq V_1$.

Proof. Since every total dominating set is a dominating set, the proof follows from Theorem 7.2.11. \qed
Theorem 7.3.6. For any two positive integers \( a, b \) with \( a < b \), there exists a graph \( G \) with \( \gamma_t(G) = a \) and \( \gamma_{gbt}(G) = b \).

Proof. Consider the complete bipartite graph \( K_{a−1, b−a+1} \), with partite sets \( U = \{u_1, u_2, \ldots, u_{a−1}\} \) and \( W = \{w_1, w_2, \ldots, w_{b−a+1}\} \). Let \( G \) be the graph obtained from \( K_{a−1, b−a+1} \) by adding new vertices \( v_1, v_2, \ldots, v_{a−1} \) and joining \( v_i \) with \( u_i \) for \( i = 1, 2, \ldots, a−1 \). Then \( \{w_1, u_1, u_2, \ldots, u_{a−1}\} \) is a \( \gamma_t \) set of \( G \). Since the vertices \( w_1, w_2, \ldots, w_{b−a+1} \) are isolated in \( \hat{G} \), the set \( W \) is a subset of every GBTDS of \( G \). Therefore, \( \{w_1, w_2, \ldots, w_{b−a+1}, u_1, u_2, \ldots, u_{a−1}\} \) is a \( \gamma_{gbt} \)-set of \( G \). 

For the graph \( G \) given in figure 7.7, \( \gamma_t(G) = 4 \) and \( \gamma_{gbt}(G) = 7 \).

Figure 7.7: Graph \( G \) with \( \gamma_t = 4 \) and \( \gamma_{gbt} = 7 \)

Theorem 7.3.7. Let \( G \) be a connected spanning subgraph of \( K_{m,n} \). Then \( \gamma_{gbt}(G) = m + n − 1 \) if and only if \( G \) is isomorphic to \( K_{m,n} − e \).

Proof. Let \( G \) be isomorphic to \( K_{m,n} − e \). Then \( \hat{G} \) consists of \( m + n − 2 \) isolated vertices and an edge \( e = uv \). So if \( S \) is a GBTDS of \( G \), then \( S \) contains all isolated vertices of \( \hat{G} \) and one of \( u \) or \( v \). Therefore, \( \gamma_{gbt}(G) \geq m + n − 1 \). Since \( V(G) − \{v\} \)
is a GBTDS of $G$, we have $\gamma_{gbt}(G) \leq m + n - 1$. Therefore, $\gamma_{gbt}(G) = m + n - 1$.

Conversely, let $\gamma_{gbt}(G) = m + n - 1$. We proved that $\gamma_{gbt}(K_{m,n}) = m + n$ and $\gamma_{gbt}(K_{m,n} - e) = m + n - 1$. If $G$ is a proper subgraph of $K_{m,n} - e$, then $\hat{G}$ has at most $m + n - 3$ isolated vertices. In that case $\hat{G}$ has a path $uvw$. Therefore, $\gamma_{gbt}(G) \leq m + n - 2$. This completes the proof.  \hfill $\square$
Global Bipartite Domination Polynomial

8.1 Introduction

In this chapter, we introduce the concept of the global bipartite domination polynomial of a connected bipartite graph and study some of its general properties. We establish some relationships between domination polynomial and global bipartite domination polynomial of certain classes of graphs.

8.2 Main results

Definition 8.2.1. Let $D_{gb}(G, i)$ be the family of global bipartite dominating sets of a simple connected bipartite graph $G$ with cardinality $i$ and let $d_{gb}(G, i) = |D_{gb}(G, i)|$. Then the global bipartite domination polynomial $D_{gb}(G, x)$ of $G$ is

\[ D_{gb}(G, x) = \sum_{i=0}^{n} d_{gb}(G, i) x^i \]

\[ = \sum_{i=0}^{n} |D_{gb}(G, i)| x^i \]

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8.2. Main results

defined as \( D_{gb}(G, x) = \sum_{i=\gamma_{gb}(G)}^{n} d_{gb}(G, i)x^i \).

**Theorem 8.2.2.** For a connected bipartite graph \( G \), if \( \hat{G} \) is also connected, then \( D_{gb}(G, x) = D_{gb}(\hat{G}, x) \).

*Proof.* Let \( S \) be a global bipartite dominating set of \( G \). Since \( \hat{G} \) is connected, \( S \) is a global bipartite dominating set of \( \hat{G} \) also. Similarly, every GBDS of \( \hat{G} \) is a GBDS of \( G \). This completes the proof. \( \square \)

**Theorem 8.2.3.** For any two positive integers \( m \) and \( n \),

(i) \( D_{gb}(K_{m,n}, x) = x^{m+n} \),

(ii) If \( K_{m,n} - e \) is connected, then \( D_{gb}(K_{m,n} - e, x) = x^{m+n-1}(x + 2) \).

*Proof.*

(i) Obviously \( \gamma_{gb}(K_{m,n}) = m + n \). Therefore, \( D_{gb}(K_{m,n}, x) = x^{m+n} \).

(ii) If \( e = uv \), then \( V(K_{m,n}) \setminus \{u\} \) and \( V(K_{m,n}) \setminus \{v\} \) are the only \( \gamma_{gb} \)-sets of \( K_{m,n} - e \). Therefore, \( \gamma_{gb}(K_{m,n} - e) = m+n-1 \) and \( d_{gb}(K_{m,n} - e, m+n-1) = 2 \).

Since \( d_{gb}(K_{m,n} - e, m+n) = 1 \), the proof follows. \( \square \)

Next, we compute the global bipartite domination polynomial of bi-star graph \( B_{m,n} \), obtained from the graph \( K_2 \) with vertices \( u \) and \( v \) by attaching \( m \) pendant edges to \( u \) and \( n \) pendant edges to \( v \).

**Theorem 8.2.4.** The global bipartite domination polynomial of bi-star graph is

\[
D_{gb}(B_{m,n}, x) = x^2 [x^m + x^n + [(1 + x)^m - 1][(1 + x)^n - 1]].
\]
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Proof. Clearly, $\gamma_{gb}(B_{m,n}) = 4$. Let $U$ and $V$ be the set of all pendant vertices at $u$ and $v$ respectively. Since the vertices $u$ and $v$ are isolated in $\hat{B}_{m,n}$, every GBDS of $B_{m,n}$ contains $u$ and $v$. Let $S$ be a subset of vertices of $B_{m,n}$ such that \{u, v\} $\subseteq S$. If $S \cap U \neq \emptyset$ and $S \cap V \neq \emptyset$, then $S$ is a GBDS of $B_{m,n}$. Also the sets $U \cup \{u, v\}$ and $V \cup \{u, v\}$ are G.B.D.S of $B_{m,n}$. This completes the proof. \qed

The next theorem follows immediately from the definition of global bipartite domination polynomial.

**Theorem 8.2.5.** For any connected spanning subgraph $G$ of $K_{m,n}$,

(i) $d_{gb}(G, m + n) = 1$,

(ii) $d_{gb}(G, i) = 0$ if and only if $i < \gamma_{gb}(G)$ or $i > m + n$,

(iii) $D_{gb}(G, x)$ has no constant term,

(iv) $D_{gb}(G, x)$ is a strictly increasing function in $[0, \infty)$,

(v) If $H$ is an induced subgraph of $G$, then $\deg(D_{gb}(G, x)) \geq \deg(D_{gb}(H, x))$,

(vi) Zero is a root of $D_{gb}(G, x)$ with multiplicity $\gamma_{gb}(G)$.

**Theorem 8.2.6.** If $G$ is an $(n - 1)$-regular connected bipartite graph with $2n$ vertices, then

$$D_{gb}(G, x) = [x(x + 2)]^n - 2nx^n.$$  

Proof. Since $G$ is $(n-1)$ regular, each component of $\hat{G}$ is $P_2$. Therefore, a G.B.D.S of $G$ contains at least one vertex from each component of $\hat{G}$. So $\gamma_{gb}(G) = n$ and for $1 \leq i \leq n$, $d_{gb}(G, n + i) = \binom{n}{i}2^{n-i}$. It follows from Theorem 7.2.11 that
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\[ d_{gb}(G, n) = 2^n - 2n. \]

Then,

\[ D_{gb}(G, x) = \binom{n}{0} 2^n x^n + \binom{n}{1} 2^{n-1} x^{n+1} + \ldots + \binom{n}{n} 2^{n-n} x^{n+n} - 2nx^n \]

\[ = x^n (x + 2)^n - 2nx^n. \]

This completes the proof. \(\Box\)

8.3 Global bipartite domination polynomials of paths

In this section, we shall study the relation between domination polynomials and global bipartite domination polynomials of paths.

We need the following:

**Lemma 8.3.1.** For a path \( P_n \) with bipartition \((X, Y)\), let \( S = V_1 \cup V_2 \) where \( V_1 \subseteq X \) and \( V_2 \subseteq Y \) be a dominating set. If \( |V_i| > 2 \), for \( i = 1, 2 \), then \( S \) is a G.B.D.S. of \( P_n \).

**Proof.** In \( P_n \) if \( |V_i| > 2 \), then \( \bigcap_{v \in V_i} N(v) = \emptyset \). Then by Theorem 7.2.11 \( S \) is a G.B.D.S. of \( P_n \). \(\Box\)

Let \( G \) be a connected bipartite graph with partite sets \( X \) and \( Y \). Let \( S = V_1 \cup V_2 \) be a GBDS of \( G \) such that \( V_1 \subseteq X \) and \( V_2 \subseteq Y \). Then by Theorem 7.2.8 we have, if \( S \cap X = \emptyset \), then \( S = Y \) and if \( S \cap Y = \emptyset \), then \( S = X \). So for \( n \geq 12 \), to find \( d(P_n, i) - d_{gb}(P_n, i) \) it suffices to consider the dominating sets \( S = V_1 \cup V_2 \) of \( P_n \) with \( 1 \leq |V_1| \leq 2 \) or \( 1 \leq |V_2| \leq 2 \). To prove theorems 8.3.2 to 8.3.5 the
8.3. Global bipartite domination polynomials of paths

partite sets of $P_{2n}$ is taken as $X = \{1, 3, 5, \ldots, 2n - 1\}$ and $Y = \{2, 4, 6, \ldots, 2n\}$ and $S = V_1 \cup V_2$, where $V_1 \subseteq X$ and $V_2 \subseteq Y$ is taken as a dominating set. Using the following theorems we can find the number of dominating sets which are not global bipartite dominating sets.

**Theorem 8.3.2.** For $|V_1| = 1$, we have

\[(i) \ d(P_{2n}, n) - d_{gb}(P_{2n}, n) = 2n - 2, \]
\[(ii) \ d(P_{2n}, n - 1) - d_{gb}(P_{2n}, n - 1) = n - 2. \]

*Proof.* Since a vertex in $X$ is adjacent to at most two vertices in $Y$, $n - 2 \leq |V_2| \leq n$. If $|V_2| = n$, then $S = V_1 \cup V_2$ is a G.B.D.S and the proof is complete. So $|V_2| = n - 2$ or $n - 1$. We consider the following cases:

**Case 1:** $V_1 = \{1\}$.

Here $V_2 = \{4, 6, 8, \ldots, 2n\}$. Since $N(1) = \{2\} \not\subseteq V_2$, $S$ is not a G.B.D.S.

**Case 2:** $V_1 = \{3\}$.

Here also $|V_2| = n - 1$ and $V_2 = \{2, 6, 8, \ldots, 2n\}$. Since $N(3) = \{2, 4\} \not\subseteq V_2$, $S$ is not a G.B.D.S.

**Case 3:** $V_1 = \{i\}, i \neq 1, 3$.

Then for each $i, V_1 \cup (Y \setminus \{i - 1, i + 1\}), V_1 \cup (Y \setminus \{i - 1\})$ and $V_1 \cup (Y \setminus \{i + 1\})$ are dominating sets of $P_{2n}$. Since $N(i) = \{i - 1, i + 1\} \not\subseteq V_2$, these are not G.B.D.S of $P_{2n}$.

In cases 1 and 2 we have two dominating sets of order $n$. In case 3 we have $2(n - 2)$ dominating sets of order $n$ and $n - 2$ dominating sets of order $n - 1$. 

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Therefore, the result follows.

**Theorem 8.3.3.** For $|V_2| = 1$, we have

(i) $d(P_{2n}, n) - d_{gb}(P_{2n}, n) = 2n - 2$,

(ii) $d(P_{2n}, n - 1) - d_{gb}(P_{2n}, n - 1) = n - 2$.

*Proof.* The proof is exactly similar to that of Theorem 8.3.2.

**Theorem 8.3.4.** For $|V_1| = 2$, we have

(i) $d(P_{2n}, n - 1) - d_{gb}(P_{2n}, n - 1) = n - 3$,

(ii) $d(P_{2n}, n) - d_{gb}(P_{2n}, n) = 2n - 4$,

(iii) $d(P_{2n}, n + 1) - d_{gb}(P_{2n}, n + 1) = n - 1$.

*Proof.* Since $|V_1| = 2$, we have $n - 3 \leq |V_2| \leq n$. If $|V_2| = n$, then $S = V_1 \cup V_2$ is a G.B.D.S. So it suffices to consider the cases $|V_2| = n - 3, n - 2$ and $n - 1$.

**Case 1:** $V_1 = \{1, 3\}$.

**Subcase 1:** $|V_2| = n - 2$.

Then $V_2 = \{6, 8, \ldots, 2n\}$. Since $N(1) \cup N(3) = \{2\} \not\subseteq V_2$, $S$ is not a G.B.D.S of $P_{2n}$.

**Subcase 2:** $|V_2| = n - 1$.

Then $V_2 = \{4, 6, 8, \ldots, 2n\}$. Since $N(1) \cup N(3) = \{2\} \not\subseteq V_2$, the dominating set $S$ is not a G.B.D.S.

**Case 2:** $V_1 = \{3, 5\}$.

As in case 1 we get two dominating sets which are not G.B.D.S of $P_{2n}$.
Case 3: \( V_1 = \{i, i + 2\}, i \neq 1, 3 \).

Subcase 1: \( |V_2| = n - 3 \).

Then \( V_2 = Y \setminus \{i - 1, i + 1, i + 3\} \).

Subcase 2: \( |V_2| = n - 2 \).

In this case we have the possibilities, \( V_2 = Y \setminus \{i - 1, i + 1\} \) and
\( V_2 = Y \setminus \{i + 1, i + 3\} \).

Subcase 3: \( |V_2| = n - 1 \).

Then \( V_2 = Y \setminus \{i + 1\} \).

In sub case 1, 2 and 3, \( S = V_1 \cup V_2 \) is a dominating set but since \( N(i) \cap N(i + 1) = \{i + 1\} \notin V_2 \), \( S \) is not a G.B.D.S of \( P_{2n} \).

In cases 1 and 2 we have two dominating sets of order \( n \) and \( n + 1 \). In case 3 we have \( n - 3 \) dominating sets of order \( n - 1 \), \( 2(n - 3) \) dominating sets of order \( n \) and \( n - 3 \) dominating sets of order \( n + 1 \). Hence the result follows.

\[ \square \]

Theorem 8.3.5. For \( |V_2| = 2 \), we have

\begin{enumerate}
\item \( d(P_{2n}, n - 1) - d_{gb}(P_{2n}, n - 1) = n - 3 \),
\item \( d(P_{2n}, n) - d_{gb}(P_{2n}, n) = 2n - 4 \),
\item \( d(P_{2n}, n + 1) - d_{gb}(P_{2n}, n + 1) = n - 1 \).
\end{enumerate}

Proof. The proof is exactly similar to that of Theorem 8.3.4. \( \square \)

Theorem 8.3.6. For \( n \geq 6 \),

\[ D(P_{2n}, x) - D_{gb}(P_{2n}, x) = (4n - 10)x^{n-1} + (8n - 12)x^n + (2n - 2)x^{n+1} . \]
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Proof. It follows from Theorems 8.3.2, 8.3.3, 8.3.4 and 8.3.5.

Next, we find the relationship between domination polynomials and global bipartite domination polynomials of $P_{2n+1}$. To prove theorems 8.3.7 to 8.3.10, we take $X = \{1, 3, 5, \ldots, 2n+1\}$ and $Y = \{2, 4, 6, \ldots, 2n\}$ as the bipartition of $P_{2n+1}$ and $S = V_1 \cup V_2$, where $V_1 \subseteq X$ and $V_2 \subseteq Y$ as a dominating set of $P_{2n+1}$.

**Theorem 8.3.7.** For $|V_1| = 1$, we have

\begin{align*}
(i) \quad & d(P_{2n+1}, n - 1) - d_{gb}(P_{2n+1}, n - 1) = n - 3, \\
(ii) \quad & d(P_{2n+1}, n) - d_{gb}(P_{2n+1}, n) = 2n - 2.
\end{align*}

Proof. Case 1: $V_1 = \{1\}$. Let $V_2 = Y \setminus \{2\}$. Since $N(1) = \{2\}$, $S = V_1 \cup V_2$ is not a G.B.D.S.

The case $V_1 = \{2n + 1\}$ is similar.

Case 2: $V_1 = \{3\}$. Let $V_2 = Y \setminus \{4\}$. Since $N(3) = \{2, 4\}$, $S = V_1 \cup V_2$ is not a G.B.D.S.

The case $V_1 = \{2n - 1\}$ is similar.

Case 3: $V_1 = \{i\}, i \notin \{1, 3, 2n-1, 2n+1\}$. In this case we have the possibilities,

$V_2 = Y \setminus \{i - 1, i + 1\}$ or $V_2 = Y \setminus \{i - 1\}$ and $V_2 = Y \setminus \{i + 1\}$. Since $N(i) = \{i - 1, i + 1\}$, $S = V_1 \cup V_2$ is not a G.B.D.S.

In cases 1 and 2 we have four dominating sets of order $n$ and in case 3 there are $n - 3$ dominating sets of order $n - 1$ and $2(n - 3)$ dominating sets of order $n$. This completes the proof.

\[\square\]
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**Theorem 8.3.8.** For $|V_2| = 1$, we have

(i) $d(P_{2n+1}, n) - d_{gb}(P_{2n+1}, n) = n$,  

(ii) $d(P_{2n+1}, n + 1) - d_{gb}(P_{2n+1}, n + 1) = 2n$.

**Proof.** Let $V_2 = \{i\}, i \in Y \Rightarrow N(i) = \{i - 1, i + 1\}$. Then $V_1$ can be $X \setminus \{i - 1\}$ or $X \setminus \{i + 1\}$ or $X \setminus \{i - 1, i + 1\}$. Since $i$ can be selected in $n$ ways, we have $2n$ dominating sets of order $n + 1$ and $n$ dominating sets of order $n$. Since $N(i) = \{i - 1, i + 1\}$, $S = V_1 \cup V_2$ is not a G.B.D.S. of $P_{2n+1}$. Hence the result follows. □

**Theorem 8.3.9.** For $|V_1| = 2$, we have

(i) $d(P_{2n+1}, n - 1) - d_{gb}(P_{2n+1}, n - 1) = n - 4$,  

(ii) $d(P_{2n+1}, n) - d_{gb}(P_{2n+1}, n) = 2n - 4$,  

(iii) $d(P_{2n+1}, n + 1) - d_{gb}(P_{2n+1}, n + 1) = n$.

**Proof.**  

**Case 1:** $V_1 = \{1, 3\}$. Then $V_2$ can be $Y \setminus \{2\}$ or $Y \setminus \{2, 3\}$. Since $N(1) \cap N(3) = \{2\}$, $S = V_1 \cup V_2$, is not a G.B.D.S.  

The case $V_1 = \{2n - 1, 2n + 1\}$ is similar.

**Case 2:** $V_1 = \{3, 5\}$. Then $V_2$ can be $Y \setminus \{4\}$ or $Y \setminus \{4, 5\}$. Since $N(3) \cap N(5) = \{4\}$, $S = V_1 \cup V_2$, is not a G.B.D.S.  

The case $V_1 = \{2n - 3, 2n - 1\}$ is similar.

**Case 3:** $V_1 = \{i, i + 2\}, i \notin \{1, 3, 2n - 3, 2n - 1\}$. Then $V_2$ can be $Y \setminus \{i - 1, i + 1, i + 3\}$ or $Y \setminus \{i - 1, i + 1\}$ or $Y \setminus \{i + 1, i + 3\}$. Since $N(i) \cap N(i + 2) =$
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\{i + 1\}, \ S = V_1 \cup V_2, \text{ is not a G.B.D.S.}

In cases 1 and 2 we have four dominating sets of order \( n \) and \( n + 1 \). In case 3 there are \( n - 4 \) dominating sets of order \( n - 1 \) and \( n + 1 \) and \( 2(n - 4) \) dominating sets of order \( n \). Thus the result follows.

**Theorem 8.3.10.** For \(|V_2| = 2\), we have

\[
\begin{align*}
(i) \quad & d(P_{2n+1},n) - d_{gb}(P_{2n+1},n) = n - 1, \\
(ii) \quad & d(P_{2n+1},n + 1) - d_{gb}(P_{2n+1},n + 1) = 2n - 2, \\
(iii) \quad & d(P_{2n+1},n + 2) - d_{gb}(P_{2n+1},n + 2) = n - 1.
\end{align*}
\]

**Proof.** Let \( V_2 = \{i, i + 2\}, \ i \in Y \Rightarrow N(i) \cap N(i + 2) = \{i + 1\} \). Then \( V_1 \) can be \( X \setminus \{i - 1, i + 1, i + 3\} \) or \( X \setminus \{i - 1, i + 1\} \) or \( X \setminus \{i + 1, i + 3\} \). Since \( V_2 \) can be selected in \( n - 1 \) ways, we have \( n - 1 \) dominating sets of order \( n \) and \( 2(n - 1) \) dominating sets of ordern + 1 and \( n - 1 \) dominating sets of order \( n + 2 \). Since \( N(i) \cap N(i + 2) = \{i + 1\} \), \( S = V_1 \cup V_2 \) is not a G.B.D.S. of \( P_{2n+1} \). This proves the result.

**Theorem 8.3.11.** For \( n \geq 6 \),

\[
D(P_{2n+1},x) - D_{gb}(P_{2n+1},x) = (2n-7)x^{n-1}+(6n-7)x^n+(5n-2)x^{n+1}+(n-1)x^{n+2}.
\]

**Proof.** It follows from Theorems 8.3.7, 8.3.8, 8.3.9 and 8.3.10.
8.4 Global bipartite domination polynomials of cycles

In this section, we find the relation between the domination polynomials and the global bipartite domination polynomials of cycles. To prove theorems 8.4.1 to 8.4.5, let \( X = \{1, 3, 5, \ldots, 2n-1\} \) and \( Y = \{2, 4, 6, \ldots, 2n\} \) be the bipartition of \( C_{2n} \) and \( S = V_1 \cup V_2 \) where \( V_1 \subseteq X \) and \( V_2 \subseteq Y \) be a dominating set of \( C_{2n} \).

**Theorem 8.4.1.** For \(|V_1| = 1\), we have

\[(i)\ d(C_{2n}, n - 1) - d_{gb}(C_{2n+1}, n - 1) = n,\]
\[(ii)\ d(C_{2n}, n) - d_{gb}(C_{2n}, n) = 2n.\]

*Proof.* Let \( V_1 = \{i\}, i \in X \). Then \( N(i) = \{i-1, i+1\} \) (if \( i = 1 \), then we take \( i - 1 = 2n \)). Then \( V_2 \) can be \( Y \setminus \{i-1, i+1\} \) or \( Y \setminus \{i-1\} \) or \( Y \setminus \{i+1\} \). Since \( i \) can be selected in \( n \) ways, we have \( n \) dominating sets of order \( n - 1 \) and \( 2n \) dominating sets of order \( n \). Since \( N(i) = \{i-1, i+1\} \), \( S = V_1 \cup V_2 \) is not a G.B.D.S. of \( C_{2n} \). Hence the result follows. \( \square \)

**Theorem 8.4.2.** For \(|V_2| = 1\), we have

\[(i)\ d(C_{2n}, n - 1) - d_{gb}(C_{2n+1}, n - 1) = n,\]
\[(ii)\ d(C_{2n}, n) - d_{gb}(C_{2n}, n) = 2n.\]

*Proof.* The proof is exactly similar to Theorem 8.4.1. \( \square \)
Theorem 8.4.3. For $|V_1| = 2$, we have

(i) $d(C_{2n}, n - 1) - d_{gb}(C_{2n}, n - 1) = n - 1,$

(ii) $d(C_{2n}, n) - d_{gb}(C_{2n}, n) = 2(n - 1),$

(iii) $d(C_{2n}, n + 1) - d_{gb}(C_{2n}, n + 1) = n - 1.$

Proof. Let $V_1 = \{i, i + 2\}$, $i \in X$. Then $N(i) \cap N(i + 2) = \{i + 1\}$ (if $i = 2n - 1$, then we take $i + 2 = 1$ and $i + 3 = 2.$) Then $V_2$ can be $Y \setminus \{i - 1, i + 1, i + 3\}$ or $Y - \{i - 1, i + 1\}$ or $Y \setminus \{i + 1\}$ or $Y \setminus \{i + 1\}$. Since $V_1$ can be selected in $n - 1$ ways, we have $(n - 1)$ dominating sets of order $n - 1, 2(n - 1)$ dominating sets of order $n$ and $n - 1$ dominating sets of order $n + 1$. Since $N(i) \cap N(i + 2) = \{i + 1\}$, $S = V_1 \cup V_2$ is not a G.B.D.S. of $C_{2n}$. Hence the result follows.

Theorem 8.4.4. For $|V_2| = 2$, we have

(i) $d(C_{2n}, n - 1) - d_{gb}(C_{2n}, n - 1) = n - 1,$

(ii) $d(C_{2n}, n) - d_{gb}(C_{2n}, n) = 2(n - 1),$

(iii) $d(C_{2n}, n + 1) - d_{gb}(C_{2n}, n + 1) = n - 1.$

Proof. The proof is exactly similar to Theorem 8.4.3.

Theorem 8.4.5. For $n \geq 6$,

$$D(C_{2n}, x) - D_{gb}(C_{2n}, x) = (4n - 2)x^{n-1} + (8n - 4)x^n + (2n - 2)x^{n+1}.$$ 

Proof. It follows from Theorems 8.4.1, 8.4.2, 8.4.3 and 8.4.4.
Further scope for research

1. Characterize non isomorphic graphs having same total domination polynomial.

2. Determine the total domination polynomial of an arbitrary Cayley graph.

3. Determine the total domination polynomial of Cartesian product of arbitrary graphs.

4. Determine the polynomials $D_{tv}(G, x)$ and $D_{tv}^c(G, x)$ of arbitrary graphs.

5. Determine the total domination polynomial of ring sum of arbitrary graphs.

6. Characterize graphs $G$ for which $\gamma(G) = \gamma_{gb}(G)$.

7. Characterize graphs $G$ for which $\gamma_{t}(G) = \gamma_{gbt}(G)$. 
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