TD-Polynomials of some graphs

4.1 Introduction

It is known that the concept of total domination in graphs can be converted to the concept of vertex cover in hypergraphs. In this chapter, we find the total domination polynomials of some graphs using the hypergraph terminology. The third section of this chapter deals with the total domination polynomials of total and middle graphs of some classes of graphs.

We need the following to proceed.

Definition 4.1.1. (see [1]) A graph $G$ is said to be an $m$-partite graph, if its vertex set can be partitioned into $m$ subsets so that no edge has both ends in any one subset. A complete $m$-partite graph, denoted by $K_{n_1,n_2,...,n_m}$, is a graph in which each vertex is joined to every vertex that is not in the same sub set. If $n_i = n$ for every $i$, then it is denoted by $K_{m[n]}$.

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Definition 4.1.2. (see [7]) The caterpillar graph $T(m_1, m_2, \ldots, m_n)$ is obtained from a path $P_n$ with $n \geq 2$, by attaching the central vertex of the star graph $K_{1, m_i}$ ($1 \leq i \leq n$) to the $i$-th vertex of the path $P_n$.

Definition 4.1.3. (see [14]) Let $\Gamma$ be a finite group with identity $e$. Let $S \subseteq \Gamma$ such that $e \notin S$ and $S = S^{-1}$, that is, $S$ is inverse closed. Then the Cayley graph $G = \text{Cay}(\Gamma, S)$, is defined as a graph with vertex set $V(G) = \Gamma$ and edge set $E(G) = \{ab : ab^{-1} \in S\}$. Let $\Gamma$ be a group and $S \subseteq \Gamma$ such that $H = S$ is a subgroup of $\Gamma$. Then $\text{Cay}(\Gamma, S)$ is called subgroup complementary Cayley graph $[14]$ denoted by $\text{SC}(\Gamma, H)$.

Definition 4.1.4. The centipede $P^*_{2n}$ with $2n$ vertices is obtained by appending a single pendant edge to each vertex of a path $P_n$.

Theorem 4.1.5. (see [5]) If a graph $G$ consists of $m$ components $G_1, G_2, \ldots, G_m$, then $D(G, x) = D(G_1, x) \ldots D(G_m, x)$.

Theorem 4.1.6. (see [1]) For a complete graph $K_n$, $D(K_n, x) = (1 + x)^n - 1$.

Theorem 4.1.7. (see [47]) For a complete graph $K_n$,

$$D_t(K_n, x) = (1 + x)^n - 1 - nx.$$ 

Theorem 4.1.8. (see [14]) Let $\Gamma$ be a finite group and $H$ be a subgroup of $\Gamma$ with $o(H) = n$. Then $\text{SC}(G, H) = K_{m[n]}$, where $m = [\Gamma : H]$. 

4.2 Main results

In this section, the total domination polynomials of some graphs are determined easily using hypergraphs. Using the hypergraph terminology, we can easily prove Theorem 4.2.1 due to [45] and Corollary 4.2.3 Theorem 4.2.9 due to [12].

Theorem 4.2.1. For a complete \(m\)-partite graph, \(K_{n_1,n_2,\ldots,n_m}\),

\[
D_t(K_{n_1,n_2,\ldots,n_m},x) = D_t(K_N,x) - \sum_{i=1}^{m} D_t(K_{n_i},x),
\]

where \(N = \sum_{i=1}^{m} n_i\).

Proof. For \(i = 1,2,\ldots,m\), let \(A_i\) be the partite sets of \(K_{n_1,n_2,\ldots,n_m}\) and let \(A = \bigcup_{i=1}^{m} A_i\). Then for each vertex \(v\), \(N(v) = A \setminus A_i\) for some \(i\). Therefore, a set \(S\) is a total dominating set if and only if \(S \cap (A \setminus A_i) \neq \emptyset\) for every \(i\). If \(S \cap (A \setminus A_i) = \emptyset\) for some \(i\), then \(S \subseteq A_i\). If \(|S| = k\), then for each \(i\), there are \(\binom{n_i}{k}\) subsets \(S\) such that \(S \subseteq A_i\). Therefore, for each \(k\), the number of non total dominating sets of the complete \(m\)-partite graph is \(\sum_{i=1}^{m} \binom{n_i}{k}\). So \(d_t(K_{n_1,n_2,\ldots,n_m},k) = d_t(K_N,k) - \sum_{i=1}^{m} d_t(K_{n_i},k)\). Therefore,

\[
D_t(K_{n_1,n_2,\ldots,n_m},x) = \sum_{k=2}^{N} d_t(K_N,k)x^k - \sum_{k=2}^{N} \left( \sum_{i=1}^{m} d_t(K_{n_i},k) \right) x^k
\]

\[
= \sum_{k=2}^{N} \binom{N}{k} x^k - \sum_{i=1}^{m} \left( \sum_{k=2}^{N} \binom{n_i}{k} \right) x^k
\]

\[
= D_t(K_N,x) - \sum_{i=1}^{m} D_t(K_{n_i},x)
\]

\[
= D_t(K_N,x) + N x - \sum_{i=1}^{m} (D_t(K_{n_i},x) + n_i x)
\]
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\[ D(K_N, x) - \sum_{i=1}^{m} D(K_{n_i}, x). \]

This completes the proof. \( \square \)

**Corollary 4.2.2.** For the complete \( m \)-partite graph \( K_{m[n]} \),

\[ D_t(K_{m[n]}, x) = D(K_{mn}, x) - mD(K_n, x) = (1 + x)^{mn} - m(1 + x)^n + m - 1. \]

**Proof.** The proof follows immediately from Theorem 4.2.1. \( \square \)

**Corollary 4.2.3.** For the complete bipartite graph \( K_{m,n} \),

\[ D_t(K_{m,n}, x) = D_t(K_{m+n}, x) - D_t(K_m, x) - D_t(K_n, x) = [(1 + x)^{m+n} - 1 - (m + n)x] - [(1 + x)^m - 1 - mx] - [(1 + x)^n - 1 - nx] = (1 + x)^{m+n} - (1 + x)^m - (1 + x)^n + 1. \]

**Proof.** The proof follows from Theorem 4.2.1. \( \square \)

**Corollary 4.2.4.** For the star graph \( K_{1,n} \), \( D_t(K_{1,n}, x) = x [(1 + x)^n - 1] \).

**Proof.** The proof follows by substituting \( m = 1 \) in Corollary 4.2.3. \( \square \)

**Theorem 4.2.5.** For an \((n-1)\)-regular bipartite graph \( G \) on \( 2n \) vertices,

\[ D_t(G, x) = [D_t(K_n, x)]^2 = [(1 + x)^n - 1 - nx]^2. \]

**Proof.** Let \( V = \{a_1, a_2, \ldots, a_n\} \cup \{b_1, b_2, \ldots, b_n\} \) and \( E = \{a_ib_j : i \neq j\} \) be the
vertex set and edge set of $G$. Let $H_1$ and $H_2$ be two complete graphs with vertex set $V = \{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ respectively. Then for $1 \leq i \leq n$, we have $N_G(a_i) = N_{H_2}(b_i)$ and $N_G(b_i) = N_{H_1}(a_i)$. Since $G$ is bipartite, the open neighborhood hypergraph of $G$ has two components. It can be observed that one component of $\text{ONH}(G)$ is isomorphic to $\text{ONH}(H_1)$ and the other is isomorphic to $\text{ONH}(H_2)$. Therefore, a set $S \subseteq V(G)$ is a total dominating set of $G$, if and only if $S$ is a total dominating set of $H_1$ and $H_2$. Therefore, by Theorems 2.1.14 and 4.1.7 we have, $D_t(G, x) = D_t(H_1, x)D_t(H_2, x) = [D_t(K_n, x)]^2$. This completes the proof.

\[ \square \]

**Corollary 4.2.6.** If $S = \{(1, 1), (1, 2), \ldots, (1, n-1)\}$, then

\[ D_t(Cay(\mathbb{Z}_2 \boxtimes \mathbb{Z}_n, S), x) = [D_t(K_n, x)]^2. \]

**Proof.** Since $Cay(\mathbb{Z}_2 \boxtimes \mathbb{Z}_n, S)$ is an $(n-1)$-regular bipartite graph, the proof follows from Theorem 4.2.5.

\[ \square \]

**Theorem 4.2.7.** Let $\Gamma$ be a group of order $n$ and $S \subseteq \Gamma$ such that $H = \overline{S}$ is a subgroup of $\Gamma$. Then the total domination polynomial of the subgroup complementary cayley graph $SC(\Gamma, H) = Cay(\Gamma, S)$ is

\[ D_t(Cay(\Gamma, S), x) = D_t(K_n, x) - mD_t(K_1|H|, x), \text{ where } m = [\Gamma : H]. \]

**Proof.** The proof follows from Theorem 4.1.8 and Corollary 4.2.2.

\[ \square \]

**Theorem 4.2.8.** Let $H$ be a subgroup of $\mathbb{Z}_n$ and $G$ be a bipartite graph with vertex set $V = \{a_0, a_1, a_2, \ldots, a_{n-1}\} \cup \{b_0, b_1, b_2, \ldots, b_{n-1}\}$ such that $a_i$ is not
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adjacent to \( b_j \) if and only if \( i, j \in xH \) for some \( x \in \mathbb{Z}_n \). Then,

\[
D_t(G, x) = [D_t(SC(\mathbb{Z}_n, H), x)]^2.
\]

Proof. We construct two graphs \( H_1 \) and \( H_2 \) with vertex sets \( V(H_1) = \{a_0, a_1, \ldots, a_{n-1}\} \) and \( V(H_2) = \{b_0, b_1, \ldots, b_{n-1}\} \) such that \( a_i \) is not adjacent to \( a_j \) and \( b_i \) is not adjacent to \( b_j \) if and only if \( i, j \in xH \) for some \( x \in \mathbb{Z}_n \). Then the graphs \( H_1 \) and \( H_2 \) are isomorphic to \( SC(\mathbb{Z}_n, H) \). Therefore, \( D_t(H_1, x) = D_t(H_2, x) = D_t(SC(\mathbb{Z}_n, H)) \). Since \( G \) is bipartite, the open neighborhood hypergraph of \( G \) has two components and they are isomorphic to the open neighborhood hypergraphs of \( H_1 \) and \( H_2 \) respectively. Therefore, \( D_t(G, x) = D_t(H_1, x)D_t(H_2, x) = [D_t(SC(\mathbb{Z}_n, H), x)]^2 \). This completes the proof.

Theorem 4.2.9. Let \( G \) be connected graph with \( n \) vertices, then

\[
D_t(G \circ K_1, x) = x^n(1 + x)^n.
\]

Proof. Let \( V(G) = \{1, 2, 3, \ldots, n\} \) and \( a_1, a_2, a_3, \ldots, a_n \) be the new vertices of \( G \circ K_1 \) such that \( N(a_i) = \{i\} \) for \( i = 1, 2, 3, \ldots, n \). So, a set \( S \) of vertices of \( G \circ K_1 \) is a total dominating set of \( G \circ K_1 \) if and only if \( \{1, 2, 3, \ldots, n\} \subseteq S \). Therefore, \( D_t(G \circ K_1, x) = x^n + \binom{n}{1}x^{n+1} + \binom{n}{2}x^{n+2} + \cdots + \binom{n}{n}x^n = x^n(1 + x)^n \). This completes the proof.

Theorem 4.2.10. Let \( G(m_1, m_2, \ldots, m_n) \) is the graph obtained from a connected graph \( G \) with \( n \geq 2 \), by attaching the root vertex of the star graph \( K_{1,m_i} \), (\( 1 \leq i \leq n \)) to the \( i \)-th vertex of the graph \( G \). If \( N = \sum_{i=1}^{n} m_i \), then the total domination polynomial of \( G(m_1, m_2, \ldots, m_n) \) is \( D_t(G(m_1, m_2, \ldots, m_n), x) = x^n(1 + x)^N \).
Proof. If \( v \) is a pendant vertex in \( G(m_1, m_2, \ldots, m_n) \) and \( u \) is the vertex adjacent to it, then \( N(v) = \{u\} \). So if \( S \) is a TD-set of \( G(m_1, m_2, \ldots, m_n) \), then \( V(G) \subseteq S \). Since \( G \) is connected, \( V(G) \) is TD-set of \( G(m_1, m_2, \ldots, m_n) \). So a TD-set with \( n+i \) elements can be selected in \( \binom{n+i}{i} \) ways. Therefore, \( D_t(G(m_1, m_2, \ldots, m_n), x) = x^n(1 + x)^N \). This completes the proof.

Corollary 4.2.11. Let \( N = \sum_{i=1}^{n} m_i \), then the TD-polynomial of the caterpillar graph \( T(m_1, m_2, \ldots, m_n) \) is, \( D_t(T(m_1, m_2, \ldots, m_n), x) = x^n(1 + x)^N \).

Proof. The caterpillar graph \( T(m_1, m_2, \ldots, m_n) \) is obtained from a path \( P_n \) with \( n \geq 2 \), by attaching the central vertex of the star graph \( K_{1,m_i} \) \( (1 \leq i \leq n) \) to the \( i \)-th vertex of the path \( P_n \). So the proof follows immediately if we take \( G \) as \( P_n \) in Theorem 4.2.10.

Corollary 4.2.12. For a graph \( G \) with \( n \) vertices, the total domination polynomial of \( G \circ K_m \) is \( x^n (1 + x)^{mn} \).

Proof. Note that the graph \( G \circ K_m \) is obtained from \( G \) and \( |V(G)| \) copies of the star graph \( K_{1,m} \) by identifying the root vertex of the \( i \)th copy of \( K_{1,m} \) with the \( i \)th vertex of \( G \). Therefore, the proof follows from Theorem 4.2.10.

Corollary 4.2.13. The total domination polynomial of the centipede \( P_n^* \) is

\[
D_t(P_n^*, x) = x^n(1 + x)^n.
\]

Proof. Observe that the centipede \( P_n^* \) is \( P_n \circ K_1 \). Therefore, the proof follows from Corollary 4.2.12.

Corollary 4.2.14. If \( B_{m,n} \) is the bi-star graph, then \( D_t(B_{m,n}, x) = x^2(1 + x)^{m+n} \).
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Proof. Let’s label the vertices of $B_{m,n}$ as shown in figure 4.1.

It can be observed that $B_{m,n}$ is obtained by identifying the vertex $u$ of the graph $K_2$ with the root vertex of the star $K_{1,m}$ and the vertex $v$ of $K_2$ with the root vertex of $K_{1,n}$. Therefore, the proof follows from Theorem 4.2.10.

Theorem 4.2.15. Let $G$ and $H$ be graphs of order $m$ and $n$, respectively, then

$$D_t(G \lor H, x) = [(1 + x)^m - 1][(1 + x)^n - 1] + D_t(G, x) + D_t(H, x).$$

Proof. If $S \subseteq V(G) \cup V(H)$, such that $S \cap V(G) \neq \emptyset$ and $S \cap V(H) \neq \emptyset$, then $S$ is a TD-set of $G \lor H$. Moreover if $S$ is a TD-set of $G$ or $H$, then $S$ is a TD-set of $G \lor H$. Therefore, $D_t(G \lor H, x) = [(1 + x)^m - 1][(1 + x)^n - 1] + D_t(G, x) + D_t(H, x).$

4.3 TD-Polynomials of total and middle graphs of some graphs

In this section, we obtain the total domination polynomials of total graph and middle graph of some graph classes.
4.3. TD-Polynomials of total and middle graphs of some graphs

Here we need the following.

**Definition 4.3.1. (see [28])** If $G$ is a graph, the total graph $T(G)$ of $G$ is the graph with vertex set $V(G) \cup E(G)$ in which two vertices are adjacent if they are either adjacent or incident in $G$. For a graph $G$, the middle graph $M(G)$ of $G$ is the graph with vertex set $V(G) \cup E(G)$ in which two vertices are adjacent if either they are adjacent edges of $G$ or one is a vertex of $G$ and the other is an edge incident with it.

**Theorem 4.3.2.** $D_t(T(K_{1,n}), x) = x \left[ (1 + x)^{2n} - 1 \right] + x^n \left[ (1 + x)^n \right]$.

**Proof.** Let $\{u\} \cup \{1, 2, 3, \ldots, n\}$ be the bipartition of the star graph $K_{1,n}$ and $E = \{e_1, e_2, e_3, \ldots, e_n\}$ be the edge set. Then the open neighborhoods of the vertices of $T(K_{1,n})$ are, for $1 \leq i \leq n$, $N(i) = \{u, e_i\}$, $N(e_i) = \{i, u\} \cup E \setminus \{e_i\}$ and $N(u) = \{1, 2, 3, \ldots, n, e_1, e_2, e_3, \ldots, e_n\}$

Let $S$ be a total dominating set of $T(K_{1,n})$. We consider the following cases.

**Case 1:** Let $u \in S$.

For any $x \in V(T(K_{1,n}))$, the set $\{u, x\}$ is a TD-set of $T(K_{1,n})$. Therefore, the number of TD-sets of cardinality $i + 1$ containing the vertex $u$ is $\binom{2n}{i}$.

**Case 2:** Let $u \notin S$.

Since $N(i) = \{u, e_i\}$, for all $i$, $S$ is a TD-set if and only if $E \subseteq S$. From the remaining $n$ vertices $\{1, 2, 3, \ldots, n\}$, $i$ vertices can be selected in $\binom{n}{i}$ ways.

Therefore,

$$D_t(T(K_{1,n}), x) = x \left[ \binom{2n}{1} x + \binom{2n}{2} x^2 + \ldots + \binom{2n}{2n} x^{2n} \right]$$
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\[ + x^n \left[ 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \ldots + \binom{n}{n} x^n \right] \]

\[ = x \left[ (1+x)^{2n} - 1 \right] + x^n [(1+x)^n]. \]

This completes the proof.

\[ \square \]

**Theorem 4.3.3.** \( D_t(M(K_{1,n}), x) = x^n (1 + x)^{n+1}. \)

**Proof.** Let \( \{u\} \cup \{1, 2, 3, \ldots, n\} \) be the bipartition of the star graph \( K_{1,n} \) and \( E = \{e_1, e_2, e_3, \ldots, e_n\} \) be the edge set. Then the open neighborhoods of the vertices of \( M(K_{1,n}) \) are, \( N(u) = \{e_1, e_2, \ldots, e_n\} \), \( N(e_i) = \{i, u\} \cup E \setminus \{e_i\} \) and \( N(i) = \{e_i\} \).

So, a set \( S \) of vertices of \( M(K_{1,n}) \) is a TD-set if and only if \( E \subseteq S \). From the remaining \( n+1 \) vertices, \( i \) vertices can be selected in \( \binom{n+1}{i} \) ways. Therefore, \( D_t(M(K_{1,n}), x) = x^n (1 + x)^{n+1} \). Thus the proof is complete.

\[ \square \]

**Theorem 4.3.4.** If \( G \) is connected graph of order \( n \) and size \( l \), then the total domination polynomial of the middle graph of \( G(m_1, m_2, \ldots, m_n) \) is

\[ D_t(M(G(m_1, m_2, \ldots, m_n)), x) = x^N (1 + x)^{N+n+l}, \text{ where } N = \sum_{i=1}^{n} m_i. \]

**Proof.** If \( v \) is a pendant vertex in \( G(m_1, m_2, \ldots, m_n) \) and \( e \), the edge incident with it, then \( N_{M(G(m_1, m_2, \ldots, m_n))}(v) = \{e\} \).

Let \( E \) be the set of all pendant edges of \( G(m_1, m_2, \ldots, m_n) \). Therefore, if \( S \) is a TD-set of \( M(G(m_1, m_2, \ldots, m_n)) \), then \( E \subseteq S \). Clearly \( E \) is a TD-set of \( M(G(m_1, m_2, \ldots, m_n)) \).

So, from the remaining \( N + n + l \) vertices in \( M(G(m_1, m_2, \ldots, m_n)) \), \( i \) vertices can be selected in \( \binom{N+n+l}{i} \) ways. Therefore, \( D_t(M(G(m_1, m_2, \ldots, m_n)), x) = x^N (1 + x)^{N+n+l} \). Thus the proof is complete.

\[ \square \]

**Corollary 4.3.5.** The TD-polynomial of middle graph of the bi-star graph \( B_{m,n} \) is \( D_t(M(B_{m,n}), x) = x^{m+n} (1 + x)^{m+n+3}. \)

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Proof. The bi-star graph $B_{m,n}$ is obtained from $K_2$ by attaching the central vertices of the star graphs $K_{1,m}$ and $K_{1,n}$ to the first and second vertices of $K_2$ respectively. Therefore, the proof follows immediately from Theorem 4.3.4.

Corollary 4.3.6. Let $G$ be a connected graph of order $m$ and size $l$ and if $n \geq 2$, then the total domination polynomial of the middle graph of $G \circ K_n$ is

$$D_t(M(G \circ K_n), x) = x^{mn}(1 + x)^{m+l+n}.$$  

Proof. The graph $G \circ K_n$ is obtained from $G$ by attaching the central vertices of $n$ copies of the star graph $K_{1,n}$ to the vertices of $G$. So replacing $m_i$ by $n$ for all $i$ in Theorem 4.3.4 we obtain the result.

We take $N = \sum_{i=1}^{n} m_i$ for Corollary 4.3.7, 4.3.8 and 4.3.9.

Corollary 4.3.7. The TD-polynomial of middle graph of the caterpillar graph is, $D_t(M(T(m_1, m_2, \ldots, m_n)), x) = x^N(1 + x)^{N+2n-1}$.

Proof. In Theorem 4.3.4 if we take $G$ as $P_n$, we obtain the result.

Corollary 4.3.8. For the caterpillar graph $T(m_1, m_2, \ldots, m_n)$,

$$D_t(M(T(m_1, m_2, \ldots, m_n)), x) = x^{N-n}(1 + x)^{2n-1}D_t(T(m_1, m_2, \ldots, m_n), x).$$

Proof. The proof follows from 4.2.11 and 4.3.7.

Corollary 4.3.9. If $G$ is the cycle $C_n$, then

$$D_t(M(G(m_1, m_2, \ldots, m_n)), x) = x^{N-n}(1 + x)^{2n}D_t(G(m_1, m_2, \ldots, m_n), x).$$
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Proof. From Theorems 4.2.10 and 4.3.4 we have, \( D_t(C_n(m_1, m_2, \ldots, m_n), x) = x^n(1 + x)^N \) and \( D_t(M(C_n(m_1, m_2, \ldots, m_n)), x) = x^N(1 + x)^{N+2n} \). Hence the result follows. \( \Box \)