Chapter 2

TD-Polynomials- A New Approach

2.1 Introduction

This chapter deals with the relation between total domination polynomials and vertex cover polynomials. For a graph $G = (V, E)$, the open neighborhood hypergraph of $G$, denoted by $ONH(G)$, is the hypergraph with vertex set $V$ and edge set $\{N_G(x) | x \in V\}$. A vertex cover in $ONH(G)$ is a set of vertices intersecting every edge of $ONH(G)$, which is equivalent to a total dominating set in $G$. Using the interplay between total dominating sets and vertex cover in hypergraphs, we determine the total domination polynomial of some classes of graphs. Here we need the following.

Definition 2.1.1. (see [44]) A graph $G$ in which a vertex is distinguished from other vertices is called a rooted graph and the vertex is called the root of $G$. Let $G$ be a rooted graph. The graph $G^{(n)}$ obtained by identifying the roots of $n$ copies

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of $G$ is called a one point union of the $n$ copies of $G$.

**Definition 2.1.2.** A one point union $C_n^{(k)}$ of $k$ copies of $C_n$ is the graph obtained by taking $v$ as a common vertex such that any two cycles $C_n^i$ and $C_n^j$ ($i \neq j$) are edge disjoint and do not have any vertex in common except $v$.

**Definition 2.1.3.** (see [38]) An $n$-gon book of $k$ pages denoted by $C_n^{2(k)}$ is the graph obtained when $k$ copies of the cycle $C_n$ share a common edge.

**Definition 2.1.4.** (see [38]) Given $k$ natural numbers $n_1, n_2, \ldots, n_k$, the generalized theta graph $\theta(n_1, n_2, \ldots, n_k)$ is obtained by connecting two vertices $u$ and $v$ by $k$ parallel paths of length $n_1 - 1, n_2 - 1, \ldots, n_k - 1$.

**Definition 2.1.5.** Let $P_{n_1+1}, P_{n_2+1}, \ldots, P_{n_k+1}$ be $k$ paths. For $i = 1, 2, \ldots, k$, let $a_i$ be a pendant vertex of the path $P_{n_i}$. Then the tree $T_{n_1, n_2, \ldots, n_k}$ is obtained by identifying the vertices $a_i$ for every $i$.

**Theorem 2.1.6.** (see [26]) The ONH of a connected bipartite graph consists of two components, while the ONH of a connected graph that is not bipartite is connected.

**Theorem 2.1.7.** (see [27]) If $G$ is a graph with no isolated vertices and $H_G$ is the ONH of $G$, then $\gamma_t(G) = \tau(H_G)$.

**Theorem 2.1.8.** (see [18]) Let $G$ be a graph and $L = \{u \in V(G) | uu \in E(G)\}$. Then $C(G, x) = x^{|L|}C(G - L, x)$.

**Theorem 2.1.9.** (see [18]) Let $G$ be a graph with $|V(G)| \geq 2$. Let $u \in V(G)$ and $d = |N_G(u)|$. If $G$ has no loops at $N_G[u]$, then $C(G, x) = xC(G - u, x) + x^dC(G - u - N_G(u), x)$.
Lemma 2.1.10. (see [18]) For \( n \geq 5 \), we have

\[
C(C_n, x) = xC(P_{n-1}, x) + x^2C(P_{n-3}, x).
\]

Theorem 2.1.11. (see [18]) Let \( G = G_1 \cup G_2 \ldots \cup G_n \) be the union of \( n \) graphs \( G_1, G_2, \ldots, G_n \). Then \( C(G, x) = C(G_1, x)C(G_2, x) \ldots C(G_n, x) \).

Theorem 2.1.12. (see [18]) For the path graph \( P_n \), where \( n \geq 2 \), we have

\[
C(P_n, x) = \sum_{i=0}^{n} \binom{i+1}{n-i} x^i.
\]

Theorem 2.1.13. (see [18]) For the cycle graph \( C_n \), where \( n \geq 3 \), we have

\[
C(C_n, x) = \sum_{i=1}^{n} \frac{n}{i} \binom{i}{n-i} x^i.
\]

Theorem 2.1.14. The total domination polynomial of a connected bipartite graph \( G \) is the product of the vertex cover polynomials of the two components of its open neighborhood hypergraph, \( H_G \), while the total domination polynomial of a connected graph that is not bipartite is the vertex cover polynomial of \( H_G \).

Proof. A set \( S \) of vertices of \( G \) is a total dominating set if and only if it is a vertex covering set of the open neighborhood hypergraph \( H_G \) of \( G \). Therefore, if \( G \) is not bipartite, the result follows. If \( G \) is bipartite, its open neighborhood hypergraph, \( H_G \), has two components and its vertex cover polynomial is the product of the vertex cover polynomials of its components. Thus the proof follows from the definitions of total domination set of \( G \) and vertex cover polynomial of \( H_G \).
2.2 TD-Polynomials of paths and cycles

We observe that using the interplay between total dominating sets in graphs and transversals in hypergraphs, several results on total domination polynomials in graphs can be obtained that appear very difficult to obtain using purely graph theoretic techniques. In this section, we study the relation between total domination polynomials and vertex cover polynomials of paths and cycles.

We need the following to prove the main results of this section.

Lemma 2.2.1. Let $P'_n$ be the graph shown in figure 2.1. Then,
\[
C(P'_n, x) = xC(P_{n-1}, x) = x \sum_{i=0}^{n-1} \left( \frac{i + 1}{n - (i + 1)} \right) x^i.
\]

Proof. The proof follows immediately from Theorem 2.1.8 and Theorem 2.1.12.

\[\]

Figure 2.1: The Graph $P'_n$

Lemma 2.2.2. Let $P''_n$ be the graph shown in Figure 2.2. Then,
\[
C(P''_n, x) = x^2C(P_{n-2}, x) = x^2 \sum_{i=0}^{n-2} \left( \frac{i + 1}{n - (i + 2)} \right) x^i.
\]

Proof. The proof follows from Theorem 2.1.8 and Theorem 2.1.12.

\[\]
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Theorem 2.2.3. For the path graph $P_n$, where $n > 2$, we have

$$C(P_n, x) = x \left[ C(P_{n-1}, x) + C(P_{n-2}, x) \right]$$

Proof. Let $(1, 2, \ldots, n)$ be the path $P_n$ and $S$ be a vertex covering set of $P_n$. If $n \in S$, then $S$ is a vertex covering set of the graph $P'_n$ shown in figure 2.1. If $n \notin S$, then the vertex $n - 1 \in S$. Therefore, $S$ is a vertex covering set of $P'_{n-1}$. Conversely, any vertex covering set of $P'_n$ or $P'_{n-1}$ is a vertex covering set of $P_n$. This completes the proof.

Theorem 2.2.4. For $n \geq 1$, $D_t(P_{2n}, x) = [C(P'_n, x)]^2$.

Proof. Let $(1, 2, \ldots, 2n)$ be the path $P_{2n}$. Since $P_{2n}$ is bipartite, the open neighborhood hypergraph of $P_{2n}, ONH(P_{2n})$ has two components say $G_1$ and $G_2$.

The edge sets of $G_1$ and $G_2$ are $E(G_1) = \{xy: x = 2i - 1 \text{ and } y = 2i + 1 \text{ where } 1 \leq i \leq n - 1\} \cup \{2n - 1\}$ and $E(G_2) = \{xy: x = 2i \text{ and } y =$
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2i + 2 where 1 ≤ i ≤ n} ∪ \{\{22\}\}. Clearly $G_1$ is isomorphic to $G_2$. Using the terminology in \cite{9}, the graph $G_2$ can be drawn as shown in figure 2.3. Since $G_2$ is isomorphic to $P_n'$ and $G_1$ is isomorphic to $G_2$, the proof follows from Theorem 2.1.8 and 2.1.11.

**Theorem 2.2.5.** For $n \geq 1$, the total domination polynomial of path $P_{2n}$ is,

$$D_t(P_{2n}, x) = x^2 \left[ \sum_{i=0}^{n-1} \left( \frac{i+1}{n-(i+1)} \right) x^i \right]^2.$$

**Proof.** The proof follows from Lemma 2.2.1 and Theorem 2.2.4.

**Theorem 2.2.6.** For $n \geq 1$, $D_t(P_{2n+1}, x) = \mathcal{C}(P_{n+1}, x)\mathcal{C}(P_n', x)$.

**Proof.** Let $(1, 2, 3, \ldots, 2n-1, 2n, 2n+1)$ be the path $P_{2n+1}$. Then the open neighborhood hypergraph of $P_{2n+1}$ has two components $G_1$ and $G_2$ with edge sets $E(G_1) = \{xy: x = 2i - 1$ and $y = 2i + 1$, where $1 \leq i \leq n\}$ and $E(G_2) = \{xy: x = 2i$ and $y = 2i + 2$, where $1 \leq i \leq n - 1\} \cup \{22, 2n2n\}$. The graph $G_2$ can be drawn as shown in figure 2.4.

![Figure 2.4: The Graph $G_2$](image)

Let $P_{n+1}$ be the path $(1, 3, 5, \ldots, 2n-1, 2n+1)$ Since $E(G_1) = E(P_{n+1})$, a set $S$ is a vertex cover of $G_1$ if and only if $S$ is a vertex cover of $P_{n+1}$. Since $G_2$ is isomorphic to $P_n'$, by Theorem 2.1.14 and Lemma 2.2.2 we have $D_t(P_{2n+1}, x) = \mathcal{C}(G_1, x)\mathcal{C}(G_2, x) = \mathcal{C}(P_{n+1}, x)\mathcal{C}(P_n', x)$. This completes the proof.
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Theorem 2.2.7. The total domination polynomial of the path $P_{2n+1}$ is,

$$D_t(P_{2n+1}, x) = x^2 \left[ \sum_{i=0}^{n+1} \binom{i+1}{n+1-i} x^i \right] \left[ \sum_{i=0}^{n-2} \binom{i+1}{n-(2+i)} x^i \right]$$

Proof. The proof follows immediately from \ref{2.1.12} \ref{2.1.14} \ref{2.2.2} and \ref{2.2.6} \hfill \Box

Theorem 2.2.8. For $n \geq 3$, we have, $D_t(C_{2n}, x) = [\mathcal{C}(C_n, x)]^2$.

Proof. Let $(1, 2, \ldots, 2n, 1)$ be the cycle $C_{2n}$. Since $C_{2n}$ is bipartite, its open neighborhood hypergraph has two components. It can be observed that the components are cycles, say $C'$ and $C''$, where $C' = (2, 4, 6, \ldots, 2n, 2)$ and $C'' = (1, 3, 5, \ldots, 2n-1, 1)$. Since the cycles $C'$ and $C''$ are isomorphic to the cycle $C_n$, the proof follows from Theorem \ref{2.1.11} \hfill \Box

Theorem 2.2.9. $D_t(C_{2n}, x) = \left[ \sum_{i=1}^{n} \frac{n}{i} \binom{i}{n-i} x^i \right]^2$.

Proof. The proof follows from Theorems \ref{2.1.13} and \ref{2.2.8} \hfill \Box

Theorem 2.2.10. If $n$ is an odd positive integer, then

$$D_t(C_n, x) = \sum_{i=1}^{n} \frac{n}{i} \binom{i}{n-i} x^i.$$ 

Proof. Since the open neighborhood hypergraph of a cycle of odd length is isomorphic to itself, the proof follows from Theorem \ref{2.1.13} \hfill \Box

Theorem 2.2.11. $D_t(C_{2n}, x) = [x\mathcal{C}(P_{n-1}, x) + x^2\mathcal{C}(P_{n-3}, x)]^2$.

Proof. From Theorem \ref{2.2.8} we have, $D_t(C_{2n}, x) = [\mathcal{C}(C_n, x)]^2$. Then the proof follows from Lemma \ref{2.1.10} \hfill \Box
Theorem 2.2.12. If \( n \) is an odd positive integer, then

\[
D_t(C_n, x) = \left[ xC(P_{n-1}, x) + x^2C(P_{n-3}, x) \right].
\]

Proof. Here, \( D_t(C_n, x) = C(C_n, x) \). Then by Lemma 2.1.10, the proof follows. \( \square \)

2.3 TD-Polynomials of some graph classes

In this section, we find the total domination polynomial of some classes of graphs using vertex cover polynomials of paths and cycles. Let \( G \) be a graph and \( A \) be a subset of the set of vertices of \( G \). We define the following classes of vertex cover polynomials to obtain the main results of this chapter.

Definition 2.3.1. Let \( C^A(G, i) = \{ S \subseteq V(G): S \in C(G, i) \text{ and } S \cap A \neq \emptyset \} \).

Then, the polynomial \( C^A(G, x) \) is defined as \( C^A(G, x) = \sum_{i=1}^{\left| V(G) \right|} c^A(G, i)x^i \), where \( c^A(G, i) = |C^A(G, i)| \).

Definition 2.3.2. Let \( C^{A^*}(G, i) = \{ S \subseteq V(G): S \in C(G, i) \text{ and } S \cap A = A \} \).

Then, the polynomial \( C^{A^*}(G, x) \) is defined as \( C^{A^*}(G, x) = \sum_{i=1}^{|V(G)|} c^{A^*}(G, i)x^i \), where \( c^{A^*}(G, i) = |C^{A^*}(G, i)| \).

Definition 2.3.3. Let \( C_A(G, i) = \{ S \subseteq V(G): S \in C(G, i) \text{ and } S \cap A = \emptyset \} \).

Then, the polynomial \( C_A(G, x) \) is defined as \( C_A(G, x) = \sum_{i=1}^{\left| V(G) \right|} c_A(G, i)x^i \), where \( c_A(G, i) = |C_A(G, i)| \).

Note 2.3.4. If \( A = \{ a \} \), then we write \( C^a(G, x) \) and \( C_a(G, x) \) instead of \( C^A(G, x) \) and \( C_A(G, x) \) respectively.
Lemma 2.3.5. Let $u$ be a vertex of degree $d$ in $G$. Let $G$ has no loops at $A \cup N_G[u]$ and $A \cap N_G[u] = \phi$. Then $C_A(G, x) = xC_A(G - u, x) + x^dC_A(G - N_G[u], x)$.

Proof. Let $S \in C_A(G, i)$. Then either $u \in S$ or $u \notin S$. Now, $u \in S$ if and only if $S \setminus \{u\} \in C_A(G - u, i - 1)$. If $u \notin S$, then by definition, $N_G(u) \subseteq S$ and $S \setminus N_G(u)$ is an $(i - d)$-vertex cover of $G - N_G[u]$. Conversely, if $S \in C_A(G - N_G[u], i - d)$, then $u \notin S$ and $S \cup N_G[u] \in C_A(G, i)$. Therefore, $c_A(G, i) = c_A(G - u, i - 1) + c_A(G - N_G[x], i - d)$. So,

$$C_A(G, x) = \sum_{i=1}^{|V(G)|} c_A(G, i)x^i$$

$$= \sum_{i=1}^{|V(G)|} [c_A(G - u, i - 1) + c_A(G - N_G[x], i - d)]x^i$$

$$= \sum_{i=1}^{|V(G)|} c_A(G - u, i - 1)x^i + \sum_{i=1}^{|V(G)|} c_A(G - N_G[x], i - d)x^i$$

$$= xC_A(G - u, x) + x^dC_A(G - N_G[u], x).$$

Thus the proof follows.

Lemma 2.3.6. If the path $P_n = (1, 2, \ldots, n)$, then

(i) $C^{(1)}(P_n, x) = xC(P_{n-1}, x)$,

(ii) $C^{(1)}_1(P_n, x) = xC(P_{n-2}, x)$,

(iii) $C^{(1,n)}(P_n, x) = x^2C(P_{n-2}, x)$,

(iv) $C^{(1,n)}_1(P_n, x) = x^2C(P_{n-4}, x)$.

Proof. (i) Let $S$ be a subset of vertices of $P_n$. It is observed that $S$ is a vertex covering set of $P_n$ containing the vertex 1 if and only if $S$ is a vertex covering
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set of the graph $H$ shown in figure 2.5. Therefore, the proof follows from Theorem 2.1.8.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.5.png}
\caption{The graph $H$.}
\end{figure}

\begin{enumerate}
\item[(ii)] If $S$ is a vertex covering set of $P_n$ and $S \cap \{1\} = \emptyset$, then $2 \in S$. So, $S$ is a vertex covering set of the graph $K$ shown in figure 2.6. Therefore, from Theorem 2.1.8 the result follows.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.6.png}
\caption{The graph $K$.}
\end{figure}

\item[(iii)] If $S$ is a vertex covering set of $P_n$ containing the vertices 1 and $n$, then $S$ is a vertex covering set of the graph $P_n''$ shown in figure 2.2. Therefore, the proof follows from theorem 2.1.8.

\item[(iv)] Let $S$ be a vertex covering set of $P_n$ such that $S \cap \{1, n\} = \emptyset$, then $S$ is a vertex covering set of the graph $K_1$ shown in figure 2.7.

Therefore, from Theorem 2.1.8 the proof follows. \□
\end{enumerate}
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Next, we find the total domination polynomial of the tree \( T_{n_1,n_2,n_3} \).

**Theorem 2.3.7.** If \( n_1, n_2, n_3 \) are even and \( T_1, T_2 \) are the components of the open neighborhood hypergraph of the tree \( T_{n_1,n_2,n_3} \), then

\[
D_t(T_{n_1,n_2,n_3}, x) = C(T_1, x)C(T_2, x),
\]
where

\[
C(T_1, x) = x^3 \prod_{i=1}^{3} C(P_{n_2}^{\frac{n_1}{2}}, x) + x^3 \prod_{i=1}^{3} C(P_{n_2}^{\frac{n_1}{2}-1}, x)
\]
and

\[
C(T_2, x) = x^3 \prod_{i=1}^{3} C(P_{n_3}^{\frac{n_2}{2}-1}, x) - x^6 \prod_{i=1}^{3} C(P_{n_3}^{\frac{n_2}{2}-3}, x).
\]

Proof. Let \( X = \{x_i : i \text{ is odd}\} \) and \( Y = \{y_j : j \text{ is even}\} \cup \{v\} \) be the partite sets of
2.3. TD-Polynomials of some graph classes

Let $T_1$ and $T_2$ be the components of the open neighborhood hypergraph of $T_{n_1,n_2,n_3}$, such that $E(T_1) = \{N(x) : x \in X\}$ and $E(T_2) = \{N(y) : y \in Y\}$. Then $T_1$ can be represented as shown in figure 2.9.

![Diagram 2.9: The graph $T_1$.](image)

From Theorem 2.1.9, we have,

$$C(T_1, x) = xC(T_1 - v, x) + x^3C(T_1 - v - \{a_2, b_2, c_2\}, x)$$

$$= xC(P_{n_1}, x)C(P_{n_2}, x)C(P_{n_3}, x)$$

$$+ x^3C(P_{n_1 - 1}, x)C(P_{n_2 - 1}, x)C(P_{n_3 - 1}, x)$$

$$= x \prod_{i=1}^{3} C(P_{n_i}, x) + x^3 \prod_{i=1}^{3} C(P_{n_i - 1}, x).$$

Next, we find the vertex cover polynomial of $T_2$. It can be observed that $E(T_2) = \{a_1, b_1, c_1\} \cup E(T_a) \cup E(T_b) \cup E(T_c)$, where the graphs $T_a, T_b$ and $T_c$ are shown in figure 2.10. Let $A = \{a_1, b_1, c_1\}$. Then a set $S$ is vertex covering set of $T_2$ if and only if $S \cap A \neq \emptyset$ and $S$ is a vertex covering set of $T_a \cup T_b \cup T_c$. In other words
\[ C(T_2, x) = C^A(T_a \cup T_b \cup T_c, x) \]. Therefore, from Theorem 2.1.11 and Lemma 2.3.6 we have,

\[
C(T_2, x) = C^A(T_a \cup T_b \cup T_c, x) = C(T_a, x)C(T_b, x)C(T_c, x) - C_{a_1}(T_a, x)C_{b_1}(T_b, x)C_{c_1}(T_c, x) = x^3C(P_{\frac{n_1}{2} - 1}, x)C(P_{\frac{n_2}{2} - 1}, x)C(P_{\frac{n_3}{2} - 1}, x) - x^6C(P_{\frac{n_1}{2} - 3}, x)C(P_{\frac{n_2}{2} - 3}, x)C(P_{\frac{n_3}{2} - 3}, x) = x^3\prod_{i=1}^3 C(P_{\frac{n_i}{2} - 1}, x) - x^6\prod_{i=1}^3 C(P_{\frac{n_i}{2} - 3}, x).
\]

This completes the proof. \[ \square \]

\[ T_a : \begin{array}{cccccc}
\bullet & & & & & \\
an_{1-1} & a_{n-3} & a_{n-5} & a_3 & a_1
\end{array} \]

\[ T_b : \begin{array}{cccccc}
\bullet & & & & & \\
b_{n-1} & b_{n-3} & b_{n-5} & b_3 & b_1
\end{array} \]

\[ T_c : \begin{array}{cccccc}
\bullet & & & & & \\
c_{n-1} & c_{n-3} & c_{n-5} & c_3 & c_1
\end{array} \]

Figure 2.10: The Graphs \( T_a, T_b \) and \( T_c \).

**Corollary 2.3.8.** If \( n_1 = n_2 = n_3 = 2n \), then

\[
D_1(T_{n_1,n_2,n_3}, x) = x^4 [C(P_n, x)C(P_{n-1}, x)]^3 + x^7 [C(P_n, x)C(P_{n-3}, x)]^3 + x^6 [C(P_n, x)]^6
\]
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\[ x^9 \left[ C(P_n-1, x) C(P_{n-3}, x) \right]^3. \]

Proof. The proof follows from Theorem 2.3.7. \( \square \)

**Theorem 2.3.9.** If \( n_1, n_2, n_3 \) are odd and \( T_1, T_2 \) are the components of the open neighborhood hypergraph of the tree \( T_{n_1,n_2,n_3} \), then

\[
D_t(T_{n_1,n_2,n_3}, x) = C(T_1, x) C(T_2, x), \text{ where }
\]

\[
C(T_1, x) = x^4 \left[ \prod_{i=1}^{3} C(P_{n_i-3}, x) + x^2 \prod_{i=1}^{3} C(P_{n_i-5}, x) \right] \]

and

\[
C(T_2, x) = \prod_{i=1}^{3} C(P_{n_i+1}, x) - x^3 \prod_{i=1}^{3} C(P_{n_i-3}, x). \]

Proof. Let \( X = \{ x_i : i \text{ is odd} \} \) and \( Y = \{ y_j : j \text{ is even} \} \cup \{ v \} \) be the bipartition of \( T_{n_1,n_2,n_3} \). Let \( T_1 \) and \( T_2 \) be the components of the open neighborhood hypergraph of \( T_{n_1,n_2,n_3} \), such that \( E(T_1) = \{ N(x) : x \in X \} \) and \( E(T_2) = \{ N(y) : y \in Y \} \). Then proceeding as in Theorem 2.3.7, we can represent \( T_1 \) as shown in figure 2.11. Let \( T_1^* = T_1 - \{ a_{n_1-1}, b_{n_2-1}, c_{n_3-1} \} \), \( T_1^{**} = T_1 - \{ v, a_{n_1-1}, b_{n_2-1}, c_{n_3-1} \} \), and \( T_1^{***} = T_1 - \{ v, a_2, b_2, c_2, a_{n_1-1}, b_{n_2-1}, c_{n_3-1} \} \). Then, from Theorem 2.1.8 and 2.1.9 we get,

\[
C(T_1, x) = x^3 C(T_1^*, x)
\]

\[
= x^3 \left[ x C(T_1^{**}, x) + x^3 C(T_1^{***}, x) \right]
\]

\[
= x^4 \left[ \prod_{i=1}^{3} C(P_{n_i-1}, x) + x^2 \prod_{i=1}^{3} C(P_{n_i-2}, x) \right]
\]

\[
= x^4 \left[ \prod_{i=1}^{3} C(P_{n_i-3}, x) + x^2 \prod_{i=1}^{3} C(P_{n_i-5}, x) \right].
\]

Let \( P_{n_1+1} = (a_1, a_3, a_5, \ldots, a_{n_1-2}, a_{n_1}) \), \( P_{n_2+1} = (b_1, b_3, b_5, \ldots, b_{n_2-2}, b_{n_2}) \) and
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Let $P_{\frac{n+1}{2}} = (c_1, c_3, c_5, \ldots, c_{n+2}, c_n)$ be three paths. Then the edge set of the graph $T_2$ is $E(T_2) = \{a_1, b_1, c_1\} \cup E(P_{\frac{n+1}{2}}) \cup E(P_{\frac{n+1}{2}}) \cup E(P_{\frac{n+1}{2}})$.

Let $A = \{a_1, b_1, c_1\}$. Then a set $S$ is vertex covering set of $T_2$ if and only if $S \cap A \neq \emptyset$ and $S$ is a vertex covering set of $P_{\frac{n+1}{2}} \cup P_{\frac{n+1}{2}} \cup P_{\frac{n+1}{2}}$. Therefore, we need to calculate the polynomial $C^A(P_{\frac{n+1}{2}} \cup P_{\frac{n+1}{2}} \cup P_{\frac{n+1}{2}}, x)$. That is $C(T_2, x) = C^A(P_{\frac{n+1}{2}} \cup P_{\frac{n+1}{2}} \cup P_{\frac{n+1}{2}}, x)$. Therefore, from Theorem 2.1.11 and Lemma 2.3.6 we have,

$$C(T_2, x) = C^A(P_{\frac{n+1}{2}} \cup P_{\frac{n+1}{2}} \cup P_{\frac{n+1}{2}}, x)$$

$$= C(P_{\frac{n+1}{2}} \cup P_{\frac{n+1}{2}} \cup P_{\frac{n+1}{2}}, x) - C_A(P_{\frac{n+1}{2}} \cup P_{\frac{n+1}{2}} \cup P_{\frac{n+1}{2}}, x)$$

$$= C(P_{\frac{n+1}{2}}, x) C(P_{\frac{n+1}{2}}, x) C(P_{\frac{n+1}{2}}, x)$$

$$- C_{a_1}(P_{\frac{n+1}{2}}, x) C_{b_1}(P_{\frac{n+1}{2}}, x) C_{c_1}(P_{\frac{n+1}{2}}, x)$$

$$= \prod_{i=1}^{n+1} C(P_{\frac{n+1}{2}}, x) - x^3 \prod_{i=1}^{n+1} C(P_{\frac{n+1}{2}}, x).$$
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Thus the result follows.

Lemma 2.3.10. If \( d \geq 3 \) is a positive integer, the vertex cover polynomial of the tree \( T_{n, n, \ldots, n} \) denoted by \( T_{n \times n \times \ldots \times n} \) is \( C(T_{n \times n \times \ldots \times n}, x) = x [C(P_n, x)]^d + x^d [C(P_{n-1}, x)]^d \).

Proof. Let \( v \) be the vertex of degree \( d \) in \( T_{n \times n \times \ldots \times n} \). Using Theorem 2.1.9 and 2.1.11 we have,

\[
C(T_{n \times n \times \ldots \times n}, x) = x [C(T_{n \times n \times \ldots \times n} - v, x)] + x^d [C(T_{n \times n \times \ldots \times n} - N_{T_{n \times n \times \ldots \times n}}[v], x)]
\]

\[
= x [C(P_n, x)]^d + x^d [C(P_{n-1}, x)]^d.
\]

Thus the proof is complete.

Lemma 2.3.11. If \( A \) is the set of all pendant vertices of the tree \( T_{n \times n \times \ldots \times n} \), then

\[
C_A(T_{n \times n \times \ldots \times n}, x) = x [xC(P_{n-2}, x)]^d + x^d [xC(P_{n-3}, x)]^d.
\]

Proof. From Lemma 2.3.5, 2.3.6 and 2.3.10 we have,

\[
C_A(T_{n \times n \times \ldots \times n}, x) = x [C_A(P_n, x)]^d + x^d [C_A(P_{n-1}, x)]^d
\]

\[
= x [xC(P_{n-2}, x)]^d + x^d [xC(P_{n-3}, x)]^d.
\]

Since \( C^A(T_{n \times n \times \ldots \times n}, x) = C(T_{n \times n \times \ldots \times n}, x) - C_A(T_{n \times n \times \ldots \times n}, x) \), from Lemma 2.3.10 we have,

\[
C^A(T_{n \times n \times \ldots \times n}, x) = x [C(P_n, x)]^d + x^d [C(P_{n-1}, x)]^d
\]

\[
- x [xC(P_{n-2}, x)]^d - x^d [xC(P_{n-3}, x)]^d.
\]
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This completes the proof. \( \square \)

Next, we find the TD-Polynomial of one point union of cycles \( C_n^{(k)} \).

**Theorem 2.3.12.** If \( n = 2m + 1 \) for some positive integer \( m \), then the TD-Polynomial of \( C_n^{(k)} \) is

\[
D_t(C_n^{(k)} , x) = x [C(P_m , x)]^{2k} + x^{2k} [C(P_{m-1} , x)]^{2k} - x [x C(P_{m-2} , x)]^{2k} - x^{2k} [x C(P_{m-3} , x)]^{2k}.
\]

**Proof.** For \( j = 1, 2, \ldots, k \) let \( v \) be the vertex common to the cycles \( C_n^{(k)} \). Let \( (v, a_1^i, a_2^i, a_3^i, \ldots, a_{n-1}^i, v) \) be the cycle \( C_n^{(k)} \). If \( G \) represents the open neighborhood hypergraph of \( C_n^{(k)} \), then \( E(G) = N_{C_n^{(k)}}(v) \cup E(T_{[m]^{2k}}) \), where the tree \( T_{[m]^{2k}} \) is shown in figure 2.12.

![Figure 2.12: The Tree \( T_{[m]^{2k}} \)](image)

If \( A \) is the set of all pendent vertices of \( T_{[m]^{2k}} \), then a set \( S \) of vertices of \( C_n^{(k)} \) is a total dominating set if and only if \( S \cap A \neq \phi \) and \( S \) is a vertex covering set of...
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Therefore, it suffices to find the polynomial \( C^A(T_{[m]^{2k}},x) \). From Lemma 2.3.11 we have,

\[
C^A(T_{[m]^{2k}},x) = x[C(P_m,x)]^{2k} + x^{2k} [C(P_{m-1},x)]^{2k} - x [xC(P_{m-2},x)]^{2k} - x^{2k} [xC(P_{m-3},x)]^{2k}.
\]

Thus the proof is complete. \( \square \)

**Theorem 2.3.13.** If \( n = 2m \), for some positive integer \( m \) and \( G_1, G_2 \) are the components of the open neighborhood hypergraph of \( C^{(k)}_n \), then

\[
C(G_1, x) = [C(P_m,x)]^k - [x^2 C(P_{m-4},x)]^k \quad \text{and} \quad C(G_2, x) = x[C(P_{m-1},x)]^k + x^{2k} [C(P_{m-3},x)]^k.
\]

**Proof.** For \( j = 1, 2, \ldots, k \) let \( v \) be the vertex common to the cycles \( C^i_j \). Let \((v, a^1_i, a^2_i, a^3_i, \ldots, a^i_{n-1}, v)\) be the cycle \( C^n_i \). Since \( n \) is even, the graph \( C^{(k)}_n \) is bipartite. Let \( X = \bigcup_{i=1}^k \{a^i_j : j \text{ is even}\} \cup \{v\} \) and \( Y = \bigcup_{i=1}^k \{a^i_j : j \text{ is odd}\} \) be the bipartition. Let \( G_1 \) and \( G_2 \) are the components of \( ONH(C^{(k)}_n) \) corresponding to \( X \) and \( Y \) respectively. Then \( E(G_1) = N_{C^{(k)}_n}(v) \cup E(H) \), where \( H \) is given in figure 2.13

Therefore, a set \( S \) of vertices of \( G_1 \) is a vertex cover if and only if \( S \) is a vertex cover of \( H \) and \( S \cap N_{C^{(k)}_n}(v) \neq \emptyset \). Since \( N_{C^{(k)}_n}(v) \), denoted here by \( N(v) \), is the set of all pendent vertices of \( H \), it suffices to find the polynomial \( C^{N(v)}(H, x) \). Since \( C^{N(v)}(H, x) \) is the polynomial in \( x \) such that the coefficient of \( x^i \) is the number of vertex covering sets of \( H \) which does not intersect with \( N(v) \), from Theorem

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From Theorems 2.1.8 and 2.1.9, we have

\[
C(G_1, x) = C^{N(v)}(H, x) = C(H, x) - C_{N(v)}(H, x) = [C(P_m, x)]^k - [x^2C(P_{m-4}, x)]^k.
\]

Figure 2.13: The graph \( H \)

Next, we find the vertex cover polynomial of the graph \( G_2 \). It can be represented as shown in figure 2.14. From Theorem 2.1.9, we have

\[
C(G_2, x) = xC(G_2 - v, x) + x^{2k}C(G_2 - v - N_{G_2}(v), x) = x[C(P_{m-1}, x)]^k + x^{2k} [C(P_{m-3}, x)]^k.
\]

This completes the proof.

\[ \square \]
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Theorem 2.3.14. If $n = 2m$ for some positive integer $m$, then

$$D_t(C_n^{(k)}, x) = \left( [C(P_m, x)]^k - [x^2C(P_{m-4}, x)]^k \right) \left( x|C(P_{m-1}, x)|^k + x^{2k}|C(P_{m-3}, x)|^k \right).$$

Proof. The proof follows immediately from Theorem 2.1.14 and 2.3.13.

Next, we compute the total domination polynomial of $C_n^{2(k)}$, the $n$-gon book of $k$ pages.

Theorem 2.3.15. If $n = 2m$ for some positive integer $m$, then the total domination polynomial of $C_n^{2(k)}$ is, $D_t(C_n^{2(k)}, x) = \left(C(T_{[m-1]^k}, x) - [x^2C(P_{m-4}, x)]^k \right)^2$.

Proof. Let $u, v$ be the vertices common to the family of cycles in $C_n^{2(k)}$. Let $C_n^i = (u, a_1^i, a_2^i, \ldots, a_{n-2}^i, v, u)$. Let $A = N_{C_n^{2(k)}}(u) = \{v, a_1^i, a_2^i, a_3^i, \ldots, a_k^i\}$. Since
n is even, $C_n^{2(k)}$ is bipartite. Clearly, the components of the open neighborhood hypergraph of $C_n^{2(k)}$ are isomorphic. Let $X = \bigcup_{i=1}^{k}\{u, a_2^i, a_4^i, a_6^i, \ldots, a_{n-2}^i\}$ be one of the partite sets. Let $H_X$ be the component of the open neighborhood hypergraph corresponding to $X$. Then $E(H_X) = A \cup E(H)$, where the graph $H$ can be represented as shown in figure 2.15. Therefore, a set $S$ is a vertex covering set of $H_X$ if and only if $S \cap A \neq \phi$ and $S$ is a vertex covering set of $H$. So we need to find the polynomial $C^A(H, x)$. Note that

$$C^A(H, x) = C(H, x) - C_A(H, x) = C(T_{[m-1]^k}, x) - [x^2C(P_{m-4}, x)]^k.$$  

This completes the proof.

![Graph H](image)

**Figure 2.15:** The graph $H$

**Theorem 2.3.16.** If $n = 2m + 1$ for some positive integer $m$, then the TD-Polynomial of $C_n^{2(k)}$ is $x^2[C(P_m, x)C(P_{m-1}, x)]^k + 2x^{2k+1}[C(P_{m-1}, x)C(P_{m-2}, x)]^k + x^{4k}[C(P_{m-2}, x)C(P_{m-3}, x)]^k$. 

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Proof. Let $C_n^i$ be the cycle $(u, a_1^i, a_2^i, a_3^i, \ldots, a_{n-2}^i, v, u)$. Since the open neighborhood hypergraph of an odd cycle is isomorphic to itself, we can represent $H_{C_n^{2(k)}}$, the open neighborhood hypergraph of $C_n^{2(k)}$ as shown in figure 2.16. Let $A = N[v]$ and $B = N[u]$. Then from Theorem 2.1.9 and 2.1.11, we have

$$D_t(C_n^{2(k)}, x) = C(H_{C_n^{2(k)}}, x)$$

$$= xC(H_{C_n^{2(k)}} - v, x) + x^{2k}C(H_{C_n^{2(k)}} - A, x)$$

$$= x\left[xC(H_{C_n^{2(k)}} - v - u, x) + x^{2k}C(H_{C_n^{2(k)}} - v - B, x)\right]$$

$$+ x^{2k}\left[xC(H_{C_n^{2(k)}} - u - A, x)\right] + x^{2k}\left[x^{2k}C(H_{C_n^{2(k)}} - A - B, x)\right]$$

$$= x^2 [C(P_m, x)]^k [C(P_{m-1}, x)]^k + x^{2k+1} [C(P_{m-1}, x)]^k [C(P_{m-2}, x)]^k$$

$$+ x^{2k+1} [C(P_{m-1}, x)]^k [C(P_{m-2}, x)]^k$$

$$+ x^{4k} [C(P_{m-2}, x)]^k [C(P_{m-3}, x)]^k$$

This completes the proof. \(\square\)

![Figure 2.16: The Graph $H_{C_n^{2(k)}}$](image-url)
Next, we find the TD-Polynomial of the theta graph \( \theta(n, n, \ldots, n) \) denoted by \( \theta(n)^k \).

**Theorem 2.3.17.** Let \( n = 2m + 2 \) for some positive integer \( m \), then the total domination polynomial of \( \theta(n)^k \) is

\[
\left[ x \left[ C(P_m, x) \right]^k + x^k \left[ C(P_{m-1}, x) \right]^k - x \left[ xC(P_{m-2}, x) \right]^k - x^k \left[ xC(P_{m-3}, x) \right]^k \right]^2.
\]

**Proof.** Let \( (u, a_1, a_2, a_3, \ldots, a_{n-2}, v) \) be the path \( P_n^k \) in \( \theta(n)^k \). Clearly, the graph \( \theta(n)^k \) is bipartite and the components of the open neighborhood hypergraph of \( \theta(n)^k \) are isomorphic to each other. Let \( X = \bigcup_{i=1}^k \{a_{2i}^i, a_{4i}^i, a_{6i}^i, \ldots, a_{n-2}^i\} \cup \{u\} \) be one of the partite sets and \( H_X \) be the open neighborhood hypergraph corresponding to \( X \). Let \( A = N_{\theta(n)^k}(u) = \{a_1^1, a_2^1, a_3^1, \ldots, a_k^1\} \). Then \( E(H_X) = A \cup E(K) \), where the graph \( K \) is shown in figure 2.17

![Figure 2.17: The graph K](image)

So it suffices to find the polynomial \( C^A(K, x) \). Since \( K \) is isomorphic to \( T_{[m]^k} \), from Lemma 2.3.11 we have,

\[
C(H_X, x) = C^A(K, x)
\]
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\[ C^A(T[m]^k, x) = x[C(P_m, x)]^k + x^k [C(P_{m-1}, x)]^k - x [xC(P_{m-2}, x)]^k - x^k [xC(P_{m-3}, x)]^k. \]

Therefore, from Theorem 2.1.14, we have \( D_t(\theta(n)^k, x) = [C(H_X, x)]^2. \)

**Theorem 2.3.18.** Let \( n = 2m + 3 \) for some positive integer \( m \). If \((X,Y)\) is the bipartition of \( \theta(n)^k \), then \( D_t(\theta(n)^k) = C(H_X)C(H_Y) \), where

\[ C(H_X, x) = x^2 [C(P_m, x)]^k + 2x^{k+1} [C(P_{m-1}, x)]^k + x^{2k} [C(P_{m-2}, x)]^k \]

and

\[ C(H_Y, x) = [C(P_{m+1}, x)]^k - x^{2k} [C(P_{m-3}, x)]^k. \]

**Proof.** Let the \( i^{th} \) path \( P_i^k \) of \( \theta(n)^k \) be \((u, a_i^1, a_i^2, a_i^3, \ldots, a_i^{n-2}, v)\). Consider the bipartition \( X = \bigcup_{i=1}^k \{a_i^1, a_i^3, \ldots, a_i^{n-2}\} \) and \( Y = \bigcup_{i=1}^k \{a_i^2, a_i^4, \ldots, a_i^n\} \cup \{u, v\} \) of \( \theta(n)^k \). Let \( H_X \) and \( H_Y \) be the components of the open neighborhood hypergraph of \( \theta(n)^k \) corresponding to \( X \) and \( Y \). Then \( H_X \) can be represented as shown in figure 2.18.

![Figure 2.18: The Graph H_X](image)

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Then by Theorem 2.1.9, we have

\[
\begin{align*}
\mathcal{C}(H, x) &= x \mathcal{C}(H - u, x) + x^k \mathcal{C}(H - u - N(u), x) \\
&= x \left[ x \mathcal{C}(H - u - v, x) + x^k \mathcal{C}(H - u - v - N(v), x) \right] \\
&+ x^k \left[ x \mathcal{C}(H - u - v - N(u), x) \right] \\
&+ x^k \left[ x^k \mathcal{C}(H - u - v - N(u) - N(v), x) \right] \\
&= x \left[ x \mathcal{C}(P_m, x) \right]^k + x^k \left[ x \mathcal{C}(P_{m-1}, x) \right]^k \\
&+ x^k \left[ x \mathcal{C}(P_{m-1}, x) \right]^k + x^k \left[ x \mathcal{C}(P_{m-2}, x) \right]^k \\
&= x^2 \mathcal{C}(P_m, x)^k + x^{k+1} \mathcal{C}(P_{m-1}, x)^k \\
&+ x^{k+1} \mathcal{C}(P_{m-1}, x)^k + x^{2k} \mathcal{C}(P_{m-2}, x)^k \\
&= x^2 \mathcal{C}(P_m, x)^k + 2x^{k+1} \mathcal{C}(P_{m-1}, x)^k + x^{2k} \mathcal{C}(P_{m-2}, x)^k.
\end{align*}
\]

Next, we determine the vertex cover polynomial of \(H_Y\).

\[
\begin{align*}
\theta(n)^k
\end{align*}
\]

Figure 2.19: The Graph \(H\)

In the graph \(\theta(n)^k\), we have \(N(u) = \{a_i^1 : i = 1, 2, \ldots, k\}\) and \(N(v) = \{a_i^{n-2} : i = 1, 2, \ldots, k\}\). Then \(E(H_Y) = N(u) \cup N(v) \cup E(H)\), where \(H\) is the
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Therefore,

\[ C(H, x) = C(H, x) - C_{N(u)\cup N(v)}(H, x) \]

\[ = [C(P_{m+1}, x)]^k - x^2 [C(P_{m-3}, x)]^k. \]

Thus the proof follows.

Next, we compute the total domination polynomial of \( K_{n,n+1}^{(k)} \), the one point union of \( k \) copies of \( K_{n,n+1} \).

**Theorem 2.3.19.** The TD-Polynomial of \( K_{n,n+1}^{(k)} \) is,

\[ D_t(K_{n,n+1}^{(k)}, x) = x [D(K_n, x)]^k + [D(K_n, x)]^{2k}. \]

**Proof.** Let \( v \) be the vertex common to the \( k \) copies of \( K_{n,n+1} \) in \( K_{n,n+1}^{(k)} \). Let \( (A_i, B_i \cup \{v\}) \) be the bipartition of the \( i^{th} \) copy of \( K_{n,n+1} \), where \( A_i = \{a_1^i, a_2^i, a_3^i, \ldots, a_n^i\} \) and \( B_i = \{b_1^i, b_2^i, b_3^i, \ldots, b_n^i\} \). Then, for \( 1 \leq i \leq k \) and \( 1 \leq t \leq n \), \( N(a_i^t) = B_i \cup \{v\} \), \( N(b_t^i) = A_i \) and \( N(v) = \bigcup_{j=1}^{k} A_j \).

Let \( S \) be the set of vertices in \( K_{n,n+1}^{(k)} \). Then we have two possibilities. Either \( v \in S \) or \( v \notin S \).

**Case 1:** Let \( v \in S \). Since \( N(v) = \bigcup_{j=1}^{k} A_j \), \( S \) is a total dominating set of \( K_{n,n+1}^{(k)} \) if and only if \( S \cap A_i \neq \emptyset \) for every \( i \). Note that \( r \) vertices can be selected from \( A_i \) in \( \binom{n}{r} \) ways. Therefore, in this case TD-Polynomial of the \( i^{th} \) copy of \( K_{n,n+1} \) in \( K_{n,n+1}^{(k)} \) is, \( x \left[ \binom{n}{1} x + \binom{n}{2} x^2 + \ldots + \binom{n}{n} x^n \right] = x [(1 + x)^n - 1]. \)
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Since there are $k$ copies of $K_{n,n+1}$, the TD-Polynomial is

$$x [(1 + x)^n - 1]^k = x [D_t (K_n, x) + nx]^k$$

$$= x [D (K_n, x)]^k.$$  

**Case 2:** Let $v \notin S$. In this case $S$ is a total dominating set of $K^{(k)}_{n,n+1}$ if and only if it is a TD-set of $K^{(k)}_{n,n+1} - v$. Since $K^{(k)}_{n,n+1} - v$ is the union of $k$ copies of $K_n$, the TD-Polynomial is $\left[\left[(1 + x)^n - 1\right]\left[(1 + x)^n - 1\right]\right]^k = [D (K_n, x)]^{2k}.$

Since the above cases are disjoint, $D_t \left(K^{(k)}_{n,n+1}, x\right) = x [D (K_n, x)]^k + [D (K_n, x)]^{2k}.$

This completes the proof.  

\[\square\]