Preliminaries

The chapter aims at listing the terminology and notation that we use in the thesis. Most of the terms used in this study belong to the standard graph theoretic terminology. Some of the terms will be introduced later. The prime source of the definition, terminology and notation is Bondy, J. A. and Murty, U.S.R, Graph theory with applications, (1976).

This chapter comprises six sections. The first section focuses on the definitions and terminologies of Graph theory, which are integral in the discussion of the topics in the forthcoming chapters. The second section deals with various graph operations that come in handy in the subsequent chapters. Basic properties of hypergraphs are discussed in the third section. Some of the fundamental ideas and basic results in domination and total domination are incorporated in section five. In section six, basic definitions and properties of domination and total domination polynomials are mentioned. Moreover, some properties of vertex covering set are discussed. The chapter is summed up by the definition of vertex cover polynomial.
1.1 Basic definitions and terminologies

A (undirected) graph \( G = (V(G), E(G)) \) consists of a nonempty set \( V(G) \) and \( E(G) \), a binary symmetric relation on \( V(G) \). The sets \( V(G) \) and \( E(G) \) are called vertex set and edge set of \( G \) respectively. If the graph \( G \) is clear from the context, we simply write \( G = (V, E) \) instead of \( G = (V(G), E(G)) \) and if \( e = \{u, v\} \), where \( e \in E \) and \( u, v \in V \), we simply write \( e = uv \). An element of \( V \) is called a vertex, and an element of \( E \) is called an edge. We draw graphs in such a way that each vertex is indicated by a point, and each edge by a line joining the points representing ends of the edge. For the graph \( G = (V, E) \), the number of vertices of \( G \) is called the order of \( G \), denoted by \( n \) and the number of edges is called the size of \( G \), denoted by \( m \).

Let \( e = uv \) be an edge of \( G \). Then, we say that the two vertices \( u \) and \( v \) are adjacent to each other and the edge \( e \) is incident with (incident to or incident at) \( u \) and \( v \). The vertices \( u \) and \( v \) are said to be the end vertices of the edge \( e \). The vertex \( v \) is called a neighbor of \( u \). We write \( u \sim v \) for ‘\( u \) adjacent to \( v \)’. Two edges are said to be adjacent if they have a common vertex. An edge with identical ends is called a loop. Two or more edges with the same pair of ends are said to be parallel edges or multiple edges and graph having multiple edges is a multigraph.

A finite graph is one in which both vertex set and edge set are finite. A graph having exactly one vertex and no edges is called a trivial graph and all other graphs are called nontrivial graphs. A graph having no loops or multiple edges is called a simple graph. Unless otherwise stated the graphs considered in this thesis are simple.
For a vertex $v$ in a graph $G$, the degree of $v$, denoted by $\text{deg } v$, is the number of edges incident with $v$. A vertex of degree one is called an end vertex or a pendant vertex and a vertex adjacent to a pendant vertex is called a support vertex. A pendant edge is the edge incident with a pendant vertex. A vertex of degree zero is called an isolated vertex. In a graph $G$, $\delta(G)$ and $\Delta(G)$ denotes the minimum and maximum degrees of vertices in $G$. A $k$-regular graph is one with $\delta(G) = \Delta(G) = k$. A 3-regular graph is also known as cubic graph.

Let $H = (V(H), E(H))$ and $K = (V(K), E(K))$ be two graphs. We say that $H$ and $K$ are isomorphic, denoted by $H \cong K$, if there exist bijections $f: V(H) \to V(K)$ and $g: E(H) \to E(K)$ such that $g(e) = f(u)f(v)$ for all $u,v$ in $V(H)$. In other words, $e = uv$ is an edge of $H$ if and only if $g(e) = f(u)f(v)$ is an edge of $K$.

A complete graph is a simple graph in which every pair of distinct vertices are adjacent. A complete graph having $n$ vertices is denoted by $K_n$. A graph is bipartite if its vertex set can be partitioned into two subsets, $X$ and $Y$ so that every edge has one end in $X$ and other end in $Y$; such a partition $(X,Y)$ is called a bipartition of the bipartite graph. A complete bipartite graph is a bipartite graph such that each vertex of $X$ is adjacent to all vertices of $Y$ and vice versa. $K_{m,n}$ denotes a complete bipartite graph with $|X| = m$ and $|Y| = n$. The complement of a simple graph $G$, denoted by $\overline{G}$, is graph with $V(\overline{G}) = V(G)$ and two vertices $u$ and $v$ are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$.

A walk in a graph $G$ is an alternating sequence $W: v_0e_1v_1e_2v_2\ldots e_nv_n$ of
1.1. Basic definitions and terminologies

vertices and edges beginning and ending with vertices in which \( v_{i-1} \) and \( v_i \) are the ends of \( e_i \); \( v_0 \) is the origin and \( v_n \) is the terminus of \( W \). We call \( W \) a \( v_0 - v_n \) walk. The \textit{length} \[1\] of a walk is the number of edges in it. If all the edges in a walk are distinct, it is called a \textit{trail} \[1\]. A \textit{path} \[1\] is a walk in which all vertices are distinct. Usually, we leave out the edges while writing a path. A \textit{cycle} \[1\] is a closed trail in which all the vertices are distinct. A cycle of length \( n \) is denoted by \( C_n \) and a path with \( n \) vertices is denoted by \( P_n \). Note that \( P_n \) has length \((n - 1)\) \[1\].

Let \( u \) and \( v \) are two vertices in a graph \( G \). If there is a \( u - v \) path in \( G \), we say that \( u \) and \( v \) are \textit{connected}. A graph \( G \) is said to be connected \[22\] if every pair of vertices of \( G \) are connected. A disconnected graph \[22\] is one which is not connected. The \textit{distance} \[22\] between \( u \) and \( v \), denoted by \( d(u, v) \), is the length of the shortest \( u-v \) path in \( G \).

A graph \( H \) is called a \textit{subgraph} \[11\] of a graph \( G \) if \( V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \). If \( H \) is a subgraph of \( G \), then \( G \) is said to be a \textit{supergraph} \[11\] of \( H \). An \textit{induced subgraph} \[13\] \( H \) of a graph \( G \) is a subgraph of \( G \) such that two vertices of \( H \) are adjacent if and only if they are adjacent in \( G \). In this case, if \( V(H) = S \), we write \( H = G[S] \) or \( H = \langle S \rangle \). A subgraph \( H \) of \( G \) is a \textit{spanning subgraph} \[19\] of \( G \), if \( V(H) = V(G) \).

A graph is \textit{acyclic} \[22\] if it has no cycles. A \textit{tree} \[22\] is a connected acyclic graph. A \textit{spanning tree} \[11\] of \( G \) is a spanning subgraph of \( G \) that is a tree.

If \( F \) is any set of edges in \( G \), then \( G - F \) is the graph \((V(G), E(G) - F)\) \[51\]. If \( F = \{e\} \), then \( G - F \) is written as \( G - e \) \[51\]. For any subset \( S \) of \( V(G) \), the graph \( G - S \) is obtained from \( G \) by deleting all the vertices in \( S \) \[51\]. If
1.2 Operations on graphs

The union \( G_1 \cup G_2 \) of two graphs \( G_1 \) and \( G_2 \) denoted by \( G_1 \cup G_2 \) is the graph with vertex set \( V(G_1) \cup V(G_2) \) and edge set \( E(G_1) \cup E(G_2) \). The join \( G_1 \join G_2 \) of two graphs \( G_1 \) and \( G_2 \) denoted by \( G_1 \join G_2 \) is the graph with vertex set \( V(G_1) \cup V(G_2) \) and edge set \( E(G_1) \cup E(G_2) \cup \{uv: u \in V(G_1) \text{ and } v \in V(G_2)\} \). The corona \( G_1 \circ G_2 \) of two graphs \( G_1 \) and \( G_2 \) denoted by \( G_1 \circ G_2 \) is the graph formed from one copy of \( G_1 \) and \( |V(G_1)| \) copies of \( G_2 \), such that the \( i \)th vertex of the copy of \( G_1 \) is adjacent to every vertex in the \( i \)th copy of \( G_2 \). The Cartesian product \( G_1 \square G_2 \) of two graphs \( G_1 \) and \( G_2 \) is the graph with vertex set \( V(G_1) \times V(G_2) \) in which two vertices \( (u_1,v_1) \) and \( (u_2,v_2) \) are adjacent if either \( u_1u_2 \in E(G_1) \) and \( v_1 = v_2 \) or \( v_1v_2 \in E(G_2) \) and \( u_1 = u_2 \).

1.3 Hypergraphs

A hypergraph \( H \) is a pair \( (V,E) \), where \( V \) is finite non-empty set called the set of vertices and \( E \) is a collection of nonempty subsets of \( V \) called hyperedges or edges. That is, \( E \) is a subset of \( \mathcal{P}(V) \setminus \{\phi\} \), where \( \mathcal{P}(V) \) is the power set of \( V \). In drawing hypergraphs, each vertex is drawn as a point in the plane. An edge \( E_1 \) with \( |E_1| > 2 \), is drawn as a curve encircling all the vertices of \( E_1 \). An edge \( E_1 \) with \( |E_1| = 2 \), is drawn as a curve connecting its two vertices and
an edge $E_1$ with $|E_1| = 1$, is drawn as a loop as in a graph \[9\].

Two vertices in a hypergraph are adjacent \[9\] if there is a hyperedge which contains both vertices. Two vertices $x$ and $y$ of a hypergraph are connected \[27\] if there is a sequence of vertices $x = v_0, v_1, \ldots, v_k = y$ such that $v_{i-1}$ is adjacent to $v_i$ for $i = 1, 2, \ldots, k$. A connected hypergraph \[27\] is a hypergraph in which every pair of vertices are connected. A maximal connected sub hypergraph of a hypergraph is called a component. Two hyperedges in a hypergraph are incident \[9\] if their intersection is nonempty. If $|E_i| = 2$ for all $i$, and if the hypergraph $H$ is simple, then $H$ is a simple graph without isolated vertices. A $k$-uniform hypergraph \[9\] or a $k$-hypergraph is a hypergraph in which every edge consists of $k$ vertices. Since every simple graph is a 2-uniform hypergraph, graphs are special hypergraphs \[49\].

1.4 Open neighbourhood hypergraph

For a graph $G$ be with vertex set $V$ and edge set $E$, the open neighborhood \[27\] of a vertex $u$ of $G$ is $N_G(u) = \{v \in V : uv \in E\}$ and its closed neighborhood is the set $N_G[u] = N_G(u) \cup \{u\}$. The open neighborhood of a set \[27\] $S \subseteq V$ is the set $N_G(S) = \cup_{u \in S} N_G(u)$ and its closed neighborhood is the set $N_G[S] = N_G(S) \cup S$. If there is no ambiguity, we simply write $N(u), N[u], N(S)$ and $N[S]$ instead of $N_G(u), N_G[u], N_G(S)$ and $N_G[S]$, respectively.

The open neighborhood hypergraph \[27\] of a graph $G$, denoted by $ONH(G)$ or $H_G$, is the hypergraph with vertex set $V(G)$ and edge set \{ $N_G(u): u \in V(G)$\} consisting of the open neighborhoods of vertices of $V$ in $G$. 
1.5 Domination in graphs

This section discusses some of the fundamental ideas and basic results in domination in graphs.

One of the fastest growing areas in Graph theory is the study of dominating sets and related properties. The study of domination in graphs began in 1960, when the problem of queen domination in an $n \times n$ chess board was studied [23].

For a graph $G$, a set $S \subseteq V(G)$ is called a dominating set [23] of $G$ if every vertex $u \in V(G)$ is either an element of $S$ or is adjacent to an element of $S$. Alternatively, we say that $S \subseteq V(G)$ is a dominating set of $G$ if every element in $V \setminus S$ is adjacent to some element in $S$. Equivalently, $N[S] = V$. If $S$ is a dominating set of a graph, then every superset of $S$ is also a dominating set. On the other hand, not every subset of $S$ is necessarily a dominating set. A dominating set $S$ of $G$ is a minimal dominating set [23] if no proper subset of $S$ is a dominating set. The domination number [23] of a graph $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$ and a $\gamma$-set is a dominating set with cardinality $\gamma(G)$.

After the introduction of the concept of domination, several types of dominating sets were introduced and studied in detail. An excellent treatment of this topic is available in [23] and [24].

The concept of total domination [16] was introduced by Cockayne, Dawes, and Hedetniemi. For a graph $G = (V, E)$, with no isolated vertices, a set $S \subseteq V$ is called a total dominating set [27] or TD-set if every vertex of $G$ is adjacent to a vertex in $S$. A total dominating set $S$ is said to be minimal [27], if no proper
subset of $S$ is a total dominating set. The total domination number $\gamma_t(G)$ of a graph $G$, denoted by $\gamma_t(G)$, is defined as the minimum cardinality of a total dominating set of $G$. If $S$ is a total dominating set of $G$ with $|S| = \gamma_t(G)$, then $S$ is called a $\gamma_t$-set of $G$. Obviously, a TD-set is a dominating set and a dominating set $S$ is a TD-set if the induced subgraph $\langle S \rangle$, has no isolated vertices.

In the next section, we discuss some graph polynomials which are essential for the study.

1.6 Graph polynomials

The concept of domination polynomial of a graph was introduced by S. Alikhani in 2009. For a graph $G$, let $D(G,i)$ be the family of dominating sets with cardinality $i$ and let $d(G,i) = |D(G,i)|$. If $\gamma(G)$ is the domination number of $G$, then the domination polynomial $D(G,x)$ of $G$ is defined as $D(G,x) = \sum_{i=\gamma(G)} d(G,i)x^i$.

**Theorem 1.6.1.** (see [1]) For every natural number $n$,

(i) $D(K_n, x) = (1 + x)^n - 1$.

(ii) $D(K_{1,n}, x) = x^n + x(1 + x)^n$.

**Theorem 1.6.2.** (see [2]) If a graph $G$ consists of $m$ components $G_1, G_2, \ldots, G_m$, then $D(G, x) = D(G_1, x)D(G_2, x) \ldots D(G_m, x)$.

In analogue to the domination polynomial, S. Sanalkumar introduced the concept of total domination polynomial of a graph. For a graph $G$, let $D_t(G, i)$ be
the family of total dominating sets with cardinality $i$ and let $d_t(G,i) = |D_t(G,i)|$ [3]. If $\gamma_t(G)$ is the total domination number of $G$, then the total domination polynomial or TD-Polynomial of $G$, denoted by $D_t(G,x)$ is defined as

$$D_t(G,x) = \sum_{i=\gamma_t(G)} d_t(G,i)x^i.$$ 

A vertex cover or transversal of a graph $G$ is a set $S$ of vertices of $G$ such that each edge in $G$ has at least one end in $S$. A vertex covering set with $k$ vertices is called a $k$-vertex cover. A minimum vertex cover is a vertex cover having the smallest possible number of vertices for a given graph [41]. The number of vertices of a minimum vertex cover of a graph $G$ is known as the vertex cover number and is denoted as $\tau(G)$ [50].

Let $C(G,i)$ be the family of vertex covering sets of a graph $G$ with cardinality $i$ and let $c(G,i) = |C(G,i)|$ [18]. The polynomial $C(G,x) = \sum_{i=\tau(G)} c(G,i)x^i$ is defined as vertex cover polynomial of $G$ [18].