Chapter-VII

ASPECTS OF INARIANT SUBMANIFOLDS OF A $f_2$-HSU MANIFOLD WITH COMPLEMENTED FRAMES
1. INTRODUCTION:

In an m-dimensional differentiable manifold M of class $C^\infty$, if a non-null tensor field of type (1,1) satisfies $f^3 - \lambda^r f = 0$, where $\lambda$ is a non-zero complex number and is constant rank p at each point of M then f is called ‘$f_\lambda$-Hsu structure of rank p’ and M with $f_\lambda$-Hsu structure an $f_\lambda$-Hsu manifold. If we put [3]

$$\ell = f^2 / \lambda^r \quad \text{and} \quad m = I - f^2 / \lambda^r$$

where I denotes the unit tensor field, then it is easy to see that

$$\ell^2 = \ell, \ m^2 = m, \ \ell + m = I, \ \ell m = m \ell = 0$$

This implies that the tensor fields $f^2 / \lambda^r$ and $I - f^2 / \lambda^r$ are complementary projection operators. Let L, M be distribution corresponding to the projection operators $f^2 / \lambda^r$ and $I - f^2 / \lambda^r$ respectively. The distributions corresponding to $f^2 / \lambda^r$ and $I - f^2 / \lambda^r$, are p and (m-p) dimensional.

Let there exist (m-p) vector fields $U_\alpha (\alpha = 1, 2, \ldots, m - p)$ spanning the distribution corresponding to $I - f^2 / \lambda^r$ and (m-p) I-form $u^\alpha$ satisfying

$$f^2 / \lambda^r = I - \sum_{\alpha=1}^{m-p} u^\alpha \otimes U_\alpha$$

and

$$f U_\alpha = 0, \ u^\alpha \circ f = 0, \ u^\alpha (U_\beta) = \delta^\alpha_\beta, \ \alpha, \beta = 1, 2, 3, \ldots$$

(m-p) where $\delta^\alpha_\beta$ is the Kronecker delta. Then we call the set

$$\{f_\lambda, U_\alpha, u^\alpha\}$$

an ‘$f_\lambda$-Hsu structure with complemented frames’
and the manifold \( M \) an \( \mathcal{f}_\lambda \)-manifold with complemented frames.

**Invariant Submanifold:**

Suppose that an \( n \)-dimensional differentiable manifold \( M \) is immersed in a manifold \( M \) by the immersion \( I: \tilde{M} \rightarrow M \). If the tangent space of \( (\tilde{M}) \) is invariant by the action of \( f \), then \( I(\tilde{M}) \) is called an invariant submanifold of \( M \).

In the present chapter, we consider a \( \mathcal{f}_\lambda \)-structure with complement frames such that \( r = m - 3 \).

2. **\( \mathcal{f}_\lambda \)-Hsu Structure with Complemented Frames:**

Let \( M \) be an \( m \)-dimensional differentiable manifold of class \( C^\infty \) and let there be given a tensor field \( \mathcal{F} \) of type \((1,1)\) and of rank \((m-2)\), two vector fields \( u, v \) and two 1-forms \( u, v \). If the set \( \{ \mathcal{f}_\lambda, U, V, W, u, v, w \} \) satisfies

\[
\begin{align*}
(2.1) \quad & f^2 / \lambda = I - u \otimes U - v \otimes V - w \otimes W \\
(2.2) \quad & (a) \quad fU = 0, fV = 0, u \circ f = 0, v \circ f = 0, w \circ f = 0 \\
& (b) \quad v(U) = 0, u(V) = 0, u(W) = 0, w(U) = 0, w(V) = 0,
\end{align*}
\]

where \( \lambda \) is any complex number not equal to zero, then we call \( \{ \mathcal{f}_\lambda, U, V, W, u, v, w \} \) an \( \mathcal{f}_\lambda \)-Hsu structure with complemented frames’ and \( M \) an \( \mathcal{f}_\lambda \)-manifold with complemented frame on an \( \mathcal{f}_\lambda \)-Hsu manifold with complemented frame \( M \).

\[
(2.3) \quad u(U) = 1, v(V) = 1, w(W) = 1.
\]

Let us define a tensor field \( S^* \) of type \((1,2,3)\) as
\[(2.4) \ S^*(X,Y) = N(X,Y) - \lambda' (du)(X,Y)U - \]
\[- \lambda' (dv)(X,Y)V - \lambda' (dw)(X,Y)W, \]
where \(du, dv, dw\) are 3-forms and \(N\) is the Nijenhuis tensor formed with, \(f\), defined by [3]

\[(2.5) \ N(X,Y) = [fX, fY] - f[fX, Y] - f[X, fY] + f^2[X, Y]. \]

**DEFINITION (2.1):**

If the tensor field \(S^*\) vanishes identically then the structure is said to be normal.

In view of (2.2), (2.4) and (2.5) we have

\[(2.6) \ S^*(X,U) = -f[X,U] + f^2[X,U] - \lambda' du(X,U)U - \lambda' dv(X,U)V - \lambda' dw(X,U)W \]

Let \(\mathcal{L}_U\) be called the Lie derivative with respect to a field \(U\). Then we have [3],

\[-f[fX, U] + f^2[X, U] = f(f[X, U]) - [fX, U] = f(\mathcal{L}_U f)X \]

and

\[du(X,U) = X(\mathcal{L}_U u) - U(\mathcal{L}_U u) - u([X, U]) \]
\[= -\{u[u[X, U] - [u(X), U]]\} = -(\mathcal{L}_U u)(X). \]

Similarly,

\[dv(X, V) = -(\mathcal{L}_v v)(X) \text{ and } dw(X, W) = -(\mathcal{L}_w w)(X) \]

Therefore from (2.6) we obtain

\[(2.7) \ S^*(X,U) = f(\mathcal{L}_U f)X + \lambda'(\mathcal{L}_U u)(X)U + \lambda'(\mathcal{L}_U v)(X)V + \lambda'(\mathcal{L}_U w)(X)W. \]
We can also prove that

\begin{equation}
(2.8) \quad S^r(X,V) = f(\mathcal{L}_v f)(X) + \lambda^r(\mathcal{L}_v u)(X)U + \\
+ \lambda^r(\mathcal{L}_v v)(X)V + \lambda^r(\mathcal{L}_v w)(X)W
\end{equation}

and

\begin{equation}
(2.9) \quad S^r(X,W) = f(\mathcal{L}_v f)(X) + \lambda^r(\mathcal{L}_v u)(X)U + \\
+ \lambda^r(\mathcal{L}_v v)(X)V + \lambda^r(\mathcal{L}_v w)(X)W.
\end{equation}

Also as a consequence of (2.2), (2.4), and (2.5), we have

\begin{equation}
(2.10) \quad u(S^r(X,Y)) = u([fX,Y]) - \lambda^r(du)(X,Y).
\end{equation}

But we have

\[ du(fX,fY) = (fX)u(fY) - (fY)u(fX) - u(fX,fY), \]

which in view of (2.2) implies that

\[ u([fX,fY]) = -(du)(fX,fY). \]

Thus from (2.10) we obtain

\begin{equation}
(2.11) \quad u(S^r(X,Y)) = -\lambda^r(du)(X,Y) - \lambda^r(du)(fX,fY).
\end{equation}

Replacing \( X \) by \( fX \) in (2.11) and using (2.1) we get

\begin{equation}
(2.12) \quad u(S^r(fX,Y)) = -\lambda^r(du)(fX,Y) - du(\lambda^r X - \\
-\lambda^r u(X)U - \lambda^r v(X)V, fY - \lambda^r W, fW) \\
= \lambda^r \{ (du)(fX,Y) + du(X,fY) - \\
- u(X)(du)(U,fY) - v(X)(dv)(V,fY) \\
- w(X)(dw)(W,fY) \}. \]

But we have
\[(du)(U, fU) = Uu(fY) - (fY)u(U) - u([U, fY])\]
\[= u\{u([fY, U]) - [u(fY), U]\}\]
\[= (\mathcal{L}_U u)(fY).\]

Similarly,
\[(dv)(V, fY) = (\mathcal{L}_v u)(fY), (dv)(W, fY)\]
\[= (\mathcal{L}_w u)(fY).\]

Hence, from (2.12) we have
\[(2.13) \quad u(S^*(fX, Y)) = -\lambda^r \{(du)(fX, Y) + (dv)(X, fY) -
\[ - u(X)(\mathcal{L}_U v)(fY) - v(X)(\mathcal{L}_U u)(fY) -
\[ - w(X)(\mathcal{L}_w u)(fY)\}\].\]

We can also see that
\[(2.14) \quad v(S^*(fX, Y)) = -\lambda^r \{(dv)(fX, Y) + (dv)(X, fY) -
\[ - v(X)(\mathcal{L}_U v)(fY) - v(X)(\mathcal{L}_v v)(fY)\}\]
\[(2.15) \quad w(S^*(fX, Y)) = -\lambda^r \{(dw)(fX, Y) + (dw)(X, fY) -
\[ - w(X)(\mathcal{L}_U v)(fY) - w(X)(\mathcal{L}_v v)(fY)\}\].\]

**THEOREM (2.1):**

If an \(f_\lambda\)-Hsu structure with complemented frames 
\[\{f_\lambda, U, V, W, \ u, v, w\}\] is normal, then
\[(2.16) \quad \mathcal{L}_U f = 0, \mathcal{L}_U u = 0, \mathcal{L}_U v = 0, \mathcal{L}_U w = 0,\]
\[(2.17) \quad \mathcal{L}_v f = 0, \mathcal{L}_v u = 0, \mathcal{L}_v v = 0, \mathcal{L}_v w = 0,\]
\[(2.18) \quad \mathcal{L}_w f = 0, \mathcal{L}_w u = 0, \mathcal{L}_w v = 0, \mathcal{L}_w w = 0,\]
\[(2.19) \quad du \pi f = 0, dv \pi f = 0, dw \pi f = 0, \ [U, V, W] = 0.\]
**PROOF:**

Let us suppose that $f_\lambda$-Hsu structure with complemented frames $\{f_\lambda, U, V, W, u, v, w\}$ is normal. Then from (2.7), we have

$$f(\mathcal{L}_U f)X + \lambda^r(\mathcal{L}_U u)(X)U + \lambda^r(\mathcal{L}_U v)(X)V +$$

$$+ \lambda^r(\mathcal{L}_U w)(X)W = 0,$$

which in view of (2.2) and (2.3) implies that

$$\mathcal{L}_U u = 0, \mathcal{L}_U v = 0, \mathcal{L}_U w = 0, f(\mathcal{L}_U f) = 0.$$

Applying $f$ to the last equation of (2.20) and using (2.1), we obtain

$$\lambda^r[(\mathcal{L}_U f) - u \circ (\mathcal{L}_U f) \otimes U - v \circ (\mathcal{L}_U f) \otimes V -$$

$$- w \circ (\mathcal{L}_U f) \otimes W] = 0$$

or

$$\lambda^r[(\mathcal{L}_U f) + \{(\mathcal{L}_U u) \circ f\} \otimes U + \{(\mathcal{L}_U v) \circ f\} \otimes V +$$

$$+ (\mathcal{L}_U w) \circ f \otimes W] = 0$$

Hence, in view of (2.20), we have

$$\mathcal{L}_U f = 0, \text{ since } \lambda \neq 0.$$

Similarly, from (2.8), we can prove that

$$\mathcal{L}_V u = 0, \mathcal{L}_v v = 0, \mathcal{L}_v w = 0, \mathcal{L}_v f = 0,$$

Let us put

$$\mathcal{L}_U (fV) = 0, \text{ we find }$$

$$\mathcal{L}_V V = 0.$$

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Applying $f$ to (2.24) and using (2.1), we get
\[
\lambda^r \left( \mathcal{L}_u V - u(\mathcal{L}_u V)U - v(\mathcal{L}_u V)V - w(\mathcal{L}_u V)W \right) = 0
\]
and \( \mathcal{L}_u V = 0 \),
which implies \( [U, V, W] = 0 \).

3. HSU-STRUCTURE F:

Let us define a tensor field $F$ of type (1,1) as follows
\[
(3.1) \ F X = fX + \lambda^{r/2} v(X)U + \lambda^{r/2} u(X)V + \lambda^{r/2} w(X)W + \\
+ \lambda^{r/2} w(X)U + \lambda^{r/2} v(X)V + \lambda^{r/2} v(X)W
\]
for an arbitrary vector field $X$.

**THEOREM (3.1):**

In order that a manifold $M$ may admit an $f_\lambda$-Hsu structure with complemented frames \( \{f_\lambda, U, V, W, u, v, w\} \), it is necessary and sufficient that the manifold admits a Hsu-structure $F$, a vector field $U$ and an 1-form $u$ such that
\[
u(U) = 1, u(FU) = 0, v(V) = 1, v(FV) = 0, w(W) = 1, \]
\[
w(FW) = 0
\]

**PROOF:**

In view of (2.1), (2.2), (2.3) and (3.1) we have
\[
F^2 X = F(FX) = f(FX + \lambda^{r/2} v(X)U + \lambda^{r/2} u(X)V + \\
+ \lambda^{r/2} w(X)U + \lambda^{r/2} w(X)V + \lambda^{r/2} u(X)W + \\
+ \lambda^{r/2} v(X)W + \lambda^{r/2} (fX + \lambda^{r/2} v(X)U + \\
+ \lambda^{r/2} u(X)V + \lambda^{r/2} w(X)U + \lambda^{r/2} w(X)W + \\
+ \lambda^{r/2} u(X)W + \lambda^{r/2} v(X)W)U + \lambda^{r/2} u(fX +
\]

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\[ + \lambda^{r/2} v(X)U + \lambda^{r/2} u(X)V + \lambda^{r/2} w(X)U + \\
+ \lambda^{r/2} w(X)V + \lambda^{r/2} u(X)W + \lambda^{r/2} v(X)W) V + \\
+ \lambda^{r/2} w(fX + \lambda^{r/2} v(X)U + \\
+ \lambda^{r/2} v(X)W) U + \lambda^{r/2} w(fX + \lambda^{r/2} v(X)U + \\
+ \lambda^{r/2} u(X)V + \lambda^{r/2} w(X)U + \lambda^{r/2} w(X)V + \\
+ \lambda^{r/2} u(X)W + \lambda^{r/2} v(X)W) V + \lambda^{r/2} u(fX + \\
+ \lambda^{r/2} v(X)U + \lambda^{r/2} u(X)V + \lambda^{r/2} w(X)U + \\
+ \lambda^{r/2} w(X)V + \lambda^{r/2} u(X)W + \lambda^{r/2} v(X)W) W + \\
+ \lambda^{r/2} v(fX + \lambda^{r/2} v(X)U + \lambda^{r/2} u(X)V + \\
+ \lambda^{r/2} w(X)U + \lambda^{r/2} w(X)V + \lambda^{r/2} u(X)W + \\
+ \lambda^{r/2} v(X)W \\
= f^2 X + \lambda^r u(X)U + \lambda^r v(X)V \\
= \lambda^r X.
\]

Therefore
\[ F^2 = \lambda^r I. \]

Thus, \( F \) defines a Hsu-structure.

Also by virtue of (2.2), (2.3) and (3.1) we can easily verify that
\[ (3.2) \quad FU = \lambda^{r/2} V, FV = \lambda^{r/2} U, FW = \lambda^{r/2} U, FW = \lambda^{r/2} V, \]
\[ F^2 = \lambda^{r/2} W, FW = \lambda^{r/2} W \]

\[ (3.3) \quad u \circ F = \lambda^{r/2} v, v \circ F = \lambda^{r/2} u, w \circ F = \lambda^{r/2} u, \]
\[ u \circ F = \lambda^{r/2} w, v \circ F = \lambda^{r/2} w, w \circ F = \lambda^{r/2} v. \]
Conversely, suppose that a manifold $M$ admits a Hsu-structure $F$, a vector field $U$ and a 1-form $u$ such that

$$(3.4)\quad u(\bar{U}) = 1, \quad u(FU) = 0.$$ 

Let us define a vector field $V$, a 1-form $v$ and a tensor field, $f$, as

$$(3.5)\quad \lambda'^{r/2}V = FU, \quad \lambda'^{r/2}V = FW$$

$$(3.6)\quad \lambda'^{r/2}v = u \circ F, \quad \lambda'^{r/2}v = w \circ F$$

$$(3.7)\quad f = F - \lambda'^{r/2}v \otimes U - \lambda'^{r/2}u \otimes V - \lambda'^{r/2}w \otimes U - \lambda'^{r/2}w \otimes V - \lambda'^{r/2}v \otimes W - \lambda'^{r/2}u \otimes W.$$ 

Now as a consequence of (3.4), (3.5) and (3.6), we have

$$(3.8)\quad u(V) = 0, \quad v(U) = 0, \quad u(W) = 0, \quad v(W) = 0, \quad w(U) = 0, \quad w(V) = 0, \quad v(V) = 1, \quad w(W) = 1.$$ 

Also in view of (3.4), (3.6), (3.7) and (3.8), we have

$$(3.9)\quad fU = 0, \quad fV = 0, \quad fW = 0,$$

$$u \circ f = 0, \quad v \circ f = 0, \quad w \circ f = 0.$$ 

Further, by virtue of (3.6) (3.7) and (3.8), we have

$$f^2X = f(fX)$$

$$= f(FX - \lambda'^{r/2}v(X)U - \lambda'^{r/2}u(X)V - \lambda'^{r/2}w(X)U - \lambda'^{r/2}w(X)V - \lambda'^{r/2}v(X)W - \lambda'^{r/2}u(X)W)$$

$$= F(FX - \lambda'^{r/2}v(X)U - \lambda'^{r/2}u(X)V - \lambda'^{r/2}w(X)U - \lambda'^{r/2}w(X)V - \lambda'^{r/2}v(X)W - \lambda'^{r/2}u(X)W)$$

$$= F^2X - \lambda'^{r/2}(v \circ F)(X)U - \lambda'^{r/2}(u \circ F)(X)V - \lambda'^{r/2}(w \circ F)(X)U - \lambda'^{r/2}(w \circ F)(X)V - \lambda'^{r/2}(v \circ F)(X)W - \lambda'^{r/2}(u \circ F)(X)W$$

$$= F^2X - \lambda'^{r}u(X)U - \lambda'^{r}v(X)V - \lambda'^{r}w(X)W$$

$$= \lambda'^{r}X - \lambda'^{r}u(X)U - \lambda'^{r}v(X)V - \lambda'^{r}w(X)W.$$
\[
\frac{f^2}{\lambda^r} = I - u \otimes U - v \otimes V - w \otimes W
\]

Equation (3.8), (3.9) and (3.10) show that M admits an \(f_{\lambda}\)-Hsu structure with complemented frames \(\{f_{\lambda}, U, V, W, u, v, w\}\).

**4. INTEGRABILITY CONDITIONS:**

In this section, we shall obtain the relation between the integrability of Hsu-structure F and that of the manifold having \(f_{\lambda}\)-Hsu structure with complemented frames.

Let \(N^*(X,Y)\) be the Nijenhuis tensor formed with the help of F. Then we have,

\[
(4.1) \quad N^*(X,Y) = [FX, FY] - F[FX, Y] - F[X, FY] + F^2 [X, Y].
\]

Now from (2.1), (3.1) and (4.1) we obtained

\[
N^*(X,Y) = [fx + \lambda^{r/2} v(X) U + \lambda^{r/2} u(X) V + \lambda^{r/2} w(X) W, fY + +\lambda^{r/2} v(Y) U + \lambda^{r/2} u(Y) V + \lambda^{r/2} w(Y) W, fY + +\lambda^{r/2} v(Y) V + \lambda^{r/2} v(Y) W + \lambda^{r/2} u(Y) V] - \]

\[-F[fX + \lambda^{r/2} v(X) U + \lambda^{r/2} u(X) V + +\lambda^{r/2} w(X) U + \lambda^{r/2} w(X) V + \lambda^{r/2} v(X) W, fY + +\lambda^{r/2} u(X) V + \lambda^{r/2} w(X) U + \lambda^{r/2} w(X) V + +\lambda^{r/2} u(X) W, fY + +\lambda^{r/2} v(X) W + \lambda^{r/2} u(X) W] + F^2 [X, Y].
\]

\[= fX + \lambda^{r/2} v(X) U + \lambda^{r/2} u(X) V + \]
\[ \begin{align*}
&+ \lambda^{r/2} w(X) U + \lambda^{r/2} w(X) V + \lambda^{r/2} v(X) W + \\
&+ \lambda^{r/2} u(X) W, fY + \lambda^{r/2} v(Y) U + \lambda^{r/2} u(Y) V + \\
&+ \lambda^{r/2} w(Y) U + \lambda^{r/2} w(Y) V + \lambda^{r/2} v(Y) W + \\
&+ \lambda^{r/2} u(Y) W ] - F[fX + \lambda^{r/2} v(X) U + \\
&+ \lambda^{r/2} u(X) V + \lambda^{r/2} w(X) U + \lambda^{r/2} w(X) V + \\
&+ \lambda^{r/2} v(X) W + \lambda^{r/2} u(X) W, Y] - F[X, fY + \\
&+ \lambda^{r/2} v(X) U + \lambda^{r/2} u(X) V + \lambda^{r/2} w(X) U + \\
&+ \lambda^{r/2} w(X) V + \lambda^{r/2} v(X) W + \lambda^{r/2} u(X) W ] + \\
&+ F^2 [X, Y] - \lambda^{r/2} v [f(X) + \lambda^{r/2} v(X) U + \\
&+ \lambda^{r/2} u(X) V + \lambda^{r/2} w(X) U + \lambda^{r/2} w(X) V + \\
&+ \lambda^{r/2} v(X) W + \lambda^{r/2} u(X) W, Y] V - \\
&- \lambda^{r/2} w[f(X) + \lambda^{r/2} v(X) U + \\
&+ \lambda^{r/2} u(X) V + \lambda^{r/2} w(X) U + \\
&+ \lambda^{r/2} w(X) V + \lambda^{r/2} v(X) W + \\
&+ \lambda^{r/2} u(X) W, Y] U - \lambda^{r/2} w[f(X) + \\
&+ \lambda^{r/2} v(X) U + \lambda^{r/2} u(X) V + \lambda^{r/2} w(X) U + \\
&+ \lambda^{r/2} w(X) V + \lambda^{r/2} v(X) W + \\
&+ \lambda^{r/2} u(X) W, Y] V - + \lambda^{r/2} v[f(X) + \\
&+ \lambda^{r/2} v(X) U + \lambda^{r/2} u(X) V + \lambda^{r/2} w(X) U + \\
&+ \lambda^{r/2} w(X) V + \lambda^{r/2} v(X) W + \\
&+ \lambda^{r/2} u(X) W, Y] W - \lambda^{r/2} u[f(X) + 
\end{align*} \]
+ \lambda r^2 v(X)U + \lambda r^2 u(X)V + \lambda r^2 w(X)U + \\
+ \lambda r^2 w(X)V + \lambda r^2 v(X)W + \\
+ \lambda r^2 u(X)W + \lambda r^2 v(X)W - f[X, fY + \lambda r^2 v(X)U + \\
+ \lambda r^2 u(X)V + \lambda r^2 w(X)U + \lambda r^2 w(X)V + \\
+ \lambda r^2 v(X)W + \lambda r^2 u(X)W] - \lambda r^2 v[X, fY + \\
+ \lambda r^2 v(X)U + \lambda r^2 u(X)V + \lambda r^2 w(X)U + \\
+ \lambda r^2 w(X)V + \lambda r^2 v(X)W + \lambda r^2 u(X)W] - \\
- \lambda r^2 v[X, fY + \lambda r^2 v(X)U + \lambda r^2 u(X)V + \\
+ \lambda r^2 w(X)U + \lambda r^2 w(X)V + \lambda r^2 v(X)W + \\
+ \lambda r^2 u(X)V + \lambda r^2 w(X)U + \lambda r^2 v(X)W + \\
+ \lambda r^2 w(X)V + \lambda r^2 v(X)W + \lambda r^2 u(X)W] - \\
- \lambda r^2 v[X, fY + \lambda r^2 v(X)U + \lambda r^2 u(X)V + \\
+ \lambda r^2 w(X)U + \lambda r^2 w(X)V + \lambda r^2 v(X)W + \\
+ \lambda r^2 u(X)V + \lambda r^2 w(X)U + \lambda r^2 w(X)V + \\
+ \lambda r^2 v(X)W + \lambda r^2 u(X)W] - \\
- \lambda r^2 u[X, fY + \lambda r^2 v(X)U + \lambda r^2 u(X)V + \\
+ \lambda r^2 w(X)U + \lambda r^2 w(X)V + \lambda r^2 v(X)W + \\
+ \lambda r^2 w(X)U + \lambda r^2 w(X)V + \lambda r^2 v(X)W +
\[ + \lambda^{r/2} u(X)W][W + f^2[X,Y] + \lambda' u(X,Y)U + \]
\[ + \lambda' v[X,Y)V + \lambda' w[X,Y]W. \]

The above equation as a consequence of (2.3), (2.5) and (2.23) reduces to

\[ (4.2) \quad \mathcal{N}^e(X,Y) = N(X,Y) - \lambda' (du)(X,Y)U - \]
\[ - \lambda' (dv)(X,Y)V - \lambda' (dw)(X,Y)W + \]
\[ + \lambda^{r/2} (dv \pi f)(X,Y)U - \lambda^{r/2} (du \pi f)(X,Y)V + \]
\[ + \lambda^{r/2} (dw \pi f)(X,Y)U - \lambda^{r/2} (dv \pi f)(X,Y)V + \]
\[ + \lambda^{r/2} (dv \pi f)(X,Y)W - \lambda^{r/2} (du \pi f)(X,Y)W - \]
\[ + \lambda^{r/2} \{v(X)(\mathcal{L} u f)Y - v(Y)(\mathcal{L} v f)X + \]
\[ + u(X)(\mathcal{L} v f)Y - u(Y)(\mathcal{L} v f)X + \]
\[ + w(X)(\mathcal{L} v f)Y - u(Y)(\mathcal{L} v f)X - \]
\[ + v(X)(\mathcal{L} w f)Y - v(Y)(\mathcal{L} w f)X \} - \]
\[ - \lambda' \{v(X)(\mathcal{L} u v)Y - v(Y)(\mathcal{L} u v)X + \]
\[ + u(X)(\mathcal{L} v v)Y - u(Y)(\mathcal{L} v v)X + \]
\[ + w(X)(\mathcal{L} v w)Y - w(Y)(\mathcal{L} v w)X \} U - \]
\[ + \lambda' \{u(X)(\mathcal{L} u u)Y - u(Y)(\mathcal{L} u u)X + \]
\[ + v(X)(\mathcal{L} u u)Y - v(Y)(\mathcal{L} u u)X + \]
\[ + w(X)(\mathcal{L} u w)Y - w(Y)(\mathcal{L} u w)X \} V - \]
\[ - \lambda' \{u(X)v(Y) - u(Y)v(X) - u(X)w(Y) - \]
\[ - v(Y)w(X) - v(X)w(Y) - \]
\[ - v(Y)w(X)\}[U,V,W]. \]
THEOREM (4.1):

If an $f_{\lambda}$-Hsu structure with complemented $\{f_{\lambda}, U, V, W, u, v, w\}$ is normal, then Hsu structure defined by (3.1) is integrable.

PROOF:

If an $f_{\lambda}$-Hsu structure with complemented frames $\{f_{\lambda}, U, V, W, u, v, w\}$ is normal, then from definition (2.1), $S^*$ is zero.

Thus by virtue of (2.4), (2.14), (2.15), (2.16) and (4.2) we have $N^*(X, Y) = 0$.

Hence the Hsu structure $F$ defined by (3.1) is integrable.

5. INVARIANT SUBMANIFOLD:

Let $\tilde{M}$ be an $n$-dimensional $(1 < n < m)$ differentiable manifold of class $C^\infty$ and suppose that $\tilde{M}$ is immersed in $M$ by the immersion $i: \tilde{M} \to M$. Let us denote by $B$ the differential $d_i$ of the immersion $i$.

Let us suppose that the vector field $U$ is tangent $i(\tilde{M})$. Then any vector tangent to $i(\tilde{M})$ annihilates the 1-form $v$, $w$ and the tangent space to $i(\tilde{M})$ is invariant by $f$.

Therefore, we have

(5.1) $U = B\tilde{U}$.

For a vector field $\tilde{U}$ of $\tilde{M}$

(5.2) $v\left(B, \tilde{X}\right) = 0$,

for any vector field $\tilde{X}$ of $\tilde{M}$ and
(5.3) \( f(B\tilde{X}) = B\tilde{f} \tilde{X} \),

for a tensor field \( \tilde{f} \) of \( \tilde{M} \) and an arbitrary vector field \( \tilde{X} \) of \( \tilde{M} \). For convenience, we call such a submanifold an invariant submanifold with respect to \( U \) and \( v \). Similarly, we can define an invariant submanifold with respect to \( V \) and \( u \).

**THEOREM (5.1):**

An invariant submanifold with respect to \( U \) and \( v \) of a manifold with \( f_\tilde{\lambda} \)-Hsu structure and complemented frames \( \{ f_\tilde{\lambda}, U, V, W, u, v, w \} \) admits a \( \{ f_\tilde{\lambda}, \tilde{U}, \tilde{u} \} \)-structure.

**PROOF:**

Let \( \tilde{M} \) be an invariant submanifold with respect to \( U \) and \( v \) of a manifold \( M \) with \( f_\tilde{\lambda} \)-Hsu structure and complemented frames \( \{ f_\tilde{\lambda}, U, V, W, u, v, w \} \).

Now applying \( f \) to (5.1) and using (2.2) and (5.3) we obtain

\[
0 = fU = f(B\tilde{U}) = B\tilde{f}\tilde{U},
\]

which gives

(5.4) \( \tilde{f}\tilde{U} = 0 \).

Applying \( f \) to (5.3) and using (2.1) we get

(5.5) \( \tilde{\lambda} r(B\tilde{X}) - u(B\tilde{X})U - v(B\tilde{X})V = B\tilde{f}^2 \tilde{X} \).

Let us put

(5.6) \( \tilde{u}(\tilde{X}) = u(B\tilde{X}) \),

then by virtue of (5.1) (5.2) and (5.6), equation (5.5) yields

(5.7) \( \tilde{f}^2 X = \lambda r(\tilde{X} - \tilde{u}(\tilde{X})\tilde{U}) \).

Also from (5.3) we have
\[ u \left( f \left( B\tilde{X} \right) \right) = u \left( B\tilde{f} \tilde{X} \right), \]

which is consequences of (2.2) and (5.6) yields

\[ (5.8) \quad \tilde{u} \left( \tilde{f}\tilde{X} \right) = 0 \]

Further from (5.1) we have

\[ \tilde{u} \left( U \right) = u \left( B\tilde{U} \right), \]

which in view of (2.3) and (5.6) gives

\[ (5.9) \quad \tilde{u} \left( \tilde{U} \right) = 1. \]

Combining (5.4), (5.7) and (5.9) we have

\[ (5.10) \quad f^2 / \lambda' = I - \tilde{u} \otimes \tilde{U}, \]

\[ \tilde{f}\tilde{U} = 0, \tilde{u} \circ \tilde{f} = 0, \tilde{u} \left( \tilde{U} \right) = 1. \]

We call a structure satisfying (5.10), \( \{ \tilde{f}, \tilde{U}, \tilde{u} \} \)-structure.

**THEOREM (5.2):**

An invariant submanifold with respect to \( V \) and \( u \) of a manifold with \( f_\lambda \)-Hsu structure and complemented frames

\[ \{ f_\lambda, U, V, W, u, v, w \} \]

admits a \( \{ \tilde{f}, \tilde{V}, \tilde{v} \} \)-structure.

**PROOF:**

The proof is similar to the proof of theorem 5.1.

**THEOREM (5.3):**

An invariant submanifold with respect to \( W \) and \( u \) of a manifold with \( f_\lambda \)-Hsu structure and complemented frames

\[ \{ f_\lambda, U, V, W, u, v, w \} \]

admits a \( \{ \tilde{f}, \tilde{W}, \tilde{w} \} \)-structure.

**PROOF:**

The proof is similar to the proof of theorem (5.1).
6. INVARIANT SUBMANIFOLDS OF A NORMAL \( f_\tilde{X} \)-HSU MANIFOLD WITH COMPLEMENTED FRAMES:

In this section we shall compute the expression \( S^*(B\tilde{X}, B\tilde{Y}) \) for an invariant submanifold with respect to \( U \), \( V \) and \( W \).

By virtue of (2.4), (2.5) and (5.3) we have

\[
S^*(B\tilde{X}, B\tilde{Y}) = \left[ fB\tilde{X}, fB\tilde{Y} \right] - f \left[ fB\tilde{X}, B\tilde{Y} \right] - \\
- f \left[ B\tilde{X}, fB\tilde{Y} \right] + f^2 \left[ B\tilde{X}, B\tilde{Y} \right] + \\
- \lambda' (du) (B\tilde{X}, B\tilde{Y}) U - \lambda' (dv) (B\tilde{X}, B\tilde{Y}) V - \\
- \lambda' (dw) (B\tilde{X}, B\tilde{Y}) W \\
= \left[ B\tilde{f}\tilde{X}, B\tilde{f}\tilde{Y} \right] - f \left[ B\tilde{f}\tilde{X}, B\tilde{Y} \right] - f \left[ B\tilde{X}, B\tilde{f}\tilde{Y} \right] + \\
+ f^2 \left[ B\tilde{X}, B\tilde{Y} \right] - \lambda^2 (du) (B\tilde{X}, B\tilde{Y}) U - \\
- \lambda' (dv) (B\tilde{X}, B\tilde{Y}) V - \lambda' (dw) (B\tilde{X}).
\]

But in view of (5.1) and (5.6), we have, therefore

\[
(du)(B\tilde{X}, B\tilde{Y}) = (du)(\tilde{X}, \tilde{Y}), (dv)(B\tilde{X}, B\tilde{Y}) = 0, \\
(dw)(B\tilde{X}, B\tilde{Y}) = 0.
\]

(6.1) \( S^*(B\tilde{X}, B\tilde{Y}) = B\left[ \tilde{f}\tilde{X}, \tilde{f}\tilde{Y} \right] - fB\left[ \tilde{f}\tilde{X}, \tilde{Y} \right] - fB\left[ \tilde{X}, \tilde{Y} \right] + \\
+ f^2 B\left[ \tilde{X}, \tilde{Y} \right] - \lambda' du\left[ \tilde{X}, \tilde{Y} \right] U.
\]

Thus, in consequence of (5.1), equation (6.1) yields

(6.2) \( S^*(B\tilde{X}, B\tilde{Y}) = B\left[ \tilde{f}\tilde{X}, \tilde{f}\tilde{Y} \right] - f\left[ \tilde{f}\tilde{X}, \tilde{Y} \right] - f\left[ \tilde{X}, \tilde{Y} \right] + \\
+ \tilde{f}^2 \left[ \tilde{X}, \tilde{Y} \right] - \lambda' du\left[ \tilde{X}, \tilde{Y} \right] U \}.
\]
THEOREM (6.1):

An invariant submanifold with respect to U, V and W of a manifold having normal $f_\lambda$-Hsu structure and complemented frames $\{f_\lambda, U, V, W, u, v, w\}$ admits a normal $\{\tilde{f}_\lambda, \tilde{U}, \tilde{u}\}$-structure.

PROOF:

If a $f_\lambda$-Hsu structure and complemented frames $\{f_\lambda, U, V, W, u, v, w\}$ is normal then $S^* = 0$. Therefore, form (6.2) we have

$$ (6.3) \left( \tilde{f}X, \tilde{f}Y \right) - \tilde{f}\left[ \tilde{f}X, \tilde{Y} \right] - \tilde{f}^2 \left[ \tilde{X}, \tilde{Y} \right] - \tilde{\lambda}' (d\tilde{u}) \left[ \tilde{X}, \tilde{Y} \right] \tilde{U} = 0. $$

Hence in view (6.3) and using theorem (5.1) we have the result.

THEOREM (6.2):

An invariant submanifold with respect to V and u of a manifold having normal $f_\lambda$-Hsu structure and complemented frames $\{f_\lambda, U, V, W, u, v, w\}$ admits a normal $\{\tilde{f}_\lambda, \tilde{V}, \tilde{v}\}$-structure.

PROOF:

The proof is similar to that of theorem (6.1).

*****
REFERENCES


