Chapter-V

ON APST RIEMANNIAN MANIFOLD WITH SECOND ORDER GENERALISED STRUCTURE
1. **INTRODUCTION:**

Let an n-dimensional Riemannian manifold $M_n$, on which there are defined a tensor field $F$ of type (1,1) a tensor field $T$, a 1-form $A$ and metric tensor $g$ satisfying for arbitrary vector field $X,Y,Z$ and $a$ is any complex number (non-zero).

(1.1) $F^2 X = a^2 X - A(X)T$

(1.2) $\bar{X} = F(X)$

(1.3) $A(T) = -a^2$

(1.4) $A(FX) = 0$

(1.5) $F(T) = 0$

(1.6) $g(T,X) = A(X)$

(1.7) $g(FX,FY) = -a^2 g(X,Y) + A(X)A(Y)$

then structure $(F,T,A,g)$ is called almost paracontact metric structure and manifold $M_n$ will be called Almost paracontact metric Riemannian manifold.

Let us call such a structure as a generalised almost contact metric structure.

Let us define

(1.8) $F(\bar{X},Y) = g(FX,Y)$

and barring $X$ in (1.8) we have

(1.9) $'F(X,Y) = g(F^2 X,Y)$

(1.9) by virtue of (1.1) gives

(1.10) $'F(X,Y) = a^2 g(X,Y) - A(X)A(Y)$

Now barring $Y$ in (1.8) we have
(1.11) \( F(X, Y) = g(FX, FY) \)

(1.11) with the help of (1.7) gives

(1.12) \( F(X, Y) = -\{a^2 g(X, Y) - A(X) A(Y)\} \)

Thus from the relation (1.10) and (1.12) we have

(1.13) \( F(X, Y) = F(X, Y) \)

Replacing X by T in equation (1.8) and making use of (1.5), we obtain-

(1.14) \( F(T, Y) = 0 \)

Barring X in equation (1.12) and making use of (1.1) and (1.14) we get

(1.15) \( F(X, Y) = -a^2 F(X, Y) \)

Now barring Y in (1.7) and making use of (1.4) and (1.5) in the resulting equation, we obtain

(1.16) \( g(FX, Y) = -g(X, FY) \)

Thus from equation (1.8) and (1.16), we have

(1.17) \( F(X, Y) = -F(Y, X) \)

2. **NIJENHUIS TENSOR**

Nijenhuis tensor is given by.

(2.1) \( N(X, Y) = \begin{bmatrix} X \end{bmatrix} Y + \begin{bmatrix} X \end{bmatrix} Y + \begin{bmatrix} X \end{bmatrix} Y + \begin{bmatrix} X \end{bmatrix} Y \)

Making use of (1.1) in (2.1), we get-

(2.2) \( N(X, Y) = \begin{bmatrix} X \end{bmatrix} Y + a^2 [X,Y] - A([X,Y]) T - \begin{bmatrix} X \end{bmatrix} Y - [X,Y] \)

Now let us put

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(2.3) \[ P(X,Y) = [\bar{X}, \bar{Y}] - [\bar{X}, \bar{Y}] \]

(2.4) \[ Q(X,Y) = [\bar{X}, \bar{Y}] - [\bar{X}, Y] \]

(2.5) \[ H(X,Y) = [\bar{X}, \bar{Y}] + a^2 [X,Y] \]

**THEOREM (2.1):**

The Nijenhuis tensor and \( P(X,Y) \) are related as-

\[
(2.6) \quad a^2 P(X,Y) - P(\bar{X}, \bar{Y}) = a^2 N(X,Y) - A(Y)[\bar{X}, T] + \\
+ a^2 A(Y)[X, T] + \\
+ A(Y) A([X, T]) T.
\]

**PROOF:**

Barring \( Y \) in (2.3) and using (1.1), we obtain-

\[
(2.7) \quad P(X, \bar{Y}) = a^2 [\bar{X}, Y] + A(Y)[\bar{X}, T] - \\
- a^2 [\bar{X}, \bar{Y}] - A(Y)[X, T].
\]

Again barring the above equation and making use of (1.1)

\[
P(\bar{X}, \bar{Y}) = a^2 [\bar{X}, Y] + A(Y)[\bar{X}, T] - \\
- A(Y) A([X, T]) T\}
\]

\[
(2.8) \quad P[\bar{X}, \bar{Y}] = a^2 [\bar{X}, Y] + A(Y)[\bar{X}, T] - a^4 [X,Y] + \\
+ A(Y) A([X,T]) T.
\]

Now from the equation (2.3) and (2.8), we obtain-
(2.9) \( a^2 P(X, Y) - P(X, Y) \) 

\[ = a^2 [\bar{X}, \bar{Y}] - a^2 [\bar{X}, Y] + a^4 [X, Y] - 
\]
\[ - A(Y)[\bar{X}, T] - a^2 A(X, Y) T + 
\]
\[ + a^2 A(Y)[X, T] - A(Y) A([X, T]) T. \]

Making use of (2.2) in (2.9) we get the result.

**COROLLARY (2.1):**

In a differentiable manifold \( M^n \), we have

(2.10) \( a^2 P(X, T) = a^2 N(X, T) + a^2 [\bar{X}, T] - 
\]
\[ - a^4 [X, T] - a^2 A([X, T]) T. \]

**PROOF:**

Putting \( T \) for \( Y \) in (2.6) and using (1.5) and (1.3), we get the result.

**THEOREM (2.2):**

In a differentiable manifold \( M^n \), we have

(2.11) \( a^2 Q(X, Y) - Q(\bar{X}, Y) = a^2 N(X, Y) - A(X)[\bar{T}, \bar{Y}] + 
\]
\[ + a^2 A(X)[T, Y] + A(X) A([T, Y]) T \]

**PROOF:**

Barring \( X \) in (2.4) and making use of (1.1), we get

(2.12) \( Q(\bar{X}, Y) = a^2 [X, \bar{Y}] + A(X) [T, \bar{Y}] - a^2 [\bar{X}, \bar{Y}] + 
\]
\[ + A(X) [\bar{T}, Y]. \]
Now barring the whole equation (2.12) and making use of (1.1), we get-

\[ Q[\bar{X}, Y] = a^2 [\bar{X}, \bar{Y}] + A(X)[\bar{T}, \bar{Y}] - 
\]

\[ -a^2 \{a^2 [X, Y] - A([X, Y])T\} + 
\]

\[ + A(X)\{a^2 [T, Y] - A([T, Y])T\}. \]

(2.13) \[ Q[\bar{X}, Y] = a^2 [\bar{X}, \bar{Y}] + A(X)[\bar{T}, \bar{Y}] - 
\]

\[ -a^4 [X, Y] + a^2 A([X, Y])T + 
\]

\[ + a^2 A(x)[T, Y] - A(x) A([T, Y])T. \]

Now from (2.4) and (2.13), we get

(2.14) \[ a^2 Q(X, Y) - Q[\bar{X}, Y] = 
\]

\[ = a^2 [\bar{X}, \bar{Y}] - a^2 [\bar{X}, \bar{Y}] - a^2 [\bar{X}, \bar{Y}] + 
\]

\[ + a^4 [X, Y] - a^2 A([X, Y])T - 
\]

\[ - A(X)[\bar{T}, \bar{Y}] - a^2 A(X)[T, Y] - 
\]

\[ - A(X) A([T, Y])T. \]

Thus from (2.2) and (2.14) we obtain the required result.

**COROLLARY (2.2):**

In a generalized almost contact metric manifolds \( M^n \), we have

(2.15) \[ a^2 Q(T, Y) = 
\]

\[ = a^2 N(T, Y) + a^2 [T, \bar{Y}] - a^4 [T, Y] 
\]

\[ = -a^2 A([T, Y])T. \]
PROOF:
Replacing $X$ by $T$ in (2.11) and using (1.3) and (1.5), we get the equation (2.15)

THEOREM (2.3):

In a generalized almost contact metric structure manifold $M^n$

\[(2.16)\ a^2 H(X, Y) - H(\overline{X}, \overline{Y}) = a^2 N(X, Y) - \]
\[-a^2 A([X, Y]) T - A(X) \overline{T, \overline{Y}}\]

PROOF:

Barring $X$ in (2.5) and making use of (1.1)

\[(2.17)\ H(\overline{X}, Y) = a^2 \overline{X, Y} + A(X) \overline{T, \overline{Y}} - a^2 (\overline{X}, Y)\]

Now barring the whole equation (2.17) and making use of (1.1)

\[(2.18)\ H(\overline{X}, \overline{Y}) = a^2 \overline{X, Y} - A(X) \overline{T, \overline{Y}} + a^2 \overline{X, \overline{Y}}\]

Thus, with the help of (2.2), (2.5) and (2.18) we get (2.16)

COROLLARY (2.3):

The equation (2.16) is equivalent to

\[(2.19)\ a^2 H(T, Y) = a^2 N(T, Y) - a^2 A([T, Y]) + a^2 \overline{T, \overline{Y}}\]

PROOF:

Replacing $X$ by $T$ in (2.16) and using the equation (1.3) and (1.5), we get the result.

THEOREM (2.4):

In a generalized almost contact metric structure manifold $M^n$, we have
(2.19) \( H(T,Y) - Q(T,Y) = a^2[T,Y] \)

**Proof:**

Equation (2.20) follows directly with the help of equation (2.15) and (2.19).

**Theorem (2.5):**

In a generalized almost contact metric manifold \( M^n \), we have

\[
(2.21) \ a^2H(X,Y) - H\left[\overline{X},Y\right] = \left\{ a^2P(X,Y) - P\left[\overline{X},\overline{Y}\right] + \right.
\]
\[
+ A(Y)\left[\overline{X},T\right] - a^2A(Y)[X,T] -
\]
\[
- A(Y)A([X,T])T - a^2T([X,Y])T -
\]
\[
- A(X)[T,\overline{Y}] .
\]

**Proof:**

Proof follows with the help of equation (2.6) and (2.16)

**Theorem (2.6):**

In order that a generalized almost contact metric manifold be completely integrable it is necessary that

\[
(2.22) \ A\left[\overline{X},\overline{Y}\right]T = 0
\]

**Proof:**

Barring \( X \) in (2.2) and with the help of equation (1.1), we get

\[
(2.23) \ N(X,Y) = a^2(X,\overline{Y}) - A(X)[T,\overline{Y}] + a^2[\overline{X},Y] -
\]
\[
- A([\overline{X},Y])T - [\overline{X},Y] - a^2[\overline{X},Y] -
\]
\[
+ A(X)[T,\overline{Y}] .
\]

Now barring the whole equation (2.23) and using (1.1), we obtain
(2.24) \[ N \left[ \bar{X}, Y \right] = a^2 \left[ X, \bar{Y} \right] - A(X) \left[ T, \bar{Y} \right] + a^2 \left[ \bar{X}, Y \right] - \]
- \[ a^2 \left[ \bar{X}, \bar{Y} \right] + A([\bar{X}, \bar{Y}])T - a^4 [X, Y] + \]
\[ + a^2 A([X, Y])T + a^2 A(X)[T, Y] + \]
\[ + A(X)A([T, Y])T. \]

Form the equation (2.2) and (2.24), we have

(2.25) \[ N \left[ \bar{X}, Y \right] + a^2 N(X, Y) \]
\[ = - A(X) \left[ T, \bar{Y} \right] + A([\bar{X}, \bar{Y}])T + \]
\[ + a^2 A(X)[T, Y] + A(X)A([T, Y])T. \]

(2.26) \[ N(T, Y) = a^2 [T, Y] + A([T, Y])T - \left[ T, \bar{Y} \right]. \]

Using (2.24) in (2.26) we obtain

(2.27) \[ N \left[ \bar{X}, Y \right] + a^2 N(X, Y) = A(X)N(T, Y) + \]
\[ + A([\bar{X}, \bar{Y}])T. \]

For completely integrable manifold equation (2.27) reduces to equation (2.22)

**THEOREM (2.7):**

In a completely integrable generalized almost contact metric structure manifold \( M^n \), we have the following result-

(2.28) \[ A(X) \left[ T, \bar{Y} \right] - \left[ T, \bar{Y} \right] + A([\bar{X}, \bar{Y}])T = \]
\[ = A(Y \left[ \bar{X}, T \right] - \left[ \bar{X}, T \right]) + A([X, \bar{Y}])T. \]

**PROOF:**

Barring X in equation (2.2) and making use of (1.1), we get
(2.29) \( N(\bar{X}, \bar{Y}) = a^2 \langle X, \bar{Y} \rangle - A(X)[T, Y] + a^2 \left[ \bar{X}, Y \right] - A\left( \left[ \bar{X}, Y \right] \right) T - \left[ \bar{X}, Y \right] - a^2 \left[ X, \bar{Y} \right] + A(X)\left[ \bar{T}, Y \right] \). 

Again barring \( Y \) in equation (2.2) and making use of (1.1), we get

(2.30) \( N(\bar{X}, \bar{Y}) = a^2 \left[ \bar{X}, Y \right] - A(Y)\left[ \bar{X}, T \right] + a^2 \left[ X, \bar{Y} \right] + A\left( \left[ X, \bar{Y} \right] \right) T - a^2 \left[ X, \bar{Y} \right] + A(Y)\left[ \bar{X}, T \right] \). 

Now from these two equation (2.29) and (2.30) and using \( N(X, Y) \), we have the required result (2.28).
REFERENCE


