Appendix I

Comments on Makila, P. M. (2006)

His strategy can be understood in the following way. Consider a step input to a lumped-parameter; linear second-order system having two poles and one zero that is represented mathematically by a linear, ordinary differential equation having constant coefficients $A$ through $E$ as:

$$\begin{align*}
A \frac{d^2 Y}{dt^2} + B \frac{dY}{dt} + CY &= D \frac{dX}{dt} + EX \\
\end{align*}$$  \(A1\)

It contains a singularity due to the derivative of the step input term. The system Eq. (A1) can be rearranged to give the following equation, where $Y^{(-n)}$ represents the $n^{th}$ integral of $Y$. Thus, for $n = 1$, $Y^{(-1)} = \int_0^t Y(t)dt$.

$$\begin{align*}
\frac{d^3}{dt^3} \left[ AY^{(-1)} + BY^{(-2)} + CY^{(-3)} - DX^{(-2)} - EX^{(-3)} \right] &= 0 \\
\end{align*}$$  \(A2\)

Let, $W(t) = AY^{(-1)} + BY^{(-2)} + CY^{(-3)} - DX^{(-2)} - EX^{(-3)}$  \(A3\)

So, $W''(t) = 0$  \(A4\)

Taking the Laplace transform of Eq. (A4), we get,

$$\begin{align*}
s^3W(s) - s^2W(0) - sW'(0) - W''(0) &= 0 \\
\end{align*}$$  \(A5\)

Let, $Z(t) = Y^{(-1)}(t)$ and $V(t) = X^{(-1)}(t)$  \(A6\)

Substituting Eq. (A6) into Eq. (A3) and taking the Laplace transform, we get,

$$\begin{align*}
W(s) &= AZ(s) + B \frac{Z(s)}{s} + C \frac{Z(s)}{s^2} - D \frac{V(s)}{s} - E \frac{V(s)}{s^2} \\
\end{align*}$$  \(A7\)

Now Eq. (A7) can be inserted into Eq. (A5) to get the Laplace transform $Y(s)$. Note that $Z(t)$ and $V(t)$ are continuous functions satisfying $Z(0) = 0$ and $V(0) = 0$. So, these
continuous functions don’t suffer from the inconsistencies contained in the term \(f(0)\) of the derivative rule. Hence, \(sZ(s) = Y(s)\) and \(sV(s) = X(s)\). Also note that \(W(0) = 0\), because all integrals in Eq. (A3) are zero at zero time. These are shown below.

\(V(0)\) is zero because \(V(t)\) is a ramp for the step input. The other integrals are zero at zero time because of the nature of the solution \(Y\), which is limited.

And \(Z(0)\) becomes,

\[
Z(0) = \int_{0^-}^{0^+} Y(t) dt = 0
\]

(A8)

So, \(Z(0) = 0\), because the integrand is limited. Similarly, all the further integrals of \(Y\) are zero at zero time.

Hence, finally we get,

\[
Y(s) = \frac{sW'(0) + W''(0) + Ds + E}{As^2 + Bs + C}
\]

(A9)

Thus, the solution can be obtained by putting either \(0^-\) or \(0^+\) in the above equation and taking the inverse Laplace transform. This method does not involve singularity contained in the derivative of discontinuous input.

However, in the following paragraph, it is shown that the above strategy works only for the step perturbation and does not work for the impulse perturbation, \(X(t) = \delta(t)\). For the impulse inputs, the new function for the integral of the input (Eq. (A6)) contains an initial discontinuity (and, hence, inconsistency), since the integral of the impulse is a step. If one tries to circumvent this by redefining the new function as the double integral of input, one again ends up in a discontinuity (due to the appearance of the initial value of the derivative of the input function) in the Laplace transform of the new function. The Laplace transform of the new function of Eq. (A6) is required in Eq. (A7). This is shown below:

Since, then:

\[
X(t) = V''(t)
\]

(A10)
\[ X(s) = s^3 V(s) - sV(0) - V'(0) \]  \hspace{1cm} (A11)

\[ X(s) = s^3 V(s) - V'(0) \]  \hspace{1cm} (A12)

Even though \( V(0) \) would then be 0, there would be discontinuity in \( V'(0) \) as \( V'(t) = X^{(-1)}(t) \) would be a step and hence,

\[ V'(0) \neq 0 \hspace{1cm} (A13) \]

So, the inconsistency can’t be avoided for the impulse input (i.e., for singular term of its derivative).
Appendix II

Algorithm for the solution of the DDE of closed-loop stirred tank heater

The presence of dead time in the model requires the program to retrieve the data obtained $\tau_d$ seconds earlier. This requires the data of the iterations to be stored in an array. However, the array can not store a data beyond its upper bound space $U$. For this purpose, the data of iterations are stored up to $U$ number of spaces, and the next data is overwritten in the first data space of the same array. Similarly, each subsequent data is stored in the second, third...., etc. space. The cycle is repeated, after every $U$, $2U$, $3U$,..., etc. numbers of values are filled. This is done in the Result (results of $dT/dt$) statement of the algorithm, where $[(X) \% U]$ means the remainder $R$ obtained after dividing $X$ by $U$. So, the data shall be stored in the $R^{th}$ space. In this statement, $U$ is chosen in such a way that, in the subsequent cycle, the un-retrieved value of the array is not overwritten, i.e., $U > \tau_d / dt$, where $dt$ is the step size of the Runge-Kutta integration algorithm.

$Y$ stands for the array for temperature, $U$ for the dimension of the array, $dt$ for the step size of time, derivative for the derivative of $Y$, $toud$ for dead time, $Y_{at\_time}$ for $Y$ at a particular instant of time, $i$ for the count of iteration number at a particular instant of time; $k_1$, $k_2$, $k_3$ & $k_4$ for the parameters of Runge-Kutta, and $N$ for the total number of iterations. $F = v_o \left( t - \tau_d \right)$ and $Q_s = Q(0^-)$ in the main text. Other symbols are defined in Section 3.7 of the main text.

![Algorithm Diagram](https://via.placeholder.com/150)

**Input:** $t_{max}$, $dt$, $N$, $K_c$, $K_1$, $K_2$, $K_3$, $toud$

**Array:** $Y[U]$

**Initial Conditions:** $time = 0^+$, $Y[0] = Y[0^+]$

**Impulse I/P at $i = 1$:** $F = F(0^-)$
derivative: \( \dot{Y}(t) = -K_1K_2Y[(\text{time - toud})/\text{dt}] \% U - K_2(Y_{\text{at time}})F + K_3F + K_4K_2Y(0^-) + K_5Q_c \)

**Fourth order Runge-Kutta numerical integration**

\[
\begin{align*}
    k_1 &= \text{fun}(t, Y[(i-1) \% U]) \ dt; \\
    k_2 &= \text{fun}(t, Y[(i-1) \% U] + k_1/2) \ dt; \\
    k_3 &= \text{fun}(t, Y[(i-1) \% U] + k_2/2) \ dt; \\
    k_4 &= \text{fun}(t, Y[(i-1) \% U] + k_3) \ dt; \\
    Y[(i) \% U] &= Y[(i-1) \% U] + (k_1 + 2k_2 + 2k_3 + k_4)/6 \\
\end{align*}
\]

While, \( i \leq N \)

While, \( i \leq N \)

Yes

\[ Y[(i) \% U] = Y[(i-1) \% U] + (k_1 + 2k_2 + 2k_3 + k_4)/6 \]

\[ i = i + 1 \]

**New values of Y are stored in the array: \( Y[1], [2], \ldots \ [U] \)**