6 Framework Applied to the Linear System Analysis

The framework of analysis and initialization proposed for singular inputs in the last chapter consisted of the comparison of two direct initialization approaches of finding the post-initial conditions, \textit{a priori}, viz. (A) including singularities in the cause, i.e., the input function, and carrying out the time integration of the model with in the limits of $0^-$ to $0^+$; and (B) including singularities in the effect, i.e., applying physical balances to the conditions of the system at time zero. Subsequently, the model is solved by using these two sets of initial conditions, and setting the singular terms zero. This framework is applied to the linear time-invariant systems to handle the inconsistency in the Laplace domain dynamics.

6.1 Inconsistency in the Laplace Transform Treatment

In the literature review Section 2.1, the inconsistency in the Laplace transform treatment was discussed at length. It is briefed here and commented upon. The source of the inconsistencies was the fine point of the use of $0^+$ and $0^-$ in the definition of the Laplace transform and the corresponding derivative rule:

$$f(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt; \quad \mathcal{L}\{f'(t)\} = sf(s) - f(0) \quad (6.1)$$

For systems perturbed by singular inputs, there are initial jump discontinuities in the output functions and/or their derivatives. Due to the discrepancy in the unilateral Laplace transform treatment, the initial values calculated from the derived responses (post-initial values $f(0^+)$) are inconsistent with the original initial conditions chosen for deriving these responses (pre-initial conditions $f(0^-)$). The $\mathcal{L}_-$ Laplace transform approach uses $0^-$ while the $\mathcal{L}_+$ Laplace transform approach uses $0^+$ in the above equation. The $\mathcal{L}_+$ Laplace transform of the $\delta$-Dirac function becomes zero from the derivative rule:

$$\mathcal{L}_+\{\delta(t)\} = su(s) - u(0^+) = s\frac{1}{s} - 1 = 0 \quad (6.2)$$
Many studies advocate the $L_-$ approach, but the resultant $0^+$ values are inconsistent with the $0^-$ values. The $L_+$ approach requires additional effort of applying physical principles to calculate the $0^+$ values, \textit{a priori}. Nevertheless, subsequent solution through the $L_+$ approach yields the same $0^+$ values. Thus, the $L_+$ form resolves the inconsistency. However, the $L_+$ transform of the $\delta$-Dirac function becomes zero, i.e., $L_+\{\delta(t)\} = 0$ (Eq. (6.2)). Many studies use the $L_+$ form, while retaining $L_+\{\delta(t)\} = 1$. Bilateral and double-sided forms were also suggested. Makila (2006), on the other hand, reported the futility of the $L_-$ and $L_+$ forms and limitations of the bilateral and double-sided forms. However, it used a convoluted strategy by transforming the ODE into a form consisting of differentials of the integrals that don’t suffer from singularity. We clearly represent this strategy in Appendix I. However, there we show that this strategy works for the step perturbation only, and not for the impulse.

As seen above, there is inconsistency prevailing on this issue. More relevantly, the inconsistency for the chemical engineering systems, which are further complicated by mixing with chemical reactions, phase changes, and the interaction of material, thermal, and mechanical energy (inertia) capabilities, e.g., CSTRs, gravity-flow tanks, condensers, etc., remain to be thoroughly investigated.

Initialization studies reported on the linear time-invariant systems frequently consider systems with terms containing differentials of the input function as they contain a singularity due to the derivative of \textit{step} input term (Brigola & Singer, 2009; Grizzle, 2004; Makila, 2006; Lundberg \textit{et al.}, 2007). Several linear and linearized chemical engineering systems also exhibit such models (Ahuja, 2010; 2011). The application of the proposed framework is illustrated below through an example of U-tube manometer system used in chemical engineering science. These results are also compared with that of an analogous system considered in Section 2.2.2 (Lundberg \textit{et al.}, 2007). Unlike the literature studies, \textit{impulse} input is also considered. So, the system also contains a singular term of the derivative of the impulse input function.
6.2 Example

Consider the U-tube manometer model of Section 3.8.1:

\[ m\ddot{u} = -a\dot{u} - bu + A(\Delta\bar{P}) \]
where \( a = 8\pi\mu LA, \quad b = 2\rho gA \) \hfill (6.3)

**Step input**

A step input in \( \Delta P \) of magnitude \( \Delta P(0^+) - \Delta P(0^-) \) is first considered. Since Eq. (6.3) must maintain equality at all times, it can be integrated within \( 0^- \) and \( t \) limits of time using the initialization Algorithm procedure of the proposed framework:

\[
\int_{0^-}^{t} m\ddot{u} dt = -\int_{0^-}^{t} a\dot{u} dt - \int_{0^-}^{t} bu dt + \int_{0^-}^{t} A\Delta\bar{P} dt \quad \text{(6.4)}
\]

\[
m\{\dot{u}(t) - \dot{u}(0^-)\} = -a\{u(t) - u(0^-)\} - \int_{0^-}^{t} bu(t) dt + A\{\Delta P(t) - \Delta P(0^-)\} \quad \text{(6.5)}
\]

The above equation applied at \( 0^+ \) value of time gives the expression for the post-initial condition for the acceleration of the column. The integrand in the third term of Eq. (6.5) is limited and the integral vanishes. So, it gives:

\[
m\{\dot{u}(0^+) - \dot{u}(0^-)\} = -a\{u(0^+) - u(0^-)\} - 0 + A\{\Delta P(0^+) - \Delta P(0^-)\} \quad \text{(6.6)}
\]

Similarly, re-integrating Eq. (6.5) within \( 0^- \) and \( 0^+ \) limits of time gives the value of the post-initial condition of the velocity:

\[
m\{\dot{u}(0^+) - \dot{u}(0^-)\} = -a\{u(0^+) - u(0^-)\} - 0 + A\{\Delta P(t) - \Delta P(0^-)\} dt \quad \text{(6.7)}
\]

\[
m\{u(0^+) - u(0^-)\} = a\{0 - 0\} + 0 + 0 = 0 \quad \text{(6.8)}
\]

All the terms on the R.H.S. of Eq. (6.7) vanish as the integrands are limited. Thus, there is no discontinuity in \( u \). Using this in Eq. (6.6), gives the following discontinuity in \( \dot{u} \) :
\[ m\dot{u}(0^+)-\dot{u}(0^-) = AM\{\Delta P(0^+)-\Delta P(0^-)\} \]

The initial conditions obtained from initialization B, i.e., the momentum balance considerations also happen to be the same. Since, imposing a step change in the pressure causes no change in the initial velocity because of the inertia of the column, the same is given by Eq. (6.8). However, this change of pressure causes a simultaneous change in the initial acceleration of the liquid column that is equal to the force applied \( A\{\Delta P(0^+)-\Delta P(0^-)\} \) per unit mass \( m \) of the column, and the same is given by Eq. (6.9).

**Impulse input**

On similar lines, it can be verified that the proposed methodology gives correct initial conditions for the impulse input. For the treatment of this case via initialization A, in the above equations, the impulse is included in the cause, i.e., the input function. The system is initially at a particular state at time \( t \to 0^- \). For an impulse input at \( t \to 0^+ \), the forcing function \( \Delta P \) changes from its pre-initial value \( \Delta P(0^-) \) (steady or unsteady) to the following equation. This equation is, then, placed in the above equations.

\[ \Delta P(t) = \Delta P(0^-) + M \delta(t-t_o) \text{ with } t_o = 0^+ \]

(6.10)

where \( M \) represents the magnitude of the impulse and \( \delta \) represents the unit-impulse function. Noting that the integral of the unit impulse, i.e., the \( \delta \)-Dirac function is one, application of the initialization A procedure to Eq. (6.7), gives the discontinuity in \( u \):

\[ m\{u(0^+)-u(0^-)\} = AM \]

(6.11)

This is the same as obtained by applying initialization B. \( A \) times the magnitude of the applied impulse equals the simultaneous change in the initial momentum of the column.

Again, Eq. (6.6) of initialization A yields the initial discontinuity in acceleration:

\[ m\dot{u}(0^+)-\dot{u}(0^-) = -a[u(0^+)-u(0^-)]-0 + A[M \delta(t-t_o)] = -aAM / m + A \delta(t-t_o) \]

(6.12)

which happens to be the same from initialization B (momentum balance). \( m \) times the discontinuity in the initial acceleration of the liquid column is equal to \( A \) times the
magnitude of the applied impulse $M$ minus the initial frictional force offered due to the sudden change in initial velocity. The restoring force term (zero term) is zero as the initial discontinuity in the displacement is zero. Eq. (6.12) exhibits a singularity in the initial acceleration because of the jump discontinuity in the initial velocity in Eq. (6.11).

Thus, the two initializations A and B yield consistent post-initial conditions, and it can be verified that the $\mathcal{L}$ Laplace transform approach also yields the same values.

$L_+$ approach: In this approach, post-initial conditions obtained above are subsequently used to solve the system Eq. (6.3). For this, Eq. (6.3) is first subtracted from the same equation at $0^-$ time; the resultant equation is $\mathcal{L}_+$ Laplace transformed to give the following equation:

$$m\left[s^2U(s) - sU(0^+) - \dot{U}(0^+)\right] + a\left[sU(s) - U(0^+)\right] + bU(s) = sA\Delta\bar{P}(s) - A\Delta\bar{P}(0^+) \quad (6.13)$$

where $U(t) = u(t) - u(0^-)$, and $\Delta\bar{P}(t) = \Delta P(t) - \Delta P(0^-) = M\delta(t - t_o)$ with $t_o = 0^+$ (6.14)

Now, for an impulse perturbation in Eq. (6.13), $\Delta\bar{P}(s) = 0$ from Eq. (6.2). Also, the last term of Eq. (6.13) is singular, as:

$$A\Delta\bar{P}(0^+) = AM\delta(t - t_o) \quad (6.15)$$

Eq. (6.15) is obtained from Eqs. (6.10) and (6.14). $\dot{U}(0^+)$ on the left of Eq. (6.13) has the same singular term (from the last term of Eq. (6.12)). Thus all the singular terms in Eq. (6.13) cancel spontaneously and its time-domain solution is feasible through inversion.

An analogous system, i.e., a car-wheel suspension is considered in Section 2.2.2, wherein the relevant details of Lundberg et al. (2007) are also given for ready reference. Applying the framework, it was seen that the post-initial values computed from initialization A were identical with that obtained using the $\mathcal{L}_+$ approach for three different pre-initial sets of conditions of the non-zero values, unsteady values, and values yielding immediate convergence to the final value of the response. This result is true in general.
Outcomes revisit

- Invariably, the post-initial conditions obtained through initialization A conform to the post-initial conditions computed through the $L_-$ approach, and both these approaches include singularities in the *cause*. The $L_-$ approach is, however, a one-step procedure.

- The $L_+$ approach (a two-step procedure) is analogous to the initialization B approach, in case, the post-initial conditions are obtained in the first step by applying physical principles to the effects of singularities. Both these approaches include singularities in the *effect*. Again, in the $L_+$ approach, the solution is obtained in the second step while all the singular input terms cancel/vanish spontaneously. The same happens in initialization B as the singular terms are set zero while carrying out the solution subsequently.

- In the above case, yet not true in general, the $L_-$ and initialization A approaches are found to be consistent with the $L_+$ and initialization B approaches.

6.3 Discussion: $L_-$ vs. $L_+$ Approach

The contributions in the literature have carried out the analysis for the step inputs to the simple cases as above. Thus, the $L_-$ approach seems to be satisfactory, and reportedly yields post-initial conditions consistent with that of the $L_+$ approach invariably (see Section 2.1, Lundberg *et al.*, 2007; Makila, 2006). However, the $L_-$ approach is inexact and inconsistent for the cases of impulse inputs to the complex processes involving mixing with reactions, phase changes, and the interaction of various capacities.

In Sections 4.1, 4.2, 4.8, 5.11, and 5.12 dealing with linear and linearized cases of nontrivial cases of chemical engineering in the preceding chapters, it was discussed that the post-initial conditions computed through the model’s considerations, viz., initialization route (a) of Chapter 4, and initialization A of Chapter 5, were inconsistent with that obtained from initialization B. Hence, impulse must be included in the *effect* rather than the *cause* as proposed in the $L_+$ and initialization B approaches. However, the $L_-$ and initialization A approaches include impulse in the *cause*. In the nontrivial cases of
chemical engineering systems, the $L_-$ approach works for the step inputs only, and gives incorrect results for the impulse inputs, and, hence, the $L_-$ and $L_+$ approaches don’t provide the same solutions always.

Therefore, in such cases, the post-initial values calculated from the $L_-$ approach and from the initial integration of the model, i.e., initialization A are inconsistent with the ones estimated from a physical balance at the initial time, i.e., initialization B, as the model can’t account for the initial discontinuities. This result is analogous to our previous findings on several cases of chemical engineering systems, viz., isothermal and non-isothermal CSTRs, condenser, flow-level tanks, etc., which indicates the inherent limitation of the model to completely represent the system affected by singularities. Nevertheless, particularly for such cases, the solution based on the $L_+$ approach, but with the initial conditions estimated from a physical balance, rather than from the initial integration procedure A, should be used for more accurate results. The post-initial conditions for several such cases were worked out through the physical balance considerations in the last chapter.

6.3.1 Resolving the inconsistency further

Further inconsistencies are now considered. The discrepancies discussed at length in Chapter 2 can be easily handled by the uniform use of the proposed methodology. Many studies report that the $L_-$ approach is better than the $L_+$ approach that requires the additional effort of applying physical balances to calculate the post-initial conditions, and involves the inconsistency of Laplace transform of the impulse function being zero. However, it is argued here that the $L_-$ approach involves several inconsistencies. The $L_+$ approach is better and the application of the additional effort can be circumvented for the simple cases as the post-initial conditions can be conveniently obtained through initialization A (Section 6.2). Subsequently, the solution can be obtained from the $L_+$ approach as the impulse input or singular terms automatically vanish from the model as they did in the U-tube manometer case above.

Since, the singular part of the response has already been included while calculating its post-initial conditions, \textit{a priori}; the effect of the impulse (or singular input
terms) has been accounted for in the post-initial conditions of the output functions. As the impulse input (or the singular input terms) occurs only for the post-initial, i.e., the $0^+$ time, and has been included in the effect, it would be erroneous to re-include it in the cause, i.e., in the input singular term. This argument also justifies the $L_+$ Laplace transform of the $\delta$-Dirac function becoming zero in Eq. (6.2). As seen in the above Example 6.2, all the singular terms cancel/vanish spontaneously in the differential equation while calculating the solution profile subsequently. Thus, the $L_+$ approach avoids taking the Laplace transform or the inversion of the singular input terms in the differential equation. So, as reasoned out, the $L_+$ approach doesn’t involve discrepancies in Eq. (6.2) that have been reported in the literature. Eq. (6.2) takes $\delta$-Dirac as the derivative of the discontinuous unit step function $u(t)$ and uses the derivative rule.

Extending the above argument for the derivative of other discontinuous functions yields interesting results. In fact, the $L_+$ Laplace transform of the differential of an initially discontinuous function is not inconsistent with that computed from the derivative rule. For, if the above logic is applied to other initially discontinuous functions $f(t)$, which are the product of a function $g(t)$ with the unit step function $u(t)$, one gets consistent results for the $L_+$ transform, while inconsistent ones for the $L_-$ transform. There is inherent inconsistency in the $L_-$ approach due to the discrepancy in the pre- and post-initial conditions. This is shown by considering the following three Cases for $g(t)$: (a) $g(t)$ is a smooth function, e.g., $\cos t$; (b) $g(t)$ is continuous, but has a derivative discontinuity, e.g., modulus function $|t|$; and (c) $g(t)$ has a jump discontinuity at the origin, e.g., $u(t)$.

$$f(t) = g(t)u(t)$$

**a.** $g(t) = \cos t$

So, $g(t)$ is a smooth function, yet $f(t)$ has an initial jump discontinuity from 0 to 1, since:

$$f(t) = \cos t \cdot u(t)$$

(6.16)

The $L_+$ Laplace transform of $f'(t)$ from the derivative rule becomes:
Again, since the derivative of Eq. (6.16) from the product rule is:

\[ f'(t) = \cos t \delta(t) - \sin t u(t) = \delta(t) - \sin t u(t) \]  
(6.18)

The singular term in Eq. (6.18) is evaluated using the property:

\[ g(t) \delta(t-a) = g(a) \delta(t-a) \quad \text{(as } g(t) \text{ is smooth)} \]  
(6.19)

The \( \mathcal{L} \) transform of \( f'(t) \) obtained through Eq. (6.18) turns out to be the same as given by Eq. (6.17), because the \( \mathcal{L} \) transform of the singular part \( \delta(t) \) vanishes, and if compared with the above derivative rule, Eq. (6.17), it has already been included in the post-initial condition \( f(0^+) \) of the derivative rule.

Anyways, the \( \mathcal{L} \) transform of \( f'(t) \) also happens to be the same through the above two ways as \( g(t) \) is smooth at the origin:

\[ \mathcal{L} \{ f'(t) \} = s^2 / (s^2 + 1) \]  
(6.20)

However, the \( \mathcal{L} \) transform doesn’t always yield consistent results from the above two ways as seen by applying the above treatment for Cases (b) and (c).

b. \( g(t) = |t| \)

which has a derivative discontinuity at origin. So, \( f(t) \) also has a derivative discontinuity (see Figs. 6.1 & 6.2 for \( g(t) \) & \( f(t) \), respectively), since:

\[ f(t) = |t| u(t) \]  
(6.21)

By referring to Fig. 6.2 (\( f(t) \) is effectively \( tu(t) \)), the \( \mathcal{L} \) Laplace transform of \( f'(t) \) from the derivative rule is:

\[ \mathcal{L} \{ f'(t) \} = s \mathcal{L} \{ f(t) \} - f(0^+) = s \frac{1}{s^2} - 0 = \frac{1}{s} \]  
(6.22)
Again, since the derivative of Eq. (6.21) from the product rule is:

\[ f'(t) = |t| \delta(t) + 1.u(t) = t \delta(t) + u(t) = 0 + u(t) \]  \hspace{1cm} (6.23)

Eq. (6.23) is obtained using Eq. (6.19), as \( g(t) = |t| = t \) can be considered smooth from the right, for \( t \geq 0^{+} \) (\( \delta(t) \) is taken \( \delta(t-0^{+}) \) in \( L_{+} \) approach above). The \( L_{+} \) transform of \( f'(t) \) obtained through the last Eq. (6.23) turns out to be the same as the above Eq. (6.22).

![Fig. 6.1](image1.png)  \hspace{1cm} \text{Fig. 6.1 Graph of } g(t) = |t| \text{ the modulus function that has a derivative discontinuity at the origin}

![Fig. 6.2](image2.png)  \hspace{1cm} \text{Fig. 6.2 Graph of } f(t) = g(t)u(t) = |t|u(t) \text{ that has a derivative discontinuity at the origin}
Now, for the $L$–approach, the transform of $f'(t)$ from the derivative rule is the same as given by Eq. (6.22). However, the derivative $f'(t)$ and, hence, the $L$–transform from the product rule becomes undefined in the domain $[0^-, \infty)$ as shown below:

\[ f'(t) = \pm \delta(t) \pm 1 \cdot u(t) = \text{undefined} \pm u(t) \]

(as $g(t) = \pm |t|$ is not smooth at origin)

(6.24)

So, the two ways don’t give the same results for the $L$–transform.

c. $g(t) = u(t)$

which has an initial jump discontinuity. So, $f(t)$ also has an initial jump discontinuity from 0 to 1, since:

\[ f(t) = g(t) \cdot u(t) \]

(6.25)

For the $L_+$ Laplace approach, the transform of $f'(t)$ from the derivative rule becomes (as $f(t)$ is effectively $u(t)$ only):

\[ L_+ \{ f'(t) \} = s \cdot L \{ f(t) \} - f(0^+) = s \cdot \frac{1}{s} - 1 = 0 \]

(6.26)

Again, since the derivative of Eq. (6.25) from the product rule is:

\[ f'(t) = u(t) \delta(t) + u(t) \delta(t) = \delta(t) + \delta(t) \]

(6.27)

Eq. (6.27) is obtained using Eq. (6.19), as $g(t) = u(t)$ can be considered smooth from the right, for $t \geq 0^+$ ($\delta(t)$ is taken $\delta(t-0^+)$ in $L_+$ approach above). The $L_+$ transform of $f'(t)$ obtained through the last Eq. (6.27) also turns out to be zero from Eq. (6.2).

Now, for the $L$–approach, the transform of $f'(t)$ from the derivative rule is:

\[ L \{ f'(t) \} = s \cdot L \{ f(t) \} - f(0^+) = s \cdot \frac{1}{s} - 0 = 1 \]

(6.28)
However, the derivative $f'(t)$ and, hence, the $\mathcal{L}$ transform from the product rule becomes undefined in the domain $[0^-, \infty)$ as shown below:

$$f'(t) = u(t)\delta(t) + u(t)\delta(t) = \text{undefined}$$  \hspace{1cm} (6.29)

(as $g(t) = u(t)$ is non-smooth at origin)

So, the two ways don’t always give the same results for the $\mathcal{L}$ transform. Also, in Case (b), the function $|t|u(t)$ is the same as $t.u(t)$, and, so, the two possess the same $\mathcal{L}_+$ Laplace transform of their derivatives. However, this is not true of the $\mathcal{L}$ Laplace transform. The same happens for Case (c). Hence, it is seen that the $\mathcal{L}$ transform depends not only upon $f(t)$ but also upon $g(t)$, i.e., on how the initial state has been achieved, whereas, the $\mathcal{L}_+$ approach is straightforward and only depends on $f(t)$ and not on its history past the origin. Hence, the $\mathcal{L}$ approach suffers from the inherent inconsistency for the systems involving initial jump discontinuities marked at the outset of Section 6.1.

### 6.3.2 Initial value theorem ($\mathcal{L}_+$ approach)

The present study further examines whether the logic of the $\mathcal{L}_+$ transform of the singular input terms becoming zero presented above is valid for the application of the initial value theorem based on the $\mathcal{L}_+$ approach for the step response, in general. The initial value theorem based on the $\mathcal{L}_+$ approach for the output variables $y$ and $\dot{y}$ becomes:

$$y(0^+) = \lim_{s \to \infty} sy(s) \quad \& \quad \dot{y}(0^+) = \lim_{s \to \infty} s\{\mathcal{L}_+\dot{y}(t)\} = \lim_{s \to \infty} s\{sy(s) - y(0^+)\}$$  \hspace{1cm} (6.30)

Note the appearance of $y(0^+)$ term here. If the U-tube manometer of Section 6.2 or the system considered in Section 2.2.2 are $\mathcal{L}_+$ Laplace transformed by setting the transform of the derivative of the step input, i.e., $\mathcal{L}_+\{\dot{x}(t)\} = 0$ according to Eq. (6.2), it can be easily verified that Eqs. (6.30) are satisfied. Application of these works out (details not shown for brevity) to the identities:

$$y(0^+) = y(0^+), \text{ and } \dot{y}(0^+) = \dot{y}(0^+)$$  \hspace{1cm} (6.31)
Hence, it is seen that the initial value theorem using the \( L^+ \) approach is satisfied. So, the \( L^+ \) approach is again consistent.

### 6.3.3 Further discussion

Some more fallacies and confusion present in the literature are discussed. Consider a system equation that is analogous to the U-tube manometer example, i.e., Eqs. (6.3) (Grizzle, 2004):

\[
1. \dot{y} = -1. \dot{x} - 1. y + 1(\dot{x}) \tag{6.32}
\]

Advocating in favor of the \( L^- \) approach in his article, this study has reported that since for a step input in \( x \), the \( L^- \) Laplace transform of \( \dot{x} \) through Eq. (6.2) is zero; the solution of the system Eq. (6.32) from the \( L^+ \) approach is identically zero as long as the initial values of the output and its derivative are zero. On this very basis, it has advocated against the use of \( L^- \) form of the Laplace transform. This is examined here.

A physical system representing the system equation is that of a U-tube manometer (but, now, \( y \) becomes the column velocity and \( x \) the pressure applied). On comparison with the U-tube manometer, the zero pre-initial values would mean initial rest conditions as \( y \) is analogous to velocity, so, \( y(0^-) = 0 \) and \( \dot{y}(0^-) = 0 \). Nevertheless, even for these pre-initial rest conditions, the discontinuity analysis of initialization A for the step input above shows that the post-initial condition \( \dot{y}(0^+) \neq 0 \) from Eq. (6.9), whose right hand side is not zero because of the step input. On subsequent solution through the \( L^+ \) approach, this non-zero post initial condition wouldn’t yield identically zero dynamic solution as has been claimed by the study in question. This would have happened only if, hypothetically, \( \dot{y}(0^+) = 0 \), which is, in fact, not so. The reason for this confusion may be the failure to keep a physical analogy in mind. This can be very easily verified from the following preliminary physical reasoning. If one starts from rest, and introduce a step input by applying pressure, then initial acceleration can’t be zero, i.e., \( \dot{y}(0^+) \neq 0 \).

Still though, if one considers the hypothesis of the post-initial acceleration becoming zero then the column doesn’t move at all (it was at rest at \( t \to 0^- \)), and, then,
again, the $L_+$ approach would predict the right “identically zero” solution. The same argument applies to an analogous RLC series circuit, with current $I$ analogous to the velocity and applied voltage $E$ to the applied pressure. If a voltage is applied, the voltage drop across the inductor element can’t be zero, so, $\dot{I}(0^+) \neq 0$. Hence, yet again the $L_+$ approach wouldn’t yield the “identically zero” solution as has been claimed by the study in question.

Lundberg et al. (2007), also, advocates the use of the $L$ approach. It affirms that, in general, the $L_+$ approach is cumbersome as the determination of the post-initial conditions either requires applications of physical balances, or the technique of impulse matching (see Section 2.1). However, for the cases studied in the literature, it is shown above that the post-initial conditions can be easily and rightly calculated from the initialization A procedure of the proposed methodology. Having obtained the post-initial conditions, the correct solution is subsequently obtained from the $L_+$ approach; thus, avoiding the inconsistencies and inaccuracy inherent in the $L$ approach (these inconsistencies are marked at the outset and are discussed in Section 6.3). Furthermore, there is no real need for carrying out the analysis presented in the recent contributions based on the demanding and complicated framework of generalized functions. Additionally, the convoluted approaches such as that of mathematical transformations of equations to eliminate the singular terms presented by Makila (2006) (see Appendix I) are also not needed.

6.4 Outcomes Revisit

It is shown that the post-initial conditions obtained through initialization A are invariably the same as those obtained through the $L$ approach. The subsequent solutions resulting through these post-initial conditions are consistent with that obtained through the $L_+$ or initialization B approach for the step inputs, and for the simple cases under impulse inputs. However, the $L$ and initialization A approaches are inconsistent and less exact, especially, for the nontrivial cases under impulse inputs.
Nevertheless, the $L_+$ approach provides consistent results and satisfies the initial value theorem even in the presence of the non-zero pre-initial values, the non steady state pre-initial conditions, the input functions containing singularities, and more relevantly for the nontrivial chemical engineering systems containing interacting material, thermal, and mechanical energy capacities under impulse inputs. So, without requiring the convoluted approaches, and the demanding machinery of generalized functions, the application of our proposed framework finds a workable solution to the problem of $0^-$ and $0^+$ in the initial conditions of the responses from a direct approach perspective. Further, it reveals the inconsistencies inherent in the $L_-$ approach for the nontrivial chemical engineering cases under impulse inputs. The treatment proposed, thus, addresses a fallacy reported in systems engineering by many authors (Grizzle, 2004; Lundberg et al., 2007; Stefani et al., 2002).

The methods proposed in this thesis are direct, avoid the complicated framework of generalized functions, address the singularities, and account for the inconsistencies in initial conditions. Further, they are general, and are equally applicable to the nonlinear and linear systems for the step and impulse inputs. The conformity of the framework of the methodology presented here to the established approaches for the linear systems once again demonstrates its validity. The same happened in Sections 5.3 and 5.4 of Chapter 5, wherein the post-initial conditions obtained from the proposed framework conformed to the definitely known post-initial conditions of the nonlinear cases.