4 Effects of Initial Discontinuities

Initial conditions of the response to the inputs with jump discontinuities are, generally, not obvious. Because of the discontinuity, the initial conditions of the response have to be selected circumspectly. An incorrect choice of the initialization route leads to a solution inconsistent with the physical realities of the system. To address and reveal these inconsistencies, a methodology based on two alternative routes of initialization is presented in this chapter. This methodology is implemented in the Laplace domain for the linear systems in Section 4.2 (Ahuja, 2010). Following this, the methodology is extended in the time domain, and is implemented to the nonlinear systems in Section 4.3 to study the qualitative effects of discontinuities on the solution profiles. These effects are validated in Sections 4.4 through 4.8 (Ahuja, 2011). However, a comprehensive, methodological framework for the analysis and direct initialization to quantitatively describe the response for the nonlinear systems to the singular inputs shall be worked out in the next chapter.

4.1 Systems with Differentials of the Input Function

Models considered in the last two chapters were the systems represented by nonlinear second-order ordinary differential equations with terms containing differentials of the input function (see, e.g., Eqs. 3.9, 3.33, 3.42, etc.). Solution profiles of these systems exhibited initial jump discontinuities under the aforesaid inputs, which are not exhibited by the systems not containing the differentials of the input term (standard second-order systems). Besides, as mentioned at the outset of Chapter 2, these models include singularities contained in the differentials of the input terms of the equation even for the step perturbation and, thus, exhibit discontinuities in the initial slopes of the response profiles. Their solution profiles were discussed at length in Section 3.9 of the last chapter and can be observed in Fig. 4.1 below. However, the methodology presented here is general and is applicable to other lumped-parameter models including the standard systems.
4.1.1 Methodology of the discontinuity analysis

A schematic diagram of the methodology for the analysis of initial discontinuities for the linear systems is outlined in Fig. 4.1 for the step and impulse inputs. Discontinuities are analyzed for the linear and the nonlinear systems by checking whether or not the input discontinuity takes into account the initial discontinuities in the output variables. As seen from Fig. 4.1, there are essentially two alternative initialization routes for addressing the 

![Schematic showing methodology for analysis and initialization of systems containing differential terms of step and impulse inputs, where X, Y are deviation variables for input and output; ∆X, M are magnitudes of step and impulse inputs, respectively; A through E are constants.](image)

**Fig. 4.1** Schematic showing methodology for analysis and initialization of systems containing differential terms of step and impulse inputs, where X, Y are deviation variables for input and output; ∆X, M are magnitudes of step and impulse inputs, respectively; A through E are constants.

4.1.1 Methodology of the discontinuity analysis

A schematic diagram of the methodology for the analysis of initial discontinuities for the linear systems is outlined in Fig. 4.1 for the step and impulse inputs. Discontinuities are analyzed for the linear and the nonlinear systems by checking whether or not the input discontinuity takes into account the initial discontinuities in the output variables. As seen from Fig. 4.1, there are essentially two alternative initialization routes for addressing the
impulse function. The first is to include impulse in the cause (i.e., the input function), i.e., Route (a) with the all the initial conditions of the input and output evaluated at $t \to 0^-$; and the second, to include impulse in the effect (The effect of the impulse is included in the output functions, and the Laplace transform of the input impulse function is zero according to Eq. (2.15)), i.e., Route (b) with the input and all the initial conditions evaluated at $t \to 0^+$. For the step response also, the initialization Route (a) implies taking all the initial conditions of the inputs, outputs, and their derivatives at $t \to 0^-$, and the initialization Route (b) implies taking all the initial conditions of the inputs, outputs, and their derivatives at $t \to 0^+$ (Note that by doing so for the step input for $X$, the $(dX/dt)$ term becomes zero and, thus, all the singularities are included in the effect). Initial conditions at $t \to 0^+$ corresponding to Route (b) are based on the application of physical balances to the system conditions at the time of singularity. For the linear systems, Route (a) corresponds to the $L_-$ approach, whereas Route (b) corresponds to the $L_+$ approach. The Laplace domain dynamics will be treated in detail separately in Chapter 6 by the application of the framework of Chapter 5.

CSTR and gravity-flow tank under feed rate perturbation, non-isothermal CSTR under jacket temperature perturbation, etc., are some nonlinear systems with derivatives of the input function, the response profiles for which would be affected by the discontinuities. For such systems, an initial discontinuity in the response identified on applying physical balances (mass/momentum etc.) through Route (b) won’t get accounted for in the initial discontinuity of the stimulus through Route (a). Hence, Route (b) won’t reduce to Route (a), and the value of this discontinuity in the response would be required for the correct solution. Simply knowing the magnitude of the input would not be sufficient to obtain the correct solution to these systems. So, these systems are said to be affected by discontinuities throughout the text as they yield inconsistent initial conditions.

However, some other systems such as the U-tube manometer, first unit of the two interacting units-in-series, jacketed stirred tank heater under feed rate perturbation, etc. also give rise to ODE with the derivative of the input. But discontinuities of such systems get accounted for as Route (b) would reduce to Route (a). A few nonlinear systems may undergo order reduction to standard linear systems (i.e., containing no derivative term of the input) upon linearization due to pole zero cancellation, and may turn into being
unaffected by discontinuities (e.g., Eqs. 3.40 and 3.77). However, some standard systems exhibit numerator-dynamics form due to the effect of initial discontinuities (Eq. 3.12). These kinds of special cases shall be treated in Section 5.12.1 of Chapter 5. The present methodology investigates the existence and validation of the qualitative effects of initial discontinuities. Validation will be done by the comparison of experimental and simulated data of non-linear models of flow-level tanks (Ahuja, 2011).

4.2 Application to the Linear Time-invariant Systems

The methodology introduced above is aimed at a simple and effective treatment of linear systems containing derivatives of the input terms, especially, the ones that are inherently of this kind (i.e., of Sections 2.1, 2.2, and 3.8). Now, the linear U-tube manometer, isothermal CSTR, and the linearized flow-level tank models will be used to illustrate the procedure. It is intended to extend the procedure later to the maiden nonlinear systems and validate it by the corresponding nonlinear flow-level tank models.

4.2.1 Example 1

**Impulse input**

Consider the U-tube manometer presented in Section 3.8.1 of the last chapter. The system is at a state at time \( t \to 0^- \). An impulse perturbation of magnitude \( M \) (Pa-s) in the applied pressure difference \( \Delta P \) is imposed on its two legs. Input function \( \Delta P(t) \) and the initial conditions need to be specified in order to solve this equation, viz., the values of \( Y, U, \) and \( \frac{dU}{dt} \) at \( t \to 0 \). Discontinuities are analyzed using the following Laplace transformed form of the system Eq. (3.42) and by referring to Fig. 4.1.

\[
m[s^2U(s) - sU(0) - \dot{U}(0)] + a[U(s) - U(0)] + bU(s) = sA\Delta \bar{P}(s) - A\Delta \bar{P}(0)
\]  

(4.1)

Applying physical balance, i.e., momentum balance in this case for Route (b); as an impulse is applied, the effect of the impulse is to give initial velocity to the liquid column. Consequently, there is discontinuity in \( U(0) \). But no such discontinuity exists in \( Y(0) \) due to inertia of the column and the resistance offered, so \( Y(0^+) = 0 \). The value of
\( U(0^+) \) is obtained by noting that the magnitude of the applied impulse force \( AM \) equals the change in the initial momentum, thus the initial discontinuity in \( U(0) \) is given by:

\[
mU(0^+) = AM \tag{4.2}
\]

As the velocity increases suddenly, its derivative at the origin would contain a singularity. To relate \( U(0^+) \) and \( \dot{U}(0^+) \), momentum balance Eq. (3.41) is used at \( t \to 0^+ \):

\[
m\dot{U}(0^+) = -aU(0^+) - bY(0^+) + A\Delta\overline{P}(0^+) = -aU(0^+) - 0 + A\Delta\overline{P}(0^+) \tag{4.3}
\]

Impulse input, thus, causes an initial discontinuity in \( U(t) \) and, hence, \( \dot{U}(t) \). The following analysis shows that these discontinuities are accounted for in the magnitude of the impulse input. The aforesaid two Routes (a) and (b) are applied to analyze Eq. (4.1):

- Route (a): take initial values \( U(0) = U(0^-) = 0 \) and \( \dot{U}(0) = \dot{U}(0^-) = 0 \), with \( \Delta\overline{P}(0^-) = 0 \), and \( \Delta\overline{P}(s) = M \)
- Route (b): take initial values as \( U(0) = U(0^+) \) and \( \dot{U}(0) = \dot{U}(0^+) \) (i.e., Eqs. (4.2) & (4.3)), with \( \Delta\overline{P}(0) = \Delta\overline{P}(0^+) \) and \( \Delta\overline{P}(t) = M\delta(t-t_o), \ t_o = 0^+, \ \Delta\overline{P}(s) = 0 \)

Now, if the given impulse perturbation is to account for all the initial discontinuities, Route (b) should reduce to Route (a). For Route (b), the terms on the left of Eq. (4.1) are evaluated in the following two equations:

\[-smU(0^+) = -sAM \quad \text{(From Eq. (4.2))}\]

\[-m\dot{U}(0^+) - aU(0^+) = +aU(0^+) - A\Delta\overline{P}(0^+) = -A\Delta\overline{P}(0^+) \quad \text{(Using Eq. (4.3))}\]

Placing these values in Eq. (4.1) and following Route (b), it is seen that Eq. (4.1) simplifies to one corresponding to Route (a). Thus, the initial discontinuities in \( U(t) \) and \( \dot{U}(t) \) are accounted for in the magnitude of the impulse input.
Step input

Analyzing the initial discontinuities for a step input in the applied pressure, again by applying momentum balance (for Route (b)) reveals that $U(0^+) = 0$. There is no discontinuity in the initial velocity due to the inertia of the liquid column. Discontinuities are there in $\Delta P(0)$ and $\dot{U}(0)$, but the force applied is equal to mass times the acceleration:

$$m\dot{U}(0^+) = A\Delta \dot{P}(0^+)$$

So, employing Route (b) of Fig. 4.1 for the step response case implies that for Eq. (4.1), the initial discontinuities in $\Delta P$ and $U$ exactly cancel each other. The same happens for Route (a) by taking all the initial conditions at $t \to 0^-$, as all the initial conditions are zero. Hence, Route (b) reduces to Route (a), and both the routes are consistent and would yield correct solution profiles.

It is, thus, observed that the initial discontinuities of the output functions are accounted for in the discontinuous input functions. Thus, the transfer function approach is valid, and the transfer function form can be written for the U-tube manometer system for the step as well as impulse inputs. The transfer function approach assumes that the initial conditions of all the input and the output functions are zero.

4.2.2 Example 2

Consider the linearized model of gravity-flow tank presented in Section 3.5.1 of the last chapter. Input function $Q(t)$ and two initial conditions need to be specified in order to solve this equation, viz. $H$ and $dH/dt$ at $t \to 0$. Discontinuities are analyzed using the Laplace transformed system Eq. (3.27) and Fig. 4.1.

The system is at a state at time $t \to 0^-$. Volumetric feed flow rate $q$ is perturbed impulsively at $t \to 0^+$. The ideal impulse input can be realized by plunging a measured amount of liquid into the tank, all in one go (with no change in the inflow rate $q$), with the magnitude of disturbance $M$ ($\text{m}^3$) equal to the volume of liquid added.

Material balance applied at this instant of perturbation for Route (b) reveals that the impulse in the volumetric feed rate equivalently results in the initial increase in the liquid holdup in tank. Hence, there are two values of level at zero time $t$, at $t \to 0^-$ and at


\( t \to 0^+ \). This leads to initial discontinuity in level \( h(0) \) and, hence, in the efflux velocity \( u(0) \) for discontinuous impulse disturbance \( q(t) \). The initial change in level \( h(0) \) equals magnitude of the impulse (i.e., volume of liquid added), divided by the area of tank:

\[
H(0^+) = \frac{M}{A_T}
\]  

(4.4)

Moving further with the physical principles, direct dependence of the outflow rate on the head is utilized. With the sudden change in level, simultaneously there is a resultant sudden increase in the efflux velocity in the pipe. Also, since there is an initial discontinuity in the level as the level increased suddenly, its derivative at the origin, \( H'(0^+) \), would contain a singularity, i.e., an impulse. The increase in the efflux velocity is related to \( H'(0^+) \) using Eq. (3.25) at \( t \to 0^+ \):

\[
H'(0^+) = \frac{Q(0^+)}{A_T} - \frac{A_r}{A_T} U(0^+)
\]  

(4.5)

Impulse input, thus, causes an initial discontinuity in \( H(t) \) and, hence, \( U(t) \) (and \( H'(t) \)). The following analysis shows that the discontinuity in \( H(0) \) is accounted for in the magnitude of the impulse input, but the same is not true for \( H'(0) \). In this system, the aforesaid two routes are applied to analyze the Laplace transformed Eq. (3.27):

- Route (a): take initial values \( H(0) = H(0^-) = 0 \) and \( H'(0) = H'(0^-) = 0 \), with \( Q(0) = Q(0^-) = 0 \) and \( Q = M\delta(t) \) or \( Q(s) = M \)

- Route (b): take initial values as \( H(0) = H(0^+) = M/A_T \) and \( H'(0) = H'(0^+) \) (i.e., Eqs. (4.4) & (4.5)), with \( Q(0) = Q(0^+) \) and \( Q(s) = 0 \)

Now, if the introduced impulse perturbation was to account for all the initial discontinuities, Route (b) would reduce to Route (a) for Eq. (3.27). This is checked by the comparison of the two routes in the following equations, where the left of the equations/in-equations corresponds to Route (b), and their right corresponds to Route (a):

\[
sH(0^+) = KsQ(s) \quad \text{(Because } A_T H(0^+) = M \text{ and } K = 1/A_T)\]

\[
\frac{B}{A} H(0^+) = K \frac{B}{A} Q(s) \quad \text{(Same reason)}
\]
Hence, the discontinuity in $H(0)$ is accounted for in the magnitude of the impulse input. Whereas,

$$H'(0^+) - \frac{Q(0^+)}{A_T} \neq 0 \text{ (From Eq. (4.5))}$$

Hence, the discontinuity in $H'(0)$ could not be accounted for in the magnitude of the impulse input. Route (a), which corresponds to the transfer function approach, assumes:

$$H'(0^+) = \frac{Q(0^+)}{A_T}, \text{ or } U(0^+) = U(0^-) = 0,$$

which are not true because of simultaneous increase in the initial level and velocity in the pipe.

Route (b), which is a valid one, yields the following equation for Eq. (3.27):

$$s^2 H(s) + \frac{B}{A} s H(s) + \frac{C}{A} H(s) = K \left( s M + \frac{B}{A} M - A_T U(0^+) \right)$$

(4.6)

It cannot be reduced to Route (a) and the transfer function form since $U(0^+) \neq 0$. Route (a) would yield inaccurate solution profiles. The transfer function approach, thus, does not work for this system under impulse perturbation. Also, the value $H'(0^+)$ or $U(0^+)$ is required for the correct solution of the system Eq. (4.6) for obtaining the impulse response, and paradoxically, it shall not be known unless the profile is obtained.

Now analyzed are the initial discontinuities for the step response of gravity-flow tank. Physical principles reveal that the discontinuities are absent in $H(0)$ and $U(0)$ because of inertia the pipe liquid. Hence, $H(0^+) = H(0^-) = 0$ and $U(0^+) = U(0^-) = 0$. Thus, discontinuity $H'(0^+)$ can be calculated from Eq. (3.25) by placing $Q = Q(0^+)$ at $t \to 0^+$.

$$H'(0^+) = (1/A_T)Q(0^+) = KQ(0^+)$$

(4.7)

Now, Eq. (3.27) is considered and is analyzed using Fig. 1. Following Route (b), all the initial values of all the input and output functions are taken at $t \to 0^+$, these values have been worked out above. Doing this, it is seen that the $H'(0)$ term cancels with the $KQ(0)$ term using Eq. (4.7). So, only the transfer function Eq. (3.28) is left. The same happens for Route (a) by taking all the initial conditions at $t \to 0^+$, as all the initial conditions are
zero. Hence, the Route (b) reduces to the Route (a), and both the routes are consistent and would yield correct solution profiles.

It is, thus, observed that the initial discontinuities of the output functions are accounted for in the discontinuous input functions. Thus, the transfer function approach is valid, and the transfer function Eq. (3.28) can be written for the gravity-flow tank system for step inputs.

4.2.3 Example 3

The first of the two interacting liquid-level tanks in series of Section 3.6 is, now, analyzed. The linearized model of Section 3.6.1 is used for the analysis.

The system is at a state at time $t \to 0^-$. Volumetric flow rate is perturbed impulsively at $t \to 0^+$. It can be introduced as in the last section, which leads to discontinuity in $H_1(0)$. The initial value of $H_1$ at $t \to 0^+$ is, in effect, the same as the magnitude of the impulse, i.e., the volume of liquid added divided by the area:

$$H_1(0^+) = \frac{M}{A_1} \tag{4.8}$$

Physical principles reveal that discontinuities are there for both $H_1(0)$ and $H_1'(0)$ but are absent in $H_2(0)$, hence, $H_2(0^+) = 0$. The dynamic lag of the first tank prevents a discontinuity in the second tank level. Impulse, however, causes a discontinuity in outflow rate from the first tank $Q_{21}$. Following analysis shows that the discontinuity of $H_1(0)$ and $H_1'(0)$ are accounted for in the magnitude of the impulse input. In this case, the two alternative routes to solve Eq. (3.34) become:

- **Route (a)**: take initial values as $H_1(0) = H_1(0^-) = 0$, $H_1'(0) = H_1'(0^-) = 0$, with $Q(0) = Q(0^-) = 0$ and $Q = M\delta(t)$ or $Q(s) = M$.

- **Route (b)**: take initial values as $H_1(0) = H_1(0^+) = M/A_1$ from Eq. (4.8) and $H_1'(0) = H_1'(0^+) = Q(0^+)/A_1 - H_1(0^+)/A_1 R_1$ (obtained using Eqs. (3.30(a)) and (3.32(a))), with $Q(0) = Q(0^+)$ and $Q(s) = 0$.
Route (b) reduces to Route (a) as seen by substituting the following two equations in Eq. (3.34), where the extreme L.H.S. of the two equations correspond to Route (b), and their extreme R.H.S. correspond to Route (a).

\[s\tau_1\tau_2 H_1(0^+) = s\tau_1\tau_2 M / A_1 = s\tau_2 R Q(s)\]

\[-\tau_2 H_1(0^+) + (\tau_1 + \tau_2 + A_1 R_2)H_1(0^+) + \tau_2 R Q(0^+) = (R_1 + R_2)Q(s)\]

From the above analysis, it is seen that the two routes are the same.

When the step response is considered, there is a discontinuity in \(Q(0)\) and \(H_1'(0)\). No discontinuity exists for \(H_1(0)\) and \(H_2(0)\). Consequently, \(H_1(0)\) and \(H_2(0)\) both are zero. To solve Eq. (3.34) through Route (b), take initial value, \(H_1'(0) = H_1'(0^+)\). Initial value of \(Q\) is \(Q(0) = Q(0^+) = A_1 H_1'(0^+)\) from Eq. (3.30(a)). This implies that the initial discontinuities in \(Q\) and \(H_1\) exactly cancel each other in Eq. (3.34). Thus, Route (b) reduces to Route (a). The system model (3.34) therefore remains unaffected by initial discontinuities both for the step and the impulse perturbations.

### 4.2.4 Example 4

Consider an impulse input to the feed stream concentration \(C_{Ao}\) to the constant density constant holdup CSTR of Section 3.2. Let \(M\) (mol)(s)/(m^3) be the magnitude of the impulse perturbation. This perturbation is equivalent to plunging \(Mv_o\) (mol) of pure liquid A into the tank suddenly. The analysis could have been carried out in the above manner. However, the analysis of Eqs. (3.7) and (3.8) are, now, done directly without combining the two equations, \textit{a priori}. This procedure would prove to be more convenient for the nonlinear systems that are to be described in the next section.

Application of material balance at the perturbation instant (for Route (b)) for component A reveals that the sudden insertion of \(Mv_o\) (mol) of pure liquid A into the tank causes the same initial increase in the number of moles of A in the tank, this increase divided by the holdup volume (constant), gives the value of initial discontinuity in the concentration of A in the tank, \(C_A\). This leads to (in terms of deviation variables):

\[\bar{C}_A(0^+) = \frac{Mv_o}{V} \quad (4.9)\]
The Laplace transformed Eqs. (3.7) and (3.8) respectively are:

\[ sC_R(s) - C_R(0) = -(v_o / V)C_R(s) + kC_A(s) \]  
\[ sC_A(s) - C_A(0) = (v_o / V)C_{Ao}(s) - (v_o / V + k)C_A(s) \]

Eq. (4.11) is first analyzed for \( C_A \) as it doesn’t have \( C_R \). It can be seen that the discontinuity in its second term \( C_A(0) \) through Route (b) (given by Eq. (4.9)), is accounted for in the magnitude of the impulse input in the third term of Eq. (4.11) through Route (a). All other terms are same in the two routes. Thus, Eq. (4.11) is not affected by the discontinuity in \( C_A \).

Eq. (4.10) is now analyzed. Moving further with physical principles (Route (b)), as the holdup is constant, concentrations of \( R \) and \( I \) fall suddenly with the sudden initial rise in the concentration of \( A \) in the tank. The following constant density condition can be applied in the deviation form at \( t \to 0^+ \) for the impulse input, considering that pure \( A \) is put directly into the tank with no change in \( C_{Ao} \) and volume \( V \), this gives:

\[ [C_R(0^+)]M_R + [C_A(0^+)]M_A + [C_I(0^+)]M_I = 0 \]  
\[ \text{or, } C_R(0^+) = -[MC_{Ao} / V]M_A / M_R - [C_I(0^+)]M_I / M_R ) \neq 0 \]

which yields a non-zero value of the initial discontinuity in the concentration of \( R \) in the tank, \( C_R \). \( M_A \), \( M_R \), and \( M_I \) are the molecular weights of \( A \), \( R \), and \( I \), respectively. This discontinuity in \( C_R(0) \), i.e., \( (C_R(0^+) - C_R(0^-)) \), in the second term of Eq. (4.10) through Route (b) is not accounted for in the initial discontinuity of impulse input through Route (a), as there is no impulse input term in Eq. (4.10). If Routes (a) and (b) were to give the same results, \( C_R(0^+) \) would have been zero, and Route (a) would have reduced to Route (b), but this is clearly not so as seen from Eq. (4.13). The fourth term of Eq. (4.10) containing \( C_A \) doesn’t influence the analysis because \( C_A \) was not affected by the discontinuity and had the same value through Routes (a) and (b) as shown above. The solution of Eq. (4.10) would, thus, be affected by the discontinuity. This discontinuity in \( C_R(0) \) results in the modified form of transfer function Eq. (3.12) for \( C_R \), which was marked by the appearance of the numerator-dynamics form.
Also, one needs to know the value of \( C_R(0^+) \) to solve the system, and paradoxically, it shall not be known unless the profile is obtained. This leads to the undetermined value of initial discontinuity in \( C_R \). Thus, the impulse response of this system, too, is affected by initial discontinuities. Following the treatment of this section, it can be shown that the step response of this system, however, remains unaffected of initial discontinuities.

### 4.3 Application to the Nonlinear Systems

Extending the methodology of Section 4.2.4 for the nonlinear systems, initial discontinuities will be analyzed again by checking whether the discontinuity in the stimulus takes into account the initial discontinuities in the response. Recall that Route (b) registered all the initial discontinuities in the output variables, whereas, Route (a) was valid only in case the discontinuous input takes all the initial discontinuities of the outputs into account. However, the Laplace transform approach is not applicable now; hence a time domain strategy is to be developed. Revision in the methodology is, hence, essential. Also, deviation variables cannot be used for the analysis in the nonlinear equations. Considering the aforesaid facts, a revised strategy is followed and is illustrated in the following cases (Ahuja, 2011).

#### 4.3.1 Example 1

First consider impulse input in volumetric feed rate \( q \), as in the Section 4.2.2, to the gravity flow tank system represented by the non-linear Eqs. (3.25) and (3.26). Corresponding to Eq. (4.4), mass balance reveals that initial change in level \( h(0) \) equals magnitude of the impulse (i.e., volume of liquid added), divided by the area of tank, thus:

\[
h(0^+) = h(0^-) + \frac{M}{A_T}
\]  

(4.14)

For the analysis of initial discontinuities, the methodology described in Fig. 1 for the case of impulse response is again followed, where the first Route (a) implies to include impulse in the cause (i.e., the input time varying forcing function), and the second Route (b) implies to include impulse in the effect (i.e., in the initial values of the output
functions). Mathematically, Route (b) implies to take initial conditions of all the output variables and their derivatives at \( t \to 0^+ \), with input \( q(t) = q(0^-) \); Route (a) implies to take the initial values of the output variables at \( t \to 0^- \), with the input at \( t \to 0^+ \) given by:

\[
q(t) = q(0^-) + M\delta(t) \tag{4.15}
\]

- Following Route (a), the disturbance variable is given by Eq. (4.15). The extent of initial discontinuity in the third term \((q/A_T)\) of Eq. (3.25) is calculated using the integral form of Eq. (4.15) at \( t \to 0^+ \),

\[
\int_{0^-}^{0^+} \frac{q(t) - q(0^-)}{A_T} dt = \int_{0^-}^{0^+} \frac{M\delta(t)}{A_T} dt = \frac{M}{A_T}
\]

The integral of the impulse input is \( [M\delta(t).dt] = M[\delta(t).dt] = M \).

- Following Route (b), the perturbation variable volumetric feed rate \( q \) becomes:

\[
q(t) = q(0^-). \]

In the integral form of Eq. (3.25), the integral of the first term at the time of introduction of input, i.e., at \( t \to 0^+ \) gives:

\[
\int_{0^-}^{0^+} \frac{dh}{dt} dt = h(t) - h(0^-) = h(0^+) - h(0^-)
\]

Thus, the extent of initial discontinuity of the level of the liquid in the tank, as given by Eq. (4.14), is:

\[
h(0^+) - h(0^-) = (M/A_T)
\]

This implies that into the integral form of Eq. (3.25), the introduction of the impulse disturbance, through Route (a), causes the third term, i.e., the feed rate term \((q/A_T)\), to change by an amount \((M/A_T)\). This happens to be the same as the initial discontinuity in the first term (i.e., the tank level from Eq. (4.14)) of Eq. (3.25) from Route (b). In other words, the initial discontinuity in the input function \( q(t) \) in the third term through Route (a), is equal to, the initial discontinuity in the output variable \( h \) of the first term through Route (b). All other terms have no discontinuity as their integrands are limited, which don’t contribute anything to the integral form of Eq. (3.25). Thus, Eq. (3.25) is unaffected by initial discontinuities and can be solved through either of the two
routes. However, since Eqs. (3.25) and (3.26) are coupled through the parameter $u_p$; analysis of Eq. (3.26) is also required. That is done as follows.

The discontinuity in $u_p(0)$, i.e., $(u_p(0^+) - u_p(0^-))$, in the first term of Eq. (3.26) through Route (b) is not accounted in the initial discontinuity of impulse input through Route (a) as there is no impulse input term in Eq. (3.26). Note that there is a change in the initial velocity in pipe as seen in Section 4.2.2, thus:

$$u_p(0^+) - u_p(0^-) \neq 0 \quad (4.16)$$

Route (b), therefore, doesn’t reduce to Route (a). Hence, the impulse response of the level and velocity of this system gets affected by the initial discontinuities. Hence, Route (a) is inconsistent with the physical principles applied to the system as expressed in Eq. (4.16). Also, one needs to know the value of $u_p(0^+)$ to solve the system, and, paradoxically, it shall not be known unless the profile is obtained.

Now, for a *step* input, there is no discontinuity in $h(0)$ and $u_p(0)$ because of inertia of liquid and resistance to flow in the pipe, i.e., the initial rise in tank level is zero, and, hence, no initial change of velocity in the pipe. But there is a discontinuity in $h'(0)$; nevertheless, discontinuity in $h'(0)$ is not needed in Eqs. (3.25) and (3.26). Hence, the step response of the system is not affected by the initial discontinuities.

Thus, only the impulse response of this system is affected by initial discontinuities and exhibits a paradox. Numerical solution of this nonlinear system is discussed in Section 4.6.

### 4.3.2 Example 2

Consider an impulse perturbation in the volumetric feed rate $q$ to the first tank of the two interacting tanks-in-series. Application of physical principles gives the following two equations. The level in the second tank is not immediately affected by the change in the level of the first tank, because of the presence of the dynamic lag in the first tank.

$$h_1(0^+) - h_1(0^-) = M/A_1 \quad (4.17)$$

$$h_2(0^+) - h_2(0^-) = 0 \quad (4.18)$$

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Following the same procedure as above, it can be shown that the initial discontinuities from the two routes balance each other, and Route (b) reduces to Route (a). Hence, the system is not affected by the initial discontinuities unlike in the last case. The same happens for the step response too. This system doesn’t exhibit the paradox noted above.

4.3.3 Example 3

Consider an impulse input in the volumetric feed rate \( \nu_o \) to the constant density constant holdup CSTR of Section 3.2. Impulse input can be brought about by plunging a measured volume \( M \) (m\(^3\)) of pure liquid \( A \) into the tank. The magnitude of impulse is equal to the magnitude of liquid added. *It is further assumed that this magnitude is small enough (or the holdup is large enough) not to cause any change in the exit volumetric flow rate.* Refer again to the impulse response case of Fig. 1 for initial conditions; there would be discontinuities in the output variables \( C_A(0) \), \( C_R(0) \) and in the input variable \( \nu_o \). The system model described by Eqs. (3.7) and (3.8) is considered.

Application of material balance for component \( A \) at the initial instant for its use in Route (b) leads to the following. Impulse disturbance of magnitude \( M \) (m\(^3\)) in feed flow rate \( \nu_o \), is equivalent to introducing \( (MC_{A_o}) \) (moles) of pure liquid \( A \) directly into the tank, all in one go (with no change in feed stream flow or feed stream concentration). The sudden increase in the number of moles of \( A \) in tank divided by the holdup volume gives the increase in \( C_A(0^+) \), i.e., the initial concentration of \( A \) in the tank.

\[
C_A(0^+) = C_A(0^-) + \frac{MC_{A_o}}{V} \quad (4.19)
\]

Concentrations of \( R \) and \( I \) fall suddenly with the sudden rise in the concentration of \( A \). The constant density reaction condition can be applied in the deviation form at \( t \to 0^+ \) for the impulse input considering that pure liquid \( A \) is put directly into the tank with no change in \( C_{A_o} \) and volume, this gives \( C_R(0^+) \) as in Section 4.2.4:

\[
C_R(0^+) = C_R(0^-) - \frac{[C_A(0^+) - C_A(0^-)]M_A}{M_R} - \frac{[C_I(0^+) - C_I(0^-)]M_I}{M_R} \quad (4.20)
\]

\[
C_R(0^+) = C_R(0^-) - \left( \frac{MC_{A_o}}{V} \right) - \frac{[C_I(0^+) - C_I(0^-)]M_I}{M_R}
\]
This equation is obtained by the combination of the constant density condition and Eq. (4.19). Discontinuities are now analyzed using Eqs. (3.7) and (3.8), by referring to Fig. 1.

- Following Route (a), the perturbation variable volumetric feed rate $v_o$ in Eq. (3.8) becomes:

$$v_o(t) = v_o(0^-) + M \delta(t)$$

(4.21)

Initial discontinuity in the second term ($v_o C_{Ao}/V$) of Eq. (3.8) comes from Eq. (4.21):

$$\int_{0^-}^{0^+} \left( v_o(t) - v_o(0^-) \right) \frac{C_{Ao}}{V} dt = \int_{0^-}^{0^+} M \delta(t) \frac{C_{Ao}}{V} dt = \frac{MC_{Ao}}{V}$$

(4.22)

- Following Route (b), the perturbation variable volumetric feed rate $v_o$ in Eq. (3.8) is:

$v_o(t) = v_o(0^-)$. In the integral form of Eq. (3.8), the integral of the first term evaluated at time $t \to 0^+$ is equal to $\int_{0^-}^{0^+} \frac{dC}{dt} dt = C_A(0^+) - C_A(0^-)$. This extent of initial discontinuity in the tank concentration is: $(C_A(0^+) - C_A(0^-)) = (MC_{Ao}/V)$, as seen in Eq. (4.19).

This implies that in the integral form of Eq. (3.8), the introduction of the impulse disturbance causes the second term, i.e., feed rate term ($v_o C_{Ao}/V$) to change by an amount $(MC_{Ao}/V)$. This happens to be the same as the initial discontinuity in the first term of the integral form of Eq. (3.8). Hence, the initial discontinuity in the feed rate in the second term through Route (a) is equal to the initial discontinuity in concentration of the first term through Route (b). Note that no initial discontinuity in the exit rate in the third term is considered by the virtue of the small magnitude (or large holdup) assumption. Thus, the initial discontinuities in the output function are accounted for in the magnitude of the impulse input function. Thus, Eq. (3.8) is not affected by the discontinuities. However, the initial discontinuity in $C_R$, i.e., $(C_R(0^+) - C_R(0^-))$ in the first term of Eq. (3.7) through Route (b) is not accounted for in the initial discontinuity of the impulse input through Route (a) as there is no impulse input term in Eq. (3.7). As seen from Eq. (4.20),

$$C_R(0^+) \neq C_R(0^-)$$

(4.23)
If it were not so, the solution would not have been affected by discontinuities. To summarize, Eq. (3.8) can be solved by either of the two routes, whereas, Eq. (3.7) only through Route (b). Hence, Route (b) does not reduce to Route (a), and Route (a) is inconsistent with the facts elicited on the application of physical balances to the system as expressed in Eq. (4.23). The value $C_R (0^+) \text{ is, thus, required to solve the system Eq. (3.7)}$ as the impulse disturbance cannot take the discontinuity in $C_R (0)$ into account. Its value shall be estimated in the framework presented in the next chapter.

The impulse response is, thus, affected by the initial discontinuities. The step response of this system, however, remains unaffected of the initial discontinuities.

### 4.4 Validation of the Effects of Discontinuities on Nonlinear Models

The inconsistencies due to singular inputs as a result of the effect of the initial discontinuities discussed above were validated by the comparison of experimental and numerical data of non-linear models of flow-level tanks. Their level responses and draining time calculations clearly revealed these initialization effects. These are described in the following sections. For validation, two nonlinear flow-level tank systems, discussed in the last chapter, were considered, viz., gravity-flow tank and interacting level-tanks. Impulse response experiments were performed on these systems. The nonlinear differential equations describing these systems cannot be solved analytically. Hence, numerical solutions were obtained. Numerical experiments simulating the above systems were performed at different initial conditions. The initial condition was guessed initially and a tentative solution profile was obtained. Further guess trials were, then, made and the impulse solution profiles, thus obtained, were compared with the experimental data to substantiate the effect of initial discontinuities. Nonlinear models were considered to avoid the errors due to linearization, so that, the error deviations of the values of the simulated profiles ($y_i$) from that of the experimental ones ($b_i$) are reliable; root mean square deviations (RMSD) were calculated:

$$RMSD = \sqrt{\frac{\sum_{i=1}^{m} (b_i - y_i)^2}{m}}$$

(4.24)
As described in Section 4.8, the effects of initial discontinuities can also be validated by the calculations of the time required for emptying these flow-level tanks. It will be shown that the impulse solution profiles can be used to calculate the time required for emptying the flow-level tanks. Not accounting for the effect of initial discontinuities, however, will yield the same calculated value of the time required to empty a gravity-flow tank, irrespective of the initial level, which is obviously incorrect. Some other interesting features of calculations of time required to empty flow-level tanks shall be discussed. The prerequisites, i.e., the calculations of the over-damped and the under-damped profiles for the systems of interest have been discussed comprehensively in Section 3.9, Chapter 3 (Ahuja, 2010; 2011).

4.5 Experimental Section

Experiments on gravity flow tank were carried out in a set up made of steel consisting of a rectangular tank of dimensions 0.4 m × 0.4 m × 0.7 m with a horizontal pipe at the outlet of internal diameter 0.024 m and length 4.27 m (Fig. 3.1). To maintain constant inflow, inlet to the tank was obtained from the downstream of an overhead tank having a mechanism to maintain water at a constant level. The system was allowed to attain a steady state. Impulse disturbance was introduced by inserting a measured amount of water into the tank, all in one go. Level response was measured from the graduated transparent tube fitted to the tank. The steady inflow rate was 2.95×10⁻⁴ m³/s, initial level was 0.202 m and the magnitude of impulse disturbance was 1.5×10⁻² m³. Reynolds number was calculated and turbulent flow in the pipe ensured.

Experiments on interacting tanks were performed in a set up consisting of two interacting transparent acrylic graduated cylindrical tanks each of internal diameter 0.101 m and height 0.3 m (Fig. 3.2). The inlet water to the first tank came from an overhead tank similar to the one mentioned above. To evaluate the constants in the flow-head relationships of Eqs. (3.31), \(k_1\) and \(k_2\), four steady state runs were performed and the constants were calculated from the slopes of the flow versus head plots. The transient experiments were carried out in similar manner as in gravity flow tank and the level response of the first tank was recorded. The initial steady state conditions were as
follows. Inflow rate was $8.33 \times 10^{-6}$ m$^3$/s, the first tank level was 0.11 m, second tank level was 0.091 m and the magnitude of the impulse input was $5.0 \times 10^{-4}$ m$^3$.

4.6 Validation by Numerical Solution

To validate the results of the inconsistencies due to singularities, the above two systems, namely, gravity-flow tank and interacting level-tanks, were numerically solved from their non-linear models. Comparing the fit of experimental data for these systems substantiates the effect of initial discontinuities in this section. Numerical solutions were carried out in the following manner.

Nonlinear Eqs. (3.25) and (3.26) were employed for the gravity-flow tank solution. The system is affected by initial discontinuities (refer to Section 4.3.1). There are initial discontinuities in level, efflux velocity and the impulse input. To simulate the above experimental procedure, $q(t)$ was taken as a constant equal to its initial steady-state value, i.e., $q(t) = q(0^-)$ as that mentioned in Route (b) in Section 4.3.1, while the experimental values of $h(0^+)$ and $u(0^+)$ were needed instead of $h(0^-)$ and $u(0^-)$ respectively. The value of $h(0^+)$ is given by Eq. (4.14). Differential equations were solved using Polymath which employs Runge-Kutta-Fehlberg algorithm (RKF45) of initial step size $= 10^{-6}$ and truncation error tolerance $= 10^{-6}$ (Gerald and Wheatley, 2004). As mentioned, a tentative solution was first obtained as the value of $u(0^+)$ was paradoxically unknown. For doing this, the initial discontinuity in the velocity of fluid in the pipe was ignored, so, the initial velocity was assumed as $u(0) = u(0^-)$, as a first estimate for the simulation. Initial discontinuity in $u(0)$ was then incorporated and successive higher values of $u(0)$ were assumed for further guess trials.

To simulate the experimental behavior of the impulse response of the first tank of interacting tanks system, Eqs. (3.30) were used. The methodology of numerical solution was the same as in the preceding paragraph. However, the values of $k_1$ and $k_2$ in the flow-head relationship Eqs. (3.31) for the first and the second tank were needed. These were obtained from the data analysis of the steady state experimental runs. The steady state data of the four runs were plotted as $q_s$ values versus $\sqrt{h_{s_1} - h_{s_2}}$ values; and $q_s$ values versus $\sqrt{h_{s_2}}$ values at steady state, and it was found that these give a straight line plot.
passing through the origin, thus confirming the flow-head relationships. These values were found to be:

\[ k_1 = 6.10 \times 10^{-5} \text{ m}^2/\text{s} \quad \text{and} \quad k_2 = 2.76 \times 10^{-5} \text{ m}^2/\text{s} \quad (4.25) \]

For the non-linear model of interacting tanks system the initial discontinuities are accounted for in the magnitude of the impulse input (refer Section 4.3.2). Initial discontinuities are there in the impulse input and the level of the first tank, and in the input to the second tank, but are absent in the level and the outflow of the second tank. To solve this model numerically for impulse input in \( q(t) \) the volumetric flow rate was taken to be \( q(0^-) \) and initial values were taken as \( h_1(0^-) \), \( h_2(0^-) \), and \( q_2(0^-) \). Eqs. (4.17) and (4.18) were used; the rest of the procedure was the same as that followed for the gravity-flow tank system.

## 4.7 Results and Discussion

Tables 4.1, 4.2, Figs. 4.2 and 4.3 present the results of the comparison of the numerical and experimental level response data for the gravity-flow tank and the first tank of the interacting tanks systems. The RMSD values of the comparison were calculated for different assumed values of the discontinuity of a particular variable mentioned above.

For the gravity-flow tank data in Table 4.1, the experimental response correspond to the initial steady state exit velocity of 0.65 m/s and the impulse of magnitude \( 1.5 \times 10^{-2} \text{ m}^3 \). For the numerical solutions, the value of \( q(t) \) was taken as \( 2.95 \times 10^{-4} \text{ m}^3/\text{s} \), and the discontinuous value of the level \( h(0) = h(0^-) \) of \( 2.96 \times 10^{-1} \text{ m} \) for all the five cases of Table 4.1. This Table shows a specific trend of RMSD values. On increasing the discontinuity in the exit velocity, the RMSD first decreased and then increased. For the case at S. No. 1, \( u(0) = u(0^-) \), as obtained through Route (a) in Section 4.3.1 was taken for simulation as an initial estimate; the corresponding RMSD value exhibited was \( 5.00 \times 10^{-3} \text{ m} \). As further trials were made at successively higher values of \( u(0) \) (as indicated through Route (b)), the RMSD value decreased considerably. For example, at S. No. 3 in Table 4.1, the RMSD value was \( 3.61 \times 10^{-3} \text{ m} \). However, when the value of \( u(0) \) further
Table 4.1 Comparison of the experimental and numerical results for the level response of gravity-flow tank for different assumed values of efflux velocity $u(0)$

<table>
<thead>
<tr>
<th>S. No.</th>
<th>Assumed values of $u(0)$ m/s</th>
<th>RMSD $\times 10^3$ m</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>0.65$^1$</td>
<td>5.00</td>
</tr>
<tr>
<td>2.</td>
<td>3</td>
<td>4.12</td>
</tr>
<tr>
<td>3.</td>
<td>8</td>
<td>3.61</td>
</tr>
<tr>
<td>4.</td>
<td>30</td>
<td>4.38</td>
</tr>
<tr>
<td>5.</td>
<td>100</td>
<td>5.53</td>
</tr>
</tbody>
</table>

$^1$Corresponding to steady flow rate of $2.95 \times 10^{-4}$ m$^3$/s

grew, the RMSD value started increasing. For example, at S. No. 5, the RMSD value was $5.53 \times 10^{-3}$ m, that is even more than that at S. No. 1 obtained by ignoring the initial discontinuity. These facts indicate that the actual value of $u(0^+)$ is higher than $u(0^-)$, and lies somewhere in between these assumed values. The aforesaid observations support the analysis carried out above that $u(0^-)$ can’t be used in place of $u(0^+)$, and, hence, it is necessary to account for the initial discontinuities in the gravity-flow tank system. However, the value of $u(0^+)$ cannot be calculated from the initial slope of the experimental level response $h'$ ($0^+$). The level changes very fast initially for the impulse response, it is difficult to accurately measure its initial rate as it is not possible to accurately draw the tangent at zero time in the absence of the left hand branch of the curve (Mickley et al., 1975).

Some corresponding numerical results are also plotted along with the experimental curve in Fig. 4.2. These are for $H$, i.e., the deviation of level from the initial steady state. The experimental curve was fitted by eye (bold curve); the actual data points are not shown here to avoid a clutter on the graph. The numerical solutions converge at the initial steady-state value at $t \to \infty$. 

Fig. 4.2 Response of gravity-flow tank for impulse input of magnitude $15 \times 10^{-3}$ m$^3$
for different assumed values of $u(0)$

- Experimental (bold);
- $u(0) = u(0^-) = 0.65$ m/s, $q(0^-) = 2.95 \times 10^{-4}$ m$^3$/s;
- $u(0) = 8$ m/s;
- $u(0) = 100$ m/s.

The numerical results for the first tank of the two interacting tanks-in-series are compared with the corresponding experimental results in Table 4.2. The experimental data corresponds to the initial steady state, i.e., inflow rate, $8.33 \times 10^{-6}$ m$^3$/s, the first tank level, $1.1 \times 10^{-1}$ m, the second tank level, $9.1 \times 10^{-2}$ m, and the magnitude of the impulse input, $5.0 \times 10^{-4}$ m$^3$. For the numerical solutions, the values of $q(t)$, $q_{21}(0^-)$, and $q_{2}(0^-)$ were taken as the corresponding initial steady state value above, while $h_{1}(0) = h_{1}(0^-)$ of $1.72 \times 10^{-2}$ m was taken for all the five cases of Table 4.2. This Table clearly shows increase in RMSD of one order when the initial value of the efflux flow rate of the first tank was changed from that calculated through physical considerations, i.e., at S. No. 1; For which, value of $q_{21}(0) = q_{21}(0^-) = 1.74 \times 10^{-5}$ m$^3$/s, obtained using Eq. (4.18) through either Routes (a) or (b) in Section 4.3.2 was used for solution. A very good agreement with an RMSD of only $4.35 \times 10^{-4}$ m was obtained. If the initial value of $h_{2}(0)$ was changed (increased or decreased, to see the effect of $q_{21}(0)$) from $h_{2}(0^-)$, the RMSD value

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Table 4.2: Comparison of the experimental and numerical results for the level response of the first tank of the two interacting liquid-level tank system for different assumed values of exit flow rate $q_{21}(0)$

<table>
<thead>
<tr>
<th>S. No.</th>
<th>Assumed values of $q_{21}(0) \times 10^5$ m$^3$/s</th>
<th>RMSD $\times 10^3$ m</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>1.74$^{1}$</td>
<td>0.435</td>
</tr>
<tr>
<td>2.</td>
<td>1.52</td>
<td>3.56</td>
</tr>
<tr>
<td>3.</td>
<td>1.69</td>
<td>1.75</td>
</tr>
<tr>
<td>4.</td>
<td>1.79</td>
<td>1.12</td>
</tr>
<tr>
<td>5.</td>
<td>1.85</td>
<td>1.81</td>
</tr>
</tbody>
</table>

$^{1}$Corresponding to steady value of level of second tank of 0.091 m

increased by one order. For example, the case at S. No. 2 exhibited (at an increased value of $q_{21}(0)$) an RMSD of $3.56 \times 10^{-3}$ m, which is one order more than that at S. No. 1. Similarly, at S. No. 5 (at a decreased value of $q_{21}(0)$), the RMSD was $1.81 \times 10^{-3}$ m, which is again one order more than that at S. No. 1. Some corresponding numerical results of the impulse level response are plotted in Fig. 4.3 along with the experimental response profile (bold curve). It supports the observations made. The curve for the case at S. No. 1 practically overlaps with the experimental curve, while other cases deviate substantially from the experimental curve. Thus, the results are just the opposite of that seen in gravity-flow tank system, as the numerical solutions deviated from the experimental findings if the initial discontinuities were incorporated. These results support the preceding analysis that there is an undesirable consequence of deviating from the value of $h_2(0)$ determined from physical principles for the interacting tanks system. Hence, this system already accounts for the initial discontinuities in $h_1(0)$ and $q_{21}(0)$, and is, thus, not affected by the same.

To summarize, it is shown that the numerical solution of the gravity-flow tank moves closer to the experimental response as the initial discontinuity in the efflux velocity of the liquid, identified upon the application of momentum balance at the initial instant of the introduction of the perturbation, is taken into account. Conversely, the numerical solution of the first of the two interacting tanks system deviates away from the experimental response as the value of an initial discontinuity is deviated away from that evaluated upon the application of the physical principles. The distinct results of the two
systems studied substantiate the analysis carried out in Sections 4.1 through 4.3. Further evidence to this effect is provided in the next section.

### 4.8 Time Required for Emptying the Tanks

For further validation, calculations of the time required for emptying (time-to-empty) flow-level tanks are now described. These calculations under gravity-powered flows can be tricky because the head changes with time (Kossik, 2000; Loiacono, 1987; Shoaei and Sommerfeld, 1989; Sommerfeld and Stallybrass, 1992). However, in the present study, the impulse responses are used indirectly to describe these calculations here. The linearized models in terms of the deviation variables defined in Sections 3.5.1 and 3.6.1 are employed to see the effect. Recall the two routes for impulse response calculations. Route (a) includes the initial discontinuities in the stimulus, while Route (b) includes them in the response and was found to be the valid route as it took all the initial
discontinuities into account. Consider impulse input to an initially empty tank with inflow and outflow equal to zero at \( t \to 0^- \), i.e., \( Q(t) = Q(0^-) = 0, H(0^-) = 0, U(0^-) = 0 \).

The impulse perturbation to the tank can be introduced by plunging a certain amount, i.e., \( M \) (m\(^3\)) of liquid suddenly into the tank, all in one go. Corresponding to Route (a), i.e., \( H(0) = H(0^-) = 0, U(0) = U(0^-) = 0, \) and \( Q(t) = M \delta(t) \). Alternatively, corresponding to Route (b), i.e., \( H(0) = H(0^+) \neq 0, U(0) = U(0^+) \neq 0 \). Linearized models are used for the calculations of time-to-empty gravity-flow tank and the first tank of the two interacting tanks system.

The effect on time-to-empty gravity-flow tank is discussed as follows. The deviation variables \( H, U \) and \( Q \) are the same as respective true values of level \( h \), efflux velocity in pipe \( u \) and inflow rate \( q \), as the initial steady state values are all zero. Subscript \( s \) represents initial steady state and superscript (') represents derivative. The rest of the symbols were defined in Chapter 3. Linearized model Eq. 3.27 is repeated here for convenience:

\[
s^2 H(s) - sH(0) - H'(0) + \frac{B}{A} [sH(s) - H(0)] + \frac{C}{A} H(s) = K [sQ(s) - Q(0)] + K \frac{B}{A} Q(s)
\]

(4.26)

where, \( K = 1/A_r \), \( A = A_r L \), \( B = (4gLQ_s)/(D_r) \), \( C = gA_p^2 / A_r \)

Giving an impulse input to initially empty tank as per Route (a), is first considered. In Eq. (4.26), \( Q(s) = M \), and all the initial conditions become zero. The equation is solved for \( H(s) \). Then, \( H(s) = 0 \) is substituted as the empty tank condition. As the left hand side of Eq. (4.26) becomes \( zero \), time-to-empty the tank is invariably independent of the amount \( M \) of liquid added (as \( M \) gets cancelled out of the equation), which is erroneous and physically impossible. Thus, Route (a) yields incorrect solutions and the system is affected by initial discontinuities as also shown in the previous sections.

However, one can calculate the right value of the time-to-empty the tank by using Route (b). Route (b), which includes the initial discontinuities in the response \( H(s) \), resulted in Eq. (4.6) above. It is repeated here for convenience:

\[
s^2 H(s) + \frac{B}{A} sH(s) + \frac{C}{A} H(s) = K \left[ sM + \frac{B}{A} M - A_p U(0^+) \right]
\]

(4.27)
Solving for $H(s)$, inverting it, and finding the value of the time when the level response $H(t)$ reaches the zero value gives the time for emptying. But the solution requires the value $U(0^+)$, which is unknown. However, the qualitative nature of the response can be looked at if the response is plotted against time. The time read from this plot as the trajectory crosses the time-axis gives the correct value of interest. For under-damped oscillatory case of gravity flow tank, there are multiple feasible solutions as the trajectory cuts the time-axis at multiple points (refer to Fig. 4.1 showing an oscillatory impulse response profile). But the tank will be empty when the trajectory cuts the time-axis at the very first value of time, so the least of these multiple values is taken as the solution. Also, starting at different initial levels in gravity-flow tank, values of $U(0^+)$ would be different. Different values of time-to-empty the tank are, thus, obtained using Eq. (4.27), which goes well with the physical fact. So, if the initial discontinuity in efflux velocity $U(0^+)$ was ignored (as in Eq. (4.26)), the calculations of time-to-empty for gravity-flow tank would result in same values irrespective of the initial level.

Now, consider the over-damped case of gravity-flow tank, calculation of the time-to-empty the tank from Route (b) gives $t \to \infty$ because the over-damped trajectory becomes asymptotic to the time-axis with the passage of time but never cuts the time-axis (as in Fig. 4.2 and 4.3). This value of time comes out to be the same irrespective of the initial level. But taking $H = 0.1$ (say) (i.e., a very small value as the empty tank condition), Eqs. (4.27) would yield different values of time-to-empty the flow-level tank for different initial levels. From Routes (a) or (b), this fact is analogously true for the first tank of a two interacting tank system (Eq. 3.34), non-interacting tanks system, or for a single tank system for that matter (as these tanks only behave over-damped). These flow-level tank systems are, thus, not affected by initial discontinuities as also shown in the sections above. For these systems, Routes (a) and (b) happened to be the same.

4.9 Outcomes Revisit

Some linear time-invariant numerator-dynamics systems and the corresponding nonlinear systems with terms containing differentials of the input, e.g., U-tube manometer, first tank of the interacting tanks, etc. are unaffected by the initial discontinuities, whereas, the
others, e.g., gravity-flow tank, constant holdup CSTR, etc. are affected by the initial discontinuities, as is studied above.

The results of Sections 4.4 through 4.8 substantially corroborate the indications of the preceding analysis carried out in Sections 4.1 through 4.3, anticipating the conformity and the closeness of the calculated solution profiles obtained through the initialization Route (b), to the physical facts of time-to-empty, and the experimental level response profiles of different flow-level tanks. The observations clearly substantiate the analysis that $u(0^-)$ cannot be used in place of $u(0^+)$, and, hence, it is necessary to account for the initial discontinuities in the gravity-flow tank system for the case of impulse response. With the introduction of the impulse input, the level of the tank suddenly increases, and there is a resultant sudden increase in the efflux velocity in the pipe. This explains the findings of the numerical simulations that as the value of the initial condition for $u(0)$ is increased above $u(0^-)$, the calculated level profiles approach the experimental level profiles. So, as remarked earlier, Route (b), based on finding initial discontinuities through the application of physical balances to the initial effects, is the correct route for the initializations of the systems perturbed by the singular inputs. Route (b), thus, yields more accurate solutions than Route (a).