Chapter 7

Ricci solitons on Para-Sasakian manifolds

7.1 Introduction

After presenting rudiments of para-contact geometry in section 7.2, in section 7.3 it is proved that if \((g, \xi, \lambda)\) is a RS on a para-Sasakian manifold \(M\) then \(M\) is a nearly quasi-Einstein manifold. Also it is shown that a RS \((g, \xi, \lambda)\) on a para-Sasakian manifold \(M\) is always expanding. We further carry on the study of RS on para-Sasakian manifolds in view of \(\nabla\) and obtained the invariance condition of RS structure under \(\nabla\) and \(\nabla\).

Finally in the last section we have constructed an example of para-Sasakian manifold whose metric is RS through which the result of this chapter is illustrated.

Note that, the content of this chapter consists of the paper [74].
7.2 Preliminaries

Let \((M^m, g), m = \text{dim}(M)\) be a para-Sasakian manifold. Then we have ([2], [71], [117]):

\[
\eta(R(X, Y)Z) = \eta(Y)g(X, Z) - \eta(X)g(Y, Z), \quad (7.2.1)
\]

\[
R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad (7.2.2)
\]

\[
R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \quad (7.2.3)
\]

\[
S(X, \xi) = -(m - 1)\eta(X), \quad Q\xi = -(m - 1)\xi \quad (7.2.4)
\]

for all \(X, Y, Z \in \chi(M)\), \(R\) is the Riemannian curvature tensor and \(S\) is the Ricci tensor of type \((0,2)\) such that \(g(QX, Y) = S(X, Y)\).

One can define quarter symmetric metric connection \(\nabla\) in a para-Sasakian manifold by ([63], [71], [117])

\[
\nabla_X Y = \nabla_X Y + H(X, Y), \quad (7.2.5)
\]

where \(H\) is a \((1,1)\) tensor given by

\[
H(X, Y) = \frac{1}{2} \left[ \tau(X, Y) + \tau'(X, Y) + \tau'(Y, X) \right], \quad (7.2.6)
\]

where the tensor \(\tau'\) and \(\tau\) are related by

\[
g(\tau'(X, Y), Z) = g(\tau(Z, X), Y). \quad (7.2.7)
\]

From (41), (7.2.6) and (7.2.7), we finally obtain [71]

\[
\nabla_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi. \quad (7.2.8)
\]
The relation between curvature tensors $R$ and $\overline{R}$ with respect to $\nabla$ and $\overline{\nabla}$ is given by ([71], [117]):

\[
\overline{R}(X, Y)Z = R(X, Y)Z + 3g(\phi X, Z)\phi Y - 3g(\phi Y, Z)\phi X + [\eta(X)Y - \eta(Y)X] \eta(Z) - [\eta(X)g(Y, Z) - \eta(Y)g(X, Z)] \xi.
\] (7.2.9)

Contracting (7.2.9) we have the relation between the Ricci tensors $S$ and $\overline{S}$ with respect to $\nabla$ and $\overline{\nabla}$ resp. given by ([71], [117]):

\[
\overline{S}(Y, Z) = S(Y, Z) + 2g(Y, Z) - (m + 1)\eta(Y)\eta(Z).
\] (7.2.10)

Contracting (7.2.10) and using (30), we get

\[
\overline{r} = r + (m - 1),
\] (7.2.11)

where $\overline{r}$ and $r$ are the scalar curvatures with respect to $\overline{\nabla}$ and $\nabla$ respectively. Again from (7.2.2) and (7.2.9), we obtain

\[
\overline{R}(X, Y)\xi = 2[\eta(X)Y - \eta(Y)X].
\] (7.2.12)

Putting $Z = \xi$ in (7.2.10) and using (7.2.4) we have

\[
\overline{S}(Y, \xi) = -2(m - 1)\eta(Y).
\] (7.2.13)
7.3 Main results

We now consider \((g, \xi, \lambda)\) is a RS on \(M^m\).

From (32) and (34) we have

\[
(\mathcal{L}_\xi g)(Y, Z) = 2g(\phi Y, Z) = 2\omega(Y, Z).
\]  \hspace{1cm} (7.3.1)

Hence from (2) and (7.3.1) we have,

\[
S(Y, Z) = -\lambda g(Y, Z) - \omega(Y, Z),
\]  \hspace{1cm} (7.3.2)

which implies that the manifold under consideration is a nearly quasi-Einstein manifold.

This leads to the following:

**Theorem 7.3.1.** If \((g, \xi, \lambda)\) is a RS on a Para-Sasakian manifold \(M^m\) then it is a nearly quasi-Einstein manifold.

Putting \(Z = \xi\) in (7.3.2) we get

\[
S(Y, \xi) = -\lambda \eta(Y).
\]  \hspace{1cm} (7.3.3)

From (7.2.4) and (7.3.3) we obtain \(\lambda = m - 1 > 0\). Thus we can state the following:

**Theorem 7.3.2.** A RS \((g, \xi, \lambda)\) on a Para-Sasakian manifold \(M^m\) is always expanding.

Let \((g, V, \lambda)\) be a RS on a para-Sasakian manifold with respect to a \(\nabla\). Then we have

\[
(\mathcal{L}_V g)(Y, Z) + 2\Sigma(Y, Z) + 2\lambda g(Y, Z) = 0,
\]  \hspace{1cm} (7.3.4)

where \(\mathcal{L}_V\) is the Lie derivative along the vector field \(V\) on \(M^m\) with respect to \(\nabla\).

Now from (7.2.8), we have

\[
(\mathcal{L}_V g)(Y, Z) = g(\nabla_Y V + \eta(V)\phi Y - g(\phi Y, V)\xi, Z)
\]  \hspace{1cm} (7.3.5)
\begin{align*}
+ & g(Y, \nabla_Z V + \eta(V)\phi Z - g(\phi Z, V)\xi) \\
= & (L_V g)(Y, Z) + 2\eta(V)g(\phi Y, Z) \\
& - [\eta(Z)g(\phi Y, V) + \eta(Y)g(\phi Z, V)].
\end{align*}

In view of (7.2.10) and (7.3.5), (7.3.4) yields

\begin{equation}
(L_V g)(Y, Z) + 2S(Y, Z) + 2\lambda g(Y, Z) + 2\eta(V)g(\phi Y, Z) + 2g(Y, Z) - (m + 1)\eta(Y)\eta(Z) - [\eta(Z)g(\phi Y, V) + \eta(Y)g(\phi Z, V)] = 0. \tag{7.3.6}
\end{equation}

If \((g, V, \lambda)\) is a RS on a para-Sasakian manifold with respect to \(\nabla\) then (2) holds. Thus from (2) and (7.3.6) we can state the following:

**Theorem 7.3.3.** A RS \((g, V, \lambda)\) on a Para-Sasakian manifold is invariant under \(\nabla\) if and only if the relation

\begin{align*}
2\eta(V)g(\phi Y, Z) + 2g(Y, Z) - (m + 1)\eta(Y)\eta(Z) \\
- [\eta(Z)g(\phi Y, V) + \eta(Y)g(\phi Z, V)] = 0
\end{align*}

holds for arbitrary vector fields \(Y, Z\).

Now, let \((g, \xi, \lambda)\) be a RS on a para-Sasakian manifold with respect to \(\nabla\). Then we have

\begin{equation}
(L_\xi g)(Y, Z) + 2S(Y, Z) + 2\lambda g(Y, Z) = 0. \tag{7.3.7}
\end{equation}

From (30), (34) and (7.2.8) we have

\begin{equation}
(\mathcal{I}_\xi g)(Y, Z) = g(\nabla_Y \xi, Z) + g(Y, \nabla_Z \xi) \tag{7.3.8}
\end{equation}
\[
\begin{align*}
&= g(\nabla_Y \xi + \phi Y, Z) + g(Y, \nabla_Z \xi + \phi Z) \\
&= 4g(\phi Y, Z) = 4\omega(Y, Z).
\end{align*}
\]

Using (7.2.10) and (7.3.8) in (7.3.7), we get

\[
S(Y, Z) = -(\lambda + 2)g(Y, Z) + (m + 1)\eta(Y)\eta(Z) - 2\omega(Y, Z).
\] (7.3.9)

From (35) and (7.3.9), we can state the following:

**Theorem 7.3.4.** If \((g, \xi, \lambda)\) is a RS on a Para-Sasakian manifold with respect to \(\nabla\) then it is pseudo \(\eta\)-Einstein manifold.

Putting \(Z = \xi\) in (7.3.9) and using (30) and (34), we get

\[
S(Y, \xi) = [(m - 1) - \lambda]\eta(Y).
\] (7.3.10)

From (7.2.4) and (7.3.10), we obtain

\[
\lambda = 2(m - 1) > 0 \quad \text{since} \ m > 1.
\] (7.3.11)

This leads to the following:

**Theorem 7.3.5.** A RS \((g, \xi, \lambda)\) on a Para-Sasakian manifold \(M^m(\phi, \xi, \eta, g)\) with respect to \(\nabla\) is always expanding.

From Theorem 7.3.1, Theorem 7.3.2, Theorem 7.3.4 and Theorem 7.3.5, we can state the following:

**Theorem 7.3.6.** Let \((g, \xi, \lambda)\) be a RS on a para-Sasakian manifold \(M\). Then the following holds:

<table>
<thead>
<tr>
<th>Connection on (M)</th>
<th>(M)</th>
<th>Nature of RS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Levi-Civita ((\nabla))</td>
<td>nearly quasi Einstein</td>
<td>expanding</td>
</tr>
<tr>
<td>Quarter symmetric ((\nabla))</td>
<td>pseudo (\eta)-Einstein</td>
<td>expanding</td>
</tr>
</tbody>
</table>
Next, we verify our result by an example.

**Example 7.3.1** Following an example of para-Sasakian manifold constructed in [137], we consider a 3-dimensional manifold $M = \mathbb{R}^3$ with standard cartesian coordinates and choose the vector fields [137]

$$
e_1 = e^x \frac{\partial}{\partial y}, e_2 = e^x (\frac{\partial}{\partial y} - \frac{\partial}{\partial z}), e_3 = -\frac{\partial}{\partial x},$$

which are linearly independent at each point of $M$. Let $g$ be the standard Euclidean metric given by $g(e_i, e_j) = \delta_{ij}$. Let $\eta$ be the 1-form defined by $\eta(Z) = g(Z, e_3)$, for any vector field $Z$ on $M$. Define the $(1,1)$ tensor field $\phi$ as $\phi(e_1) = e_1, \phi(e_2) = e_2$ and $\phi(e_3) = 0$. One can verify easily that with $\xi = e_3$, $(\phi, \xi, \eta, g)$ defines an almost paracontact structure on $M$ [137]. Let $\nabla$ be the Levi-Civita connection with respect to $g$. Then we have

$$[e_1, e_2] = 0, [e_1, e_3] = e_1, [e_2, e_3] = e_2.$$ 

Taking $e_3 = \xi$ and using Koszul’s formula, we obtain

$$\nabla_{e_1} e_2 = 0, \nabla_{e_1} e_3 = e_1, \nabla_{e_1} e_1 = -e_3,$$

$$\nabla_{e_2} e_3 = e_2, \nabla_{e_2} e_2 = -e_3, \nabla_{e_2} e_1 = 0,$$

$$\nabla_{e_3} e_3 = 0, \nabla_{e_3} e_2 = 0, \nabla_{e_3} e_1 = 0.$$ 

From the above it can be easily seen that $(\phi, \xi, \eta, g)$ is a Para-Sasakian structure on $M$. Hence $M(\phi, \xi, \eta, g)$ is a 3-dimensional Para-Sasakian manifold [137]. After some straightforward calculations, the non-vanishing components of the curvature tensor is given by:

$$R(e_1, e_i) e_i = -e_1, \quad i = 2, 3,$$
\[ R(e_2, e_j)e_j = -e_2, \quad j = 1, 3, \]
\[ R(e_3, e_l)e_l = -e_3, \quad l = 1, 2. \]

Tracing the Riemann curvature tensor, the non-vanishing components of the Ricci tensor is given by:
\[ S(e_i, e_i) = -2, \quad i = 1, 2, 3. \]

That is \( S(X, Y) = -2g(X, Y) \).

Now from (7.2.8),
\[
\nabla_{e_1}e_1 = -2e_3, \quad \nabla_{e_1}e_2 = 0, \quad \nabla_{e_1}e_3 = 2e_1, \\
\nabla_{e_2}e_1 = 0, \quad \nabla_{e_2}e_2 = -2e_3, \quad \nabla_{e_2}e_3 = 2e_2, \\
\nabla_{e_3}e_1 = 0, \quad \nabla_{e_3}e_2 = 0, \quad \nabla_{e_3}e_3 = 0,
\]
implies
\[ \nabla_X Y = 2\nabla_X Y, \quad \text{(By linearity of \( \nabla \))}. \]

Again from (7.2.10) the only non-vanishing component of the Ricci tensor w.r.to \( \nabla \) is given by
\[ \overline{S}(e_3, e_3) = -4. \]

Let
\[
V = \sum_{i=1}^{3} v_i e_i, \\
Y = \sum_{i=1}^{3} y_i e_i, \\
Z = \sum_{i=1}^{3} z_i e_i,
\]
where \( v_i, z_i, y_i \in C^\infty(\mathbb{R}), i = 1, 2, 3 \). Then,

\[
\nabla_Y V = \nabla_{y_1e_1 + y_2e_2 + y_3e_3}(v_1e_1 + v_2e_2 + v_3e_3) \\
= v_3y_1e_1 - v_2y_2e_3 + v_3y_2e_2 - y_1v_1e_3 \\
= v_3y_1e_1 + v_3y_2e_2 - (v_2y_2 + v_1y_1)e_3.
\]

Now,

\[
(\mathcal{L}_V g)(Y, Z) = 2(\mathcal{L}_V g)(Y, Z), \text{ (By linearity of } \nabla),
\]

and

\[
S(Y, Z) = S(Y, Z) + 2g(Y, Z) - 4\eta(Y)\eta(Z) \\
= -2g(Y, Z) + 2g(Y, Z) - 4\eta(Y)\eta(Z) \\
= -4\eta(Y)\eta(Z) \\
= -4yz_3.
\]

Hence,

\[
2\eta(V)g(\phi Y, Z) + 2g(Y, Z) - 4\eta(Y)\eta(Z) \tag{7.3.12}
\]

\[
- [\eta(Z)g(\phi Y, V) + \eta(Y)g(\phi Z, V)] \\
= 2v_3(y_1z_1 + y_2z_2) + 4(y_1z_1 + y_2z_2 + y_3z_3) \\
- 8y_3z_3 - [z_3(y_1v_1 + y_2v_2) + y_3(z_1v_1 + z_2v_2)] \\
= -v_1(y_1z_3 + y_3z_1) - v_2(y_2z_3 + y_3z_2) + 2v_3(y_1z_1 + y_2z_2) \\
+ 4(y_1z_1 + y_2z_2 - y_3z_3).
\]
Now let

\[ H_1(Y, Z) = (E_V g)(Y, Z) + 2S(Y, Z) + 2\lambda g(Y, Z) \]

and

\[ H_2(Y, Z) = (\mathcal{E}_V g)(Y, Z) + 2S(Y, Z) + 2\lambda g(Y, Z). \]

Then

\[
(H_2 - H_1)(Y, Z) = (E_V g)(Y, Z) - 2S(Y, Z) - 8y_3z_3
= g(\nabla_Y V, Z) + g(Y, \nabla_Z V) - 2S(Y, Z) - 8y_3z_3
= 2v_3(y_1 z_1 + y_2 z_2) + 4(y_1 z_1 + y_2 z_2 - y_3 z_3)
- v_1(y_1 z_3 + y_3 z_1) - v_2(y_2 z_3 + y_3 z_2).
\]

Hence the required condition for the RS to be invariant under \( \nabla \) is that:

\[
(H_2 - H_1)(Y, Z) = 0,
\]

\[
=> 2v_3(y_1 z_1 + y_2 z_2) + 4(y_1 z_1 + y_2 z_2 - y_3 z_3)
- v_1(y_1 z_3 + y_3 z_1) - v_2(y_2 z_3 + y_3 z_2) = 0.
\]

and this is also verified from (7.3.12) and (7.3.13) and hence our Theorem 7.3.3 is verified. With this interesting example, we close the thesis.

* * * * *