Chapter 5

Ricci almost solitons on generalized Sasakian-space-forms

5.1 Introduction

In this chapter we propose the study of $RAS$ structure on $GSSF$. In section 5.2 we have presented some essential background materials to dig into our results. Section 5.3 captures the study of $RAS$ on our proposed structure. It is proved that in this setup the manifold becomes Einstein, Ricci symmetric and Ricci semisymmetric respectively. Finally in section 5.4 we have introduced the study of $\eta$-$RAS$ and proved that if we choose the connection to be semisymmetric, then the structure turns out to be pseudo $\eta$-Einstein with respect to $\nabla$.

The contents of this chapter consists of the papers ([72], [87]).
5.2 Preliminaries

Definition 5.2.1. A vector field $V$ is said to be a conformal Killing vector field [126] if it satisfies

$$(\mathcal{L}_V g)(U,V) = 2\rho g(U,V)$$

(5.2.1)

for all $U, V \in \chi(M^n(f_1, f_2, f_3))$, where $\rho \in C^\infty(M^n)$.
In particular, if $\rho$ is constant then $V$ is called homothetic and if $\rho = 0$ then $V$ is called isometric as well as Killing vector field.

We also recall the following:

Definition 5.2.2. A GSSF $M^n(f_1, f_2, f_3)$ is called $\eta$-Ricci recurrent [43] if the following relation holds:

$$(\nabla_X S)(\phi Y, \phi Z) = \alpha(X) S(Y, Z),$$

(5.2.2)

where $\alpha$ is a no-where vanishing 1-form.
In particular, if $\alpha$ vanishes identically then $M^n(f_1, f_2, f_3)$ is called $\eta$-Ricci parallel.

Theorem 5.2.1. [43] If a GSSF $M^n(f_1, f_2, f_3)$ is $\eta$-Ricci recurrent then $(n-1)f_1 + 3f_2 - f_3$ can never be a non-zero constant.

Theorem 5.2.2. [43] If a GSSF $M^n(f_1, f_2, f_3)$ is $\eta$-Ricci parallel then $(n-1)f_1 + 3f_2 - f_3$ is constant.

There are so many weaker version Ricci symmetry such as Ricci recurrent manifold [110], Ricci semisymmetric manifold [140] etc. It is known that every Ricci symmetric manifold is Ricci semisymmetric but not conversely [140]. However, in this paper it is proved that if $(g, V, \lambda)$ is a RAS on a GSSF $M^n(f_1, f_2, f_3)$, where $V$ is conformal Killing vector field then $M^n(f_1, f_2, f_3)$ is Ricci semisymmetric if and only if it is Ricci symmetric.

In 1970 Pokhariyal and Mishra [116] were introduced new tensor fields, called $W_2$
and $E$ tensor fields, in a Riemannian manifold and studied their properties. According to them a $W_2$-curvature tensor on a $GSSF$ $M^n(f_1, f_2, f_3)$, $n > 3$, is defined by [116]

$$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{(n-1)} [g(X,Z)QY - g(Y,Z)QX], \quad (5.2.3)$$

where $Q$ is the Ricci-operator. A $GSSF$ $M^n(f_1, f_2, f_3)$ is called $W_2$-flat if $W_2(X,Y)Z$ vanishes identically for all $X$, $Y$ and $Z \in \chi(M^n)$.

### 5.3 Ricci almost solitons with conformal Killing vector field

This section deals with the study of of $GSSF$ whose metric is $RAS$ with a conformal Killing vector field and we prove the following:

**Theorem 5.3.1.** Let $(g, V, \lambda)$ be a $RAS$ on a $GSSF$ $M^n(f_1, f_2, f_3)$. If $V$ is conformal Killing vector field then the followings are equivalent:

(i) $M^n(f_1, f_2, f_3)$ is Einstein.
(ii) $\lambda + \rho$ is constant though $\lambda$ and $\rho$ are smooth functions.
(iii) $M^n(f_1, f_2, f_3)$ is Ricci symmetric.
(iv) $M^n(f_1, f_2, f_3)$ is Ricci semisymmetric.

**Proof.** Since $(g, V, \lambda)$ is a $RAS$ on a $GSSF$ $M^n(f_1, f_2, f_3)$ with $V$ is conformal Killing vector field then by virtue of (5.2.1) we obtain from (2) that

$$S(X,Y) = - (\lambda + \rho) g(X,Y), \quad (5.3.1)$$

which implies that the manifold under consideration is Einstein, i.e. (i) holds and hence $\lambda + \rho$ is always constant by Bianchi’s identity, though $\lambda$ and $\rho$ are smooth functions, i.e. (ii) holds.
As $M^n(f_1, f_2, f_3)$ is Einstein, its Ricci tensor is parallel, i.e. $M^n(f_1, f_2, f_3)$ is Ricci symmetric, which implies (iii). Again in [59] it is proved that if the Ricci tensor of a GSSF $M^n(f_1, f_2, f_3)$ with $f_1 \neq f_3$ is parallel then $M^n(f_1, f_2, f_3)$ is Einstein, i.e. (iii) implies (i).

Now for any $X, Y, Z, U \in \chi(M^n(f_1, f_2, f_3))$, we have

$$
(R(X, Y) \cdot S)(Z, U) = -S(R(X, Y)Z, U) - S(Z, R(X, Y)U).
$$

Using (5.3.1) in (5.3.2), we obtain

$$
(R(X, Y) \cdot S)(Z, U) = (\lambda + \rho)[g(R(X, Y)Z, U) + g(Z, R(X, Y)U)] = 0,
$$

which implies that the manifold under consideration is Ricci semisymmetric, i.e. (iv) holds.

Now from (4.2.3) and (5.3.1), we get

$$
-(\lambda + \rho) = (n - 1)f_1 + 3f_2 - f_3,
$$

$$
3f_2 + (n - 2)f_3 = 0.
$$

From (5.3.4) and (5.4.1) we obtain $\lambda = -[\rho + (n - 1)(f_1 - f_3)]$.

This leads to the following:

**Theorem 5.3.2.** In a GSSF $M^n(f_1, f_2, f_3)$, a RAS $(g, V, \lambda)$ with $V$ is a conformal Killing vector field is (i) shrinking for $\rho + (n - 1)(f_1 - f_3) > 0$, (ii) steady for $\rho + (n - 1)(f_1 - f_3) = 0$, and (iii) expanding for $\rho + (n - 1)(f_1 - f_3) < 0$ respectively.

**Corollary 5.3.1.** A RAS $(g, V, \lambda)$, where $V$ is Killing vector field in a GSSF, is shrinking, steady and expanding according as $(f_1 - f_3) > 0$, $= 0$ and $< 0$, respectively.

**Corollary 5.3.2.** A RAS $(g, V, \lambda)$, where $V$ is a Killing vector field in a Sasakian-space-form, is always shrinking.

In connection to the study of $W_2$-curvature tensor field in a GSSF, Hui and Sarkar found [83]
Theorem 5.3.3. [83] Every GSSF $M^n(f_1, f_2, f_3)$ is $W_2$-flat if and only if $3f_2 + (n - 2)f_3 = 0$.

So by virtue of (5.4.1) and Theorem 5.3.3, we can state the following:

**Theorem 5.3.4.** If $(g, V, \lambda)$ is a RAS on a GSSF $M^n(f_1, f_2, f_3)$ such that the potential vector field $V$ is conformal Killing, then $M^n(f_1, f_2, f_3)$ is $W_2$-flat.

### 5.4 $\eta$-Ricci almost solitons

This section deals with the study of $\eta$-RAS on GSSF.

Let $M^n(f_1, f_2, f_3)$ be a GSSF. From (9) and (4.2.1), we get

\[
(L_\xi g)(X,Y) = \frac{d}{dt}g(L_\xi X,Y) + g(L_\xi X,\nabla_Y \xi) - g(X,\nabla_Y \xi) = -(f_1 - f_3)g(\phi X,Y) - (f_1 - f_3)g(X,\phi Y) = 0. 
\] (5.4.1)

From (4.2.3) and (5.4.1), we obtain

\[
(L_\xi g)(X,Y) + 2S(X,Y) + 2\bar{\lambda}g(X,Y) + 2\bar{\mu}g(Y,X) = 0
\] (5.4.2)

for all $X, Y, Z \in \chi(M^n(f_1, f_2, f_3))$, where $\bar{\lambda} = -(n - 1)f_1 + 3f_2 - f_3$ and $\bar{\mu} = 3f_2 + (n - 2)f_3$.

If $M^n(f_1, f_2, f_3)$ is $\eta$-Ricci recurrent then from Theorem 5.2.1 it follows that $(n - 1)f_1 + 3f_2 - f_3$ can never be a non-zero constant, i.e. $\bar{\lambda}$ is a smooth function as $f_1$, $f_2$ and $f_3$ are smooth functions. Thus we can state the following:
Theorem 5.4.1. If a GSSF $M^n(f_1, f_2, f_3)$ is $\eta$-Ricci recurrent, then $(g, \xi, \bar{\lambda}, \bar{\mu})$ yields an $\eta$-RAS, provided $\bar{\mu}$ is a non-constant smooth function.

Again if $M^n(f_1, f_2, f_3)$ is $\eta$-Ricci parallel then from Theorem 5.2.2 it follows that $(n - 1)f_1 + 3f_2 - f_3$ is constant, i.e. $\bar{\lambda}$ is a constant. Thus we can state the following:

Theorem 5.4.2. If a GSSF $M^n(f_1, f_2, f_3)$ is $\eta$-Ricci parallel, then $(g, \xi, \bar{\lambda}, \bar{\mu})$ yields an $\eta$-RS, provided $\bar{\mu}$ is a constant.

Since in a Sasakian-space-form, $\bar{\lambda}$ and $(f_1 - f_3)$ are always constants, we can state the following:

Corollary 5.4.1. In a Sasakian-space-form, $(g, \xi, \bar{\lambda}, \bar{\mu})$ yields an $\eta$-RS.

Theorem 5.4.3. [83] Every GSSF $M^n(f_1, f_2, f_3)$ is $W_2$-flat if and only if $3f_2 + (n - 2)f_3 = 0$.

Theorem 5.4.4. [83] A GSSF $M^n(f_1, f_2, f_3)$ is $W_2$-flat if and only if it is projectively flat.

So, if the GSSF $M^n(f_1, f_2, f_3)$ is $W_2$-flat (or projectively flat) then from Theorem 5.4.3 and Theorem 5.4.4, we get $\bar{\mu} = 3f_2 + (n - 2)f_3 = 0$ and hence we can state the following:

Theorem 5.4.5. In a $W_2$-flat (or projectively flat) GSSF $M^n(f_1, f_2, f_3)$, there are no $\eta$-RAS as well as $\eta$-RS.

By virtue of Theorem 5.2.1, Theorem 5.4.3 and Theorem 5.4.4 we can state the following:

Theorem 5.4.6. If a $W_2$-flat (or projectively flat) GSSF $M^n(f_1, f_2, f_3)$ is $\eta$-Ricci recurrent, then $(g, \xi, \bar{\lambda})$ yields an RAS.

Also in view of Theorem 5.2.2, Theorem 5.4.3 and Theorem 5.4.4 we can state the following:
**Theorem 5.4.7.** If a $W_2$-flat (or projectively flat) GSSF $M^n(f_1, f_2, f_3)$ is $\eta$-Ricci parallel, then $(g, \xi, \lambda)$ yields an RS.

Let $(g, \xi, \lambda, \mu)$ be an $\eta$-RAS on a GSSF $M^n(f_1, f_2, f_3)$ with respect to $\nabla^s$. Then we have

$$\left(\mathcal{L}_\xi^s g\right)(Y, Z) + 2S^s(Y, Z) + 2\lambda g(Y, Z) + 2\mu\eta(Y)\eta(Z) = 0.$$  \hspace{1cm} (5.4.3)

Now we have

$$\left(\mathcal{L}_\xi^s g\right)(Y, Z) = g(\nabla^s_Y \xi, Z) + g(Y, \nabla^s_Z \xi)$$  \hspace{1cm} (5.4.4)

$$= g(\nabla_Y \xi + Y - \eta(Y)\xi, Z) + g(Y, \nabla_Z \xi + Z - \eta(Z)\xi)$$

$$= 2[g(Y, Z) - \eta(Y)\eta(Z)].$$

Using (48) and (5.4.4) in (5.4.3), we get

$$S(Y, Z) = [a - \lambda - 1]g(Y, Z) - (\mu - 1)\eta(Y)\eta(Z) + (n - 2)\alpha(Y, Z),$$  \hspace{1cm} (5.4.5)

which implies that the manifold under consideration is pseudo $\eta$-Einstein [122].

This leads to the following:

**Theorem 5.4.8.** If $(g, \xi, \lambda, \mu)$ is an $\eta$-RAS on a GSSF $M^n(f_1, f_2, f_3)$ with respect to $\nabla^s$ then $M$ is pseudo $\eta$-Einstein.

We now consider $(g, V, \lambda, \mu)$ is an $\eta$-RAS on a GSSF $M^n(f_1, f_2, f_3)$ with respect to $\nabla^s$. Then we have

$$\left(\mathcal{L}_V^s g\right)(Y, Z) + 2S^s(Y, Z) + 2\lambda g(Y, Z) + 2\mu\eta(Y)\eta(Z) = 0.$$  \hspace{1cm} (5.4.6)

By virtue of (40), we have

$$\left(\mathcal{L}_V^s g\right)(Y, Z) = g(\nabla^s_Y V, Z) + g(Y, \nabla^s_Z V).$$  \hspace{1cm} (5.4.7)
Using (48) and (5.4.7) in (5.4.6), we get

\[
(\mathcal{L}_V g)(Y, Z) + 2 S(Y, Z) + 2 \lambda g(Y, Z) + 2 \mu \eta(Y) \eta(Z)
\]

\[+ 2 \{\eta(V) - a\} g(Y, Z) - 2((n - 2)) \alpha(Y, Z)
\]

\[- [\eta(Z) g(Y, V) + \eta(Y) g(Z, V)] = 0.
\]

If \((g, V, \lambda, \mu)\) is an \(\eta\)-RAS on a GSSF \(M^n(f_1, f_2, f_3)\) with respect to \(\nabla\) then (4) holds. Thus from (4) and (5.4.8), we can state the following:

**Theorem 5.4.9.** An \(\eta\)-RAS \((g, V, \lambda, \mu)\) on a GSSF \(M^n(f_1, f_2, f_3)\) is invariant under \(\nabla^*\) if and only if the relation

\[2 \{\eta(V) - a\} g(Y, Z) - 2((n - 2)) \alpha(Y, Z) - [\eta(Z) g(Y, V) + \eta(Y) g(Z, V)] = 0
\]

holds for arbitrary vector fields \(Y\) and \(Z\).