Chapter 3

Ricci solitons on $\beta$-Kenmotsu manifolds

3.1 Introduction

In this chapter we have studied $RS$s on a more generalized class of Kenmotsu manifolds, namely $\beta$-Kenmotsu manifolds. After presenting preliminary facts in section 3.2, we have studied $RAS$ on concircular Ricci pseudosymmetric $\beta$-Kenmotsu manifolds in section 3.3. For a Ricci pseudosymmetric manifold, there is a naturally defined function denoted by $L_S$. We have found the critical value of this function imposing $RAS$ structure on the manifold under consideration and characterize the $RAS$ according to this critical value. We have also constructed an explicit example to verify our results. Next in section 3.4 we have applied $D$-homothetic deformation on our adapted $RS$ structure and observed that the said structure transforms into an $\eta$-$RS$ structure. In section 3.5 we have studied $RAS$ on $\beta$-Kenmotsu manifold with respect to $\nabla^*$ and proved that the structure remains invariant with respect to $\nabla$ for a non-regular $\beta$-Kenmotsu manifold. Finally, we have proved that if the metric of a $\beta$-Kenmotsu
The contents of this chapter consists of the papers ([75], [35]).

3.2 Preliminaries

In a $\beta$-Kenmotsu manifold, the following relations hold ([90], [108], [124]):

\[
(\nabla_X \eta)(Y) = \beta [g(X,Y) - \eta(X)\eta(Y)], \quad (3.2.1)
\]

\[
R(X,Y)\xi = -\beta^2 [\eta(Y)X - \eta(X)Y]
+ (X\beta)\{Y - \eta(Y)\xi\} - (Y\beta)\{X - \eta(X)\xi\}, \quad (3.2.2)
\]

\[
R(\xi,X)Y = [\beta^2 + (\xi\beta)][\eta(Y)X - g(X,Y)\xi], \quad (3.2.3)
\]

\[
\eta(R(X,Y)Z) = \beta^2 [\eta(Y)g(X,Z) - \eta(X)g(Y,Z)]
- (X\beta)\{g(Y,Z) - \eta(Y)\eta(Z)\}
+ (Y\beta)\{g(X,Z) - \eta(Z)\eta(X)\}, \quad (3.2.4)
\]

\[
S(X,\xi) = -(n-1)\beta^2 + (\xi\beta)\eta(X) - (n-1-1)(X\beta), \quad (3.2.5)
\]

for all $X, Y, Z \in \chi(M)$.

Let $(g, \xi, \lambda)$ be a $RAS$ on a $\beta$-Kenmotsu manifold $M^n(\phi, \xi, \eta, g)$. Then from (10), we get

\[
(\mathcal{L}_\xi g)(X,Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi)
\]
\[
\begin{align*}
&= \beta[g(X - \eta(X)\xi, Y) + g(X, Y - \eta(Y)\xi)] \\
&= 2\beta[g(X, Y) - \eta(X)\eta(Y)],
\end{align*}
\]

i.e.

\[
\frac{1}{2}(\mathcal{L}_\xi g)(X, Y) = \beta\{g(X, Y) - \eta(X)\eta(Y)\}. \tag{3.2.6}
\]

From (2) and (3.2.6) we have

\[
S(X, Y) = -(\beta + \lambda)g(X, Y) + \beta\eta(X)\eta(Y), \tag{3.2.7}
\]

which yields

\[
QX = -(\beta + \lambda)X + \beta\eta(X)\xi, \tag{3.2.8}
\]

\[
S(X, \xi) = -\lambda\eta(X), \tag{3.2.9}
\]

\[
r = -n\lambda - (n - 1)\beta, \tag{3.2.10}
\]

where \(Q\) is the Ricci operator and \(r\) is the scalar curvature of \(M^n(\phi, \xi, \eta, g)\).

Let \(M(\phi, \xi, \eta, g)\) be an almost contact metric manifold. Then we have two naturally defined distribution in the tangent bundle \(TM\) of \(M\) as follows [121]

\[
\mathcal{H} = \ker(\eta), \mathcal{V} = \text{span}(\xi).
\]

Then we have \(\mathcal{H} \oplus \mathcal{V} = TM, \mathcal{H} \cap \mathcal{V} = 0\) and \(\mathcal{H} \perp \mathcal{V}\). This decomposition allows one to define the \(\nabla^*\) over an almost contact metric structure. The \(\nabla^*\) on a \(\beta\)-Kenmotsu manifold with respect to \(\nabla\) is defined by [158]

\[
\nabla^*_X Y = \nabla_X Y + \beta[g(X, Y)\xi - \eta(Y)X]. \tag{3.2.11}
\]
If \( R \) and \( R^\ast \), \( S \) and \( S^\ast \) and \( r \) and \( r^\ast \) be the Riemann curvature tensor, Ricci curvature and scalar curvature in a 3-dimensional \( \beta \)-Kenmotsu manifold with respect to \( \nabla \) and \( \nabla^\ast \) respectively, then we have [158]

\[
R^\ast(X,Y)Z = R(X,Y)Z + \beta^2 \{ g(Y,Z)X - g(X,Z)Y \} \\
+ \dot{\beta} \{ g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\
+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \},
\]

\[
S^\ast(X,Y) = S(X,Y) + (2\beta^2 + \dot{\beta})g(Y,Z) + \dot{\beta} \eta(Y)\eta(Z),
\]

and

\[
r^\ast = r + 6\beta^2 + 4\dot{\beta}.
\]

for all \( X, Y, Z \in \chi(M) \), where we are assuming \( \dot{\beta} = \xi \beta \). A \( \beta \)-Kenmotsu manifold is said to be regular if \( \beta^2 + \dot{\beta} \neq 0 \). The interesting fact about the connection \( \nabla^\ast \) is that the \( g, \xi \) and \( \eta \) are all parallel with respect to this connection.

### 3.3 Concircular Ricci pseudosymmetry and Ricci almost solitons

This section deals with the study of RASs on concircular Ricci pseudosymmetric \( \beta \)-Kenmotsu manifolds. A concircular curvature tensor is an interesting invariant of a concircular transformation. A transformation of a \( \beta \)-Kenmotsu manifold \( M^n(\phi, \xi, \eta, g) \), which transforms every geodesic circle of \( M \) into a geodesic circle, is called a concircular transformation [155]. A concircular transformation is always a conformal transformation. Here geodesic circle means a curve in \( M \) whose first curvature is constant and
whose second curvature is identically zero. Thus the geometry of concircular transfor-
mations, that is, the concircular geometry, is a generalization of inversive geometry
in the sense that the change of metric is more general than that induced by a circle
preserving diffeomorphism. The interesting invariant of a concircular transformation
is the concircular curvature tensor ̃\(C\), which is defined by [155]
\[
\tilde{C}(X, Y)Z = R(X, Y)Z - \frac{r}{n(n - 1)} [g(Y, Z)X - g(X, Z)Y],
\]  
(3.3.1)
where \(R\) is the curvature tensor and \(r\) is the scalar curvature of the \(\beta\)-Kenmotsu
manifold \(M^n(\phi, \xi, \eta, g)\).

Using (5), (8), (9), (3.2.2) and (3.2.4), we get
\[
\tilde{C}(X, Y)\xi = -[\beta^2 + \frac{r}{n(n - 1)}][\eta(Y)X - \eta(X)Y]
+ (X\beta)\{Y - \eta(Y)\xi\} - (Y\beta)\{X - \eta(X)\xi\},
\]  
(3.3.2)
\[
\eta(\tilde{C}(X, Y)U) = [\beta^2 + \frac{r}{n(n - 1)}][\eta(Y)g(X, U) - \eta(X)g(Y, U)]
- (X\beta)\{g(Y, U) - \eta(Y)\eta(U)\}
+ (Y\beta)\{g(X, U) - \eta(X)\eta(U)\}.  
\]  
(3.3.3)

A \(\beta\)-Kenmotsu manifold \(M^n(\phi, \xi, \eta, g)\), is said to be concircular Ricci pseudosym-
metric if its concircular curvature tensor \(\tilde{C}\) satisfies
\[
(\tilde{C}(X, Y) \cdot S)(Z, U) = L_S Q(g, S)(Z, U; X, Y)
\]  
(3.3.4)
on \(U_S = \{x \in M : S \neq \xi g\text{ at } x\}\), where \(L_S\) is some function on \(U_S\).

Let us take a concircular Ricci pseudosymmetric \(\beta\)-Kenmotsu manifold \(M^n(\phi, \xi, \eta, g)\)
whose metric is $RAS$. Then by virtue of (3.3.4) that
\[
S(\tilde{C}(X,Y)Z,U) + S(Z,\tilde{C}(X,Y)U) = L_S[g(Y,Z)S(X,U) \tag{3.3.5}
\]
\[
-g(X,Z)S(Y,U) + g(Y,U)S(X,Z) - g(X,U)S(Y,Z).
\]

By virtue of (3.2.7) it follows from (3.3.5) that
\[
\eta(\tilde{C}(X,Y)Z)\eta(U) + \eta(Z)\eta(\tilde{C}(X,Y)U) \tag{3.3.6}
\]
\[
= L_S[g(Y,Z)\eta(X)\eta(U) - g(X,Z)\eta(Y)\eta(U)
\]
\[
+ g(Y,U)\eta(X)\eta(Z) - g(X,U)\eta(Y)\eta(Z)].
\]

Setting $Z = \xi$ in (3.3.6) and using (3.3.2) and (3.3.3), we get
\[
[L_S + \beta^2 + \frac{r}{n(n-1)}][\eta(Y)g(X,U) - \eta(X)g(Y,U)] \tag{3.3.7}
\]
\[
- (X\beta)\{g(Y,U) - \eta(Y)\eta(U)\} + (Y\beta)\{g(X,U) - \eta(X)\eta(U)\} = 0.
\]

Putting $Y = \xi$ in (3.3.7) and using (5), (8), (9), and (3.2.10), we get
\[
[L_S + \beta^2 + (\xi\beta) - \frac{\lambda}{n-1} - \frac{\beta}{n}][g(X,U) + \eta(X)\eta(U)] = 0 \tag{3.3.8}
\]
for all vector fields $X$ and $U$, from which it follows that
\[
L_S = -\beta^2 - (\xi\beta) + \frac{\lambda}{n-1} + \frac{\beta}{n}. \tag{3.3.9}
\]

This leads to the following :

**Theorem 3.3.1.** If $(g,\xi,\lambda)$ is a RAS on a concircular Ricci pseudosymmetric $\beta$-Kenmotsu manifold $M^n(\phi,\xi,\eta,g)$, then $L_S = -\beta^2 - (\xi\beta) + \frac{\lambda}{n-1} + \frac{\beta}{n}$. 
We call this value of $L_S$ as the \textbf{critical value} for $L_S$.

\textbf{Corollary 3.3.1.} In a concircular Ricci pseudosymmetric Kenmotsu manifold $M^n(\phi, \xi, \eta, g)$, the RAS $(g, \xi, \lambda)$ is shrinking, steady and expanding according as $L_S + \frac{n-1}{n} < 0$, $L_S + \frac{n-1}{n} = 0$ and $L_S + \frac{n-1}{n} > 0$, respectively.

\textbf{Example 3.3.1} Following [125], let us consider a 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$, where $(x, y, z)$ are the standard coordinates in $\mathbb{R}^3$. Let \{E_1, E_2, E_3\} be a linearly independent global frame on $M$ given by

\[ E_1 = z^2 \frac{\partial}{\partial x}, E_2 = z^2 \frac{\partial}{\partial y}, E_3 = \frac{\partial}{\partial z}. \]

Let $g$ be the Riemannian metric defined by $g(E_1, E_2) = g(E_1, E_3) = g(E_2, E_3) = 0$, $g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1$. Let $\eta$ be the 1-form defined by $\eta(U) = g(U, E_3)$ for any $U \in \chi(M)$. Let $\phi$ be the $(1, 1)$ tensor field defined by $\phi E_1 = -E_2, \phi E_2 = E_1$ and $\phi E_3 = 0$. Then using the linearity of $\phi$ and $g$ we have

\[ \eta(E_3) = 1, \phi^2 U = -U + \eta(U)E_3 \]

and

\[ g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W) \]

for any $U, W \in \chi(M)$. Thus for $E_3 = \xi$, $(\phi, \xi, \eta, g)$ defines an almost contact metric structure on $M$.

Let $\nabla$ be the $\nabla$ of $g$. Then we have

\[ [E_1, E_2] = 0, [E_1, E_3] = -\frac{2}{z} E_1, [E_2, E_3] = -\frac{2}{z} E_2. \]

Using Koszul formula for the Riemannian metric $g$, we can easily calculate

\[ \nabla_{E_1} E_1 = \frac{2}{z} E_3, \nabla_{E_1} E_2 = 0, \nabla_{E_1} E_3 = -\frac{2}{z} E_1, \]
\[ \nabla_{E_2}E_1 = 0, \nabla_{E_2}E_2 = \frac{2}{z}E_3, \nabla_{E_2}E_3 = -\frac{2}{z}E_2, \]
\[ \nabla_{E_3}E_1 = \nabla_{E_3}E_2 = \nabla_{E_3}E_3 = 0. \]

From the above it can be easily seen that for \( E_3 = \xi, (\phi, \xi, \eta, g) \) is a \( \beta \)-Kenmotsu structure on \( M \). Consequently \( M^3(\phi, \xi, \eta, g) \) is a \( \beta \)-Kenmotsu manifold with \( \beta = -\frac{2}{z} \) [125].

Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:
\[ R(E_1, E_2)E_1 = \frac{4}{z^2}E_2, R(E_1, E_2)E_2 = -\frac{4}{z^2}E_1, \]
\[ R(E_1, E_3)E_1 = \frac{6}{z^2}E_3, R(E_1, E_3)E_3 = -\frac{6}{z^2}E_1, \]
\[ R(E_2, E_3)E_2 = \frac{6}{z^2}E_3, R(E_2, E_3)E_3 = -\frac{6}{z^2}E_2 \]
and the components which can be obtained from these by the symmetry properties from which, we can easily calculate the non-vanishing components of the Ricci tensor as follows:
\[ S(E_1, E_1) = S(E_2, E_2) = -\frac{10}{z^2}, S(E_3, E_3) = -\frac{12}{z^2}. \]

Also the scalar curvature \( r \) is given by:
\[ r = -\frac{32}{z^2}. \]

Since \( \{E_1, E_2, E_3\} \) forms a basis of the 3-dimensional \( \beta \)-Kenmotsu manifold, any vector field \( X, Y, Z, U \in \chi(M) \) can be written as
\[ X = a_1E_1 + b_1E_2 + c_1E_3, \]
\[ Y = a_2E_1 + b_2E_2 + c_2E_3, \]
\[ Z = a_3 E_1 + b_3 E_2 + c_3 E_3, \]

\[ U = a_4 E_1 + b_4 E_2 + c_4 E_3, \]

where \( a_i, b_i, c_i \in \mathbb{R}^+ \) for all \( i = 1, 2, 3 \) such that \( a_i, b_i, c_i \) are not proportional. Then

\[
R(X, Y)Z = -2z^2 \left\{ 2b_3(a_1b_2 - a_2b_1) + 3c_3(a_1c_2 - a_2c_1) \right\} E_1 \tag{3.3.10}
+ \frac{2}{z^2} \left\{ 2a_3(a_1b_2 - a_2b_1) - 3c_3(b_1c_2 - b_2c_1) \right\} E_2
+ \frac{6}{z^2} \left\{ b_3(b_1c_2 - b_2c_1) + a_3(a_1c_2 - a_2c_1) \right\} E_3,
\]

\[
R(X, Y)U = -2z^2 \left\{ 2b_4(a_1b_2 - a_2b_1) + 3c_4(a_1c_2 - a_2c_1) \right\} E_1 \tag{3.3.11}
+ \frac{2}{z^2} \left\{ 2a_4(a_1b_2 - a_2b_1) - 3c_4(b_1c_2 - b_2c_1) \right\} E_2
+ \frac{6}{z^2} \left\{ b_4(b_1c_2 - b_2c_1) + a_4(a_1c_2 - a_2c_1) \right\} E_3.
\]

In view of (3.3.10) we have from (3.3.1) that

\[
\tilde{C}(X, Y)Z = R(X, Y)Z - \frac{r}{6} [g(Y, Z)X - g(X, Z)Y]
\]

\[
= -2z^2 \left\{ 2b_3(a_1b_2 - a_2b_1) + 3c_3(a_1c_2 - a_2c_1) \right\}
- \frac{8}{3} \left\{ a_1(b_2b_3 + c_2c_3) - a_2(b_1b_3 + c_3c_1) \right\} E_1
+ \frac{2}{z^2} \left\{ 2a_3(a_1b_2 - a_2b_1) - 3c_3(b_1c_2 - b_2c_1) \right\} E_2
+ \frac{8}{3} \left\{ b_1(a_2a_3 + c_2c_3) - b_2(a_1a_3 + c_3c_1) \right\} E_2
+ \frac{2}{z^2} \left\{ 3b_3(b_1c_2 - b_2c_1) + a_3(a_1c_2 - a_2c_1) \right\} E_3
+ \frac{8}{3} \left\{ c_1(a_2a_3 + b_2b_3) - c_2(a_1a_3 + b_1b_3) \right\} E_3.
\]
Hence

\[
S(\tilde{C}(X,Y)Z, U) \tag{3.3.12}
\]
\[
= \frac{20a_4}{z^4} (2b_3(a_1b_2 - a_2b_1) + 3c_3(a_1c_2 - a_2c_1)
- \frac{8}{3} \{a_1(b_2b_3 + c_2c_3) - a_2(b_1b_3 + c_3c_1)\}
- \frac{20b_4}{z^4} [2a_3(a_1b_2 - a_2b_1) - 3c_3(b_1c_2 - b_2c_1)
+ \frac{8}{3} \{b_1(a_2a_3 + c_2c_3) - b_2(a_1a_3 + c_3c_1)\}
- \frac{24c_4}{z^4} [3\{b_3(b_1c_2 - b_2c_1) + a_3(a_1c_2 - a_2c_1)\]
+ \frac{8}{3} \{c_1(a_2a_3 + b_2b_3) - c_2(a_1a_3 + b_1b_3)\}].
\]

Similarly we obtain

\[
S(Z, \tilde{C}(X,Y)U) \tag{3.3.13}
\]
\[
= \frac{20a_3}{z^4} (2b_4(a_1b_2 - a_2b_1) + 3c_4(a_1c_2 - a_2c_1)
- \frac{8}{3} \{a_1(b_2b_4 + c_2c_4) - a_2(b_1b_4 - c_1c_4)\}
- \frac{20b_3}{z^4} [2a_4(a_1b_2 - a_2b_1) - 3c_4(b_1c_2 - b_2c_1)
+ \frac{8}{3} \{b_1(a_2a_4 + c_2c_4) - b_2(a_1a_4 + c_3c_4)\}
- \frac{24c_3}{z^4} [3\{b_4(b_1c_2 - b_2c_1) + a_4(a_1c_2 - a_2c_1)\]
+ \frac{8}{3} \{c_1(a_2a_4 + b_2b_4) - c_2(a_1a_4 + b_1b_4)\}].
\]
Now we have

\[
\begin{align*}
  g(Y, Z) &= a_2a_3 + b_2b_3 + c_2c_3, \\
  g(X, Z) &= a_1a_3 + b_1b_3 + c_1c_3, \\
  g(Y, U) &= a_2a_4 + b_2b_4 + c_2c_4, \\
  g(X, U) &= a_1a_4 + b_1b_4 + c_1c_4.
\end{align*}
\]

(3.3.14)

Also we have

\[
\begin{align*}
  S(Y, Z) &= -\frac{2}{z^2}(5a_2a_3 + 5b_2b_3 + 6c_2c_3), \\
  S(X, Z) &= -\frac{2}{z^2}(5a_1a_3 + 5b_1b_3 + 6c_1c_3), \\
  S(Y, U) &= -\frac{2}{z^2}(5a_2a_4 + 5b_2b_4 + 6c_2c_4), \\
  S(X, U) &= -\frac{2}{z^2}(5a_1a_4 + 5b_1b_4 + 6c_1c_4).
\end{align*}
\]

(3.3.15)

Therefore from (3.3.14) and (3.3.15) we have

\[
\begin{align*}
  g(Y, Z)S(X, U) - g(X, Z)S(Y, U) &
  + g(Y, U)S(X, Z) - g(X, U)S(Y, Z) \\
  &= \frac{2}{z^2}[(a_1c_2 - a_2c_1)(a_3c_4 + a_4c_3) + (b_1c_2 - b_2c_1)(b_3c_4 + b_4c_3)] \\
  &\neq 0,
\end{align*}
\]

since \(a_i, b_i, c_i\) are not proportional and assume that \((a_1c_2 - a_2c_1)(a_3c_4 + a_4c_3) + (b_1c_2 - b_2c_1)(b_3c_4 + b_4c_3) \neq 0\).

Also from (3.3.12) and (3.3.13) we get

\[
S(\tilde{C}(X,Y)Z, U) + S(Z, \tilde{C}(X,Y)U) \]
\[ \frac{44}{3z^4}[(a_1c_2 - a_2c_1)(a_3c_4 + a_4c_3) + (b_1c_2 - b_2c_1)(b_3c_4 + b_4c_3)] \neq 0. \]

Let us consider the function

\[ L_S = \frac{22}{3z^2} \]  \hspace{1cm} (3.3.18)

By virtue of (3.3.18) we have from (3.3.16) and (3.3.17) that

\[
S(\check{C}(X,Y)Z,U) + S(Z,\check{C}(X,Y)U) = L_S[g(Y,Z)S(X,U) - g(X,Z)S(Y,U) - g(X,U)S(Y,Z)].
\]

Hence the \( \beta \)-Kenmotsu manifold \( M^3(\phi, \xi, \eta, g) \) is concircular Ricci pseudosymmetric. If \( (g, \xi, \lambda) \) is a RAS on this \( \beta \)-Kenmotsu manifold \( M^3(\phi, \xi, \eta, g) \), then from (3.2.10) we get

\[ r = -3\lambda - 2\beta, \]

i.e.,

\[ -\frac{32}{z^2} = -3\lambda + \frac{4}{z}, \]

i.e.,

\[ \lambda = \frac{4}{3} \left( \frac{1}{z} + \frac{8}{z^2} \right) \]

and hence from (3.3.9) we get

\[
L_S = -\beta^2 - (\xi\beta) + \frac{\lambda}{2} + \frac{\beta}{3} = \frac{22}{3z^2}, \quad \text{as} \quad \beta = -\frac{2}{z}, \xi = E_3 = \frac{\partial}{\partial z},
\]
which satisfies (3.3.18). Thus Theorem (3.3.1) is verified.

### 3.4 $D$-homothetic deformation and Ricci solitons

Let $M(\phi, \xi, \eta, g)$ be an almost contact metric manifold with $\dim M = n$. Then $\eta = 0$ defines a $(n - 1)$-dimensional distribution $D$ on $M$. By an $D$-homothetic deformation [143], we mean a change of structure tensor of the form

$$\begin{align*}
\tilde{\phi} &= \phi, \\
\tilde{\xi} &= \frac{1}{a}\xi, \\
\tilde{\eta} &= a\eta, \\
\tilde{g} &= ag + a(a - 1)\eta \otimes \eta,
\end{align*}
$$

(3.4.1)

where $a$ is a non-zero positive constant. This is to be noted that if $M(\phi, \xi, \eta, g)$ is an almost contact metric manifold then $\tilde{M}(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is also an almost contact metric manifold. Now we have from (10) that

$$
(\mathcal{L}_{\xi}g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi)
$$

(3.4.2)

$$
= 2\beta[g(X, Y) - \eta(X)\eta(Y)].
$$

Since the Lie derivative operator only depends on the smooth structure of the underlying manifold so we have

$$
(\mathcal{L}_{\xi}g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi)
$$

(3.4.3)

$$
= 2\beta[g(X, Y) - \eta(X)\eta(Y)]
$$

$$
= (\mathcal{L}_{\xi}\tilde{g})(X, Y).
$$
Again from (2) and (3.4.2) we have

$$S(X,Y) = -(\beta + \lambda)g(X,Y) + \beta \eta(X)\eta(Y). \quad (3.4.4)$$

Now applying $D$-homothetic deformation we have from (3.4.4) that

$$\bar{S}(X,Y) = -\frac{\lambda + \beta}{a}\bar{g}(X,Y) + \frac{(a - 1)(\lambda + \beta) + 1}{a^2}\bar{\eta}(X)\bar{\eta}(Y). \quad (3.4.5)$$

Next from (3.4.3) and (3.4.5) we have

$$(\mathcal{L}_{\xi}\bar{g})(X,Y) + 2\bar{S}(X,Y) + \bar{\lambda}\bar{g}(X,Y) + \bar{\mu}\bar{\eta}(X)\bar{\eta}(Y) = 0$$

for all $X, Y, Z \in \chi(M)$ with $\bar{\lambda} = \frac{\lambda - \beta}{a}$ and $\bar{\mu} = \frac{a\beta - 3a\lambda - \beta}{a^2}$. This shows that the structure $(\bar{g}, \bar{\xi}, \bar{\lambda}, \bar{\mu})$ is an $\eta$-RS structure on $\bar{M}$. Hence we can state the following:

**Theorem 3.4.1.** If $(g, \xi, \lambda)$ is a RS structure on $M$, then the $D$-homothetic deformation transforms the RS structure into an $\eta$-RS structure.

### 3.5 Schouten-van Kampen connection and Ricci solitons

Let the metric $g$ of a 3-dimensional $\beta$-Kenmotsu manifold be a RAS with respect to $\nabla$. Now

$$(\mathcal{L}_{\xi}g)(X,Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) \quad (3.5.1)$$

$$= g(\nabla^*_X \xi - \beta\{\eta(X)\xi - \xi\}), Y)$$

$$+ g(X, \nabla^*_Y \xi - \beta\{\eta(Y)\xi - \xi\})$$

$$= g(\nabla^*_X \xi, Y) + g(X, \nabla^*_Y \xi) - \beta g(\eta(X)\xi, Y)$$

$$= \beta \eta(Y) + \beta \eta(X) - \beta g(X, \eta(Y)\xi)$$
Now from (2) and (3.5.1) we have

\[(L_\xi^* g)(X, Y) + 2S^*(X, Y) + 2\lambda g(X, Y) = (2\beta - 2\dot{\beta})\eta(X)\eta(Y) - \beta \{\eta(X) + \eta(Y)\} - 2(2\beta^2 + \dot{\beta})g(X, Y).\]  

If \((g, \xi, \lambda)\) is a RAS structure on \(M\) with respect to \(\nabla\) then the RAS structure is preserved for the \(\nabla^*\) if and only if

\[(2\beta - 2\dot{\beta})\eta(X)\eta(Y) = \beta \{\eta(X) + \eta(Y)\} + 2(2\beta^2 + \dot{\beta})g(X, Y) \tag{3.5.3}\]

holds for arbitrary \(X, Y \in \chi(M)\).

So in particular putting \(X = Y = \xi\) in (3.5.3), we have \(\beta^2 + \dot{\beta} = 0\). This leads to the following:

**Theorem 3.5.1.** Let \((g, \xi, \lambda)\) be a RAS structure on non-regular \(\beta\)-Kenmotsu manifold \(M\) with respect to \(\nabla\) then the RAS structure is preserved for \(\nabla^*\).

Next, let \((g, \xi, \lambda)\) be a RAS on \(M\) with respect to \(\nabla^*\). Then we have

\[(L_\xi^* g)(X, Y) + 2S^*(X, Y) + 2\lambda g(X, Y) = 0.\]  

But from (3.2.11), we have

\[(L_\xi^* g)(X, Y) = 0,\]

which shows that the Reeb vector field is Killing. Hence (3.5.4) implies that

\[S^*(X, Y) = -\lambda g(X, Y).\]

Thus we can state the following:
Theorem 3.5.2. If \((g, \xi, \lambda)\) is a RS on a \(\beta\)-Kenmotsu manifold with respect to \(\nabla^*\), then the manifold is \(K\)-contact Einstein.