Chapter 2

\(\eta\)-Ricci solitons on \(\eta\)-Einstein Kenmotsu manifolds

2.1 Introduction

The notion of Einstein manifolds comes naturally as the solution of the vacuum Einstein field equation in general relativity and this kind of metric has been studied extensively in the framework of contact metric geometry. As a generalization of Einstein metrics Okumura studied \(\eta\)-Einstein manifolds on contact structures [106]. This kind of metrics has been studied by Boyer in the context of Sasakian geometry [27]. In [90] Kenmotsu introduced a new class of non-compact almost contact metric structure which are now days called Kenmotsu manifolds and studied \(\eta\)-Einstein geometry on this structure. In this connection we note that in dimension three the notion on Kenmotsu and \(\eta\)-Einstein manifolds are same. The distinction occurs on dimension \(> 3\). In this chapter we have discussed \(\eta\)-\(RS\) on \(\eta\)-Einstein Kenmotsu manifolds.

After giving necessary preliminary results in section 2.2, our main results have been discussed in section 2.3, where it is proved that if \(\xi\) is a recurrent torse forming
\( \eta\)-RS on an \( \eta\)-Einstein Kenmotsu manifold \((M, g, \xi, \lambda, \mu, a, b)\) then \( \xi \) is (i) concurrent and (ii) Killing vector field.

The contents of this chapter has been published in the paper [85].

### 2.2 Preliminaries

**Proposition 2.2.1.** In an \( \eta\)-Einstein Kenmotsu manifold, the following relations hold:

\[
S(\phi X, Y) = a g(\phi X, Y) = -a g(X, \phi Y) = -S(X, \phi Y), \quad (2.2.1)
\]

\[
S(X, \xi) = (a + b) \eta(X), \quad S(\xi, \xi) = (a + b), \quad (2.2.2)
\]

\[
S(\phi X, \phi Y) = a[g(X, Y) - \eta(X) \eta(Y)]. \quad (2.2.3)
\]

**Proof.** In view of (5) - (9), the proposition follows.

**Definition 2.2.1.** A vector field \( \xi \) is called torse forming ([21], [101]) if it satisfies

\[
\nabla_X \xi = fX + \gamma(X) \xi, \quad (2.2.4)
\]

for all vector fields \( X \) on \( M \), where \( f \in C^\infty(M) \) and \( \gamma \) is a 1-form.

A torse forming vector field \( \xi \) is called recurrent if \( f = 0 \).

**Definition 2.2.2.** A vector field \( V \) is called concurrent [38] if it satisfies

\[
\nabla_X V = 0 \quad (2.2.5)
\]

for any vector field \( X \in \chi(M) \).

### 2.3 Main results

In this section, we study \( \eta\)-RS on \( \eta\)-Einstein Kenmotsu manifolds \((M, g, \xi, \lambda, \mu, a, b)\) and in similar to Proposition 2.2 of the paper [21], we prove the following:
Theorem 2.3.1. If \((M, g, \xi, \lambda, \mu, a, b)\) is an \(\eta\)-RS on an \(\eta\)-Einstein Kenmotsu manifold, then

(i) \(a + b + \lambda + \mu = 0\),

(ii) \(\xi\) is a geodesic vector field.

Proof. Let \((M, g, \xi, \lambda, \mu, a, b)\) be an \(\eta\)-RS on an \(\eta\)-Einstein Kenmotsu manifold. In view of (35) we have from (4) that

\[ g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) + 2[(a + \lambda)g(X, Y) + 2(b + \mu)\eta(X)\eta(Y)] = 0. \] (2.3.1)

Putting \(X = Y = \xi\) in (2.3.1) and using (5), (6), (8) and (9) we obtain \(g(\nabla_\xi \xi, \xi) = -(a + b + \lambda + \mu)\), but \(g(\nabla_X \xi, \xi) = 0\) for any vector field \(X\) on \(M\), since \(\xi\) has a constant norm. Hence we get (i).

Consequently (2.3.1) becomes

\[ g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) + 2(a + \lambda)[g(X, Y) - \eta(X)\eta(Y)] = 0. \] (2.3.2)

Setting \(Y = \xi\) in (2.3.2) and using (5), (8), (9) and (10) we get \(g(\nabla_\xi \xi, X) = 0\) for any vector field \(X\) on \(M\) and hence we have \(\nabla_\xi \xi = 0\), i.e., \(\xi\) is a geodesic vector field. Thus we get (ii).

Also in similar to Proposition 2.3 of the paper [21], we prove the following:

Theorem 2.3.2. If \(\xi\) is a torse forming \(\eta\)-RS on an \(\eta\)-Einstein Kenmotsu manifold \((M, g, \xi, \lambda, \mu, a, b)\) then \(f = -(a + \lambda)\), \(\eta\) is closed, \(b = -a + (n - 1)(a + \lambda)^2\) and \(\mu = -\lambda - (n - 1)(a + \lambda)^2\).

Proof. Let \(\xi\) be a torse forming \(\eta\)-RS on an \(\eta\)-Einstein Kenmotsu manifold \((M, g, \xi, \lambda, \mu, a, b)\).

Then we have from (2.2.4) that \(g(\nabla_X \xi, \xi) = f\eta(X) + \gamma(X)\) and hence we get \(\gamma = -f\eta\).

Consequently (2.2.4) becomes

\[ \nabla_X \xi = f[X - \eta(X)\xi]. \] (2.3.3)
Using (2.3.3) in (2.3.2), we get

\[(f + a + \lambda)[g(X, Y) - \eta(X)\eta(Y)] = 0 \quad (2.3.4)\]

for all \(X\) and \(Y \in \chi(M)\) and hence it follows that \(f = -(a + \lambda)\). Thus we get from (2.3.3) that

\[\nabla_X \xi = -(a + \lambda)[X - \eta(X)\xi], \quad (2.3.5)\]

which means that \(\nabla_X \xi\) is collinear to \(\phi^2 X\) for all \(X\) and hence we get \(d\eta = 0\), i.e., \(\eta\) is closed.

It is known that

\[R(X, Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X, Y]} \xi. \quad (2.3.6)\]

In view of (2.3.5), (2.3.6) yields

\[R(X, Y)\xi = (a + \lambda)^2[\eta(X)Y - \eta(Y)X]. \quad (2.3.7)\]

From (2.3.7), we get

\[S(X, \xi) = (n - 1)(a + \lambda)^2\eta(X). \quad (2.3.8)\]

From (2.2.2) and (2.3.8), we get \(b = -a + (n - 1)(a + \lambda)^2\) and \(\mu = -\lambda - (n - 1)(a + \lambda)^2\).

Thus we get the theorem.

**Corollary 2.3.1.** If \(\xi\) is a torse forming RS on an \(\eta\)-Einstein Kenmotsu manifold \((M, g, \xi, \lambda, \mu, a, b)\) then the RS is shrinking, steady and expanding according as \(a + b > 0\), \(a + b = 0\) and \(a + b < 0\) respectively.

**Proof.** In particular, if \(\mu = 0\) then from Theorem 2.3.1 and Theorem 2.3.2, we get \(\lambda + (n - 1)(a + \lambda)^2 = 0\) and hence we obtain \(\lambda = -(a + b)\). Hence the proof is complete.
Corollary 2.3.2. If $\xi$ is a recurrent torse forming $\eta$-RS on an $\eta$-Einstein Kenmotsu manifold $(M, g, \xi, \lambda, \mu, a, b)$ then $\xi$ is (i) concurrent and (ii) Killing vector field.

Proof. Since $\xi$ is recurrent, therefore $f = 0$ and hence $a + \lambda = 0$. So, by virtue of (2.3.5) we get $\nabla_X \xi = 0$, for all $X$ on $M$, which means that $\xi$ is concurrent vector field. Also in that case
\[(\mathcal{L}_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 0\]
for all $X, Y \in \chi(M)$ that means $\xi$ is Killing vector field.

As a generalization of $\phi$-Ricci symmetric Kenmotsu manifold introduced by Shukla and Shukla [137], recently Hui [83] introduced the notion of $\phi$-pseudo Ricci symmetric Kenmotsu manifolds.

Definition 2.3.1. A Kenmotsu manifold $M^n(\phi, \xi, \eta, g)$ is said to be $\phi$-pseudo Ricci symmetric [83] if the Ricci operator $Q$ satisfies
\[\phi^2((\nabla_X Q)(Y)) = 2A(X)QY + A(Y)QX + S(Y, X)\rho\] (2.3.9)
for any vector fields $X, Y \in \chi(M)$, where $A$ is a non-zero 1-form.

If, in particular, $A = 0$, then (2.3.9) turns into the notion of $\phi$-Ricci symmetric Kenmotsu manifold introduced by Shukla and Shukla [137].

In [83], it is proved that every $\phi$-pseudo Ricci symmetric Kenmotsu manifold is an $\eta$-Einstein manifold and its Ricci tensor $S$ is of the form (35), where $a = \frac{(n-1)A(\xi)}{A(\xi)-1}$ and $b = \frac{(n-1)A(\xi)}{1-A(\xi)}$, provided $1 - A(\xi) \neq 0$. So $a + b = -(n - 1)$ and hence by Theorem 2.3.1, we get $\lambda + \mu = (n - 1)$. Thus by virtue of Theorem 2.3.2, we can state the following:

Theorem 2.3.3. If $\xi$ is a torse forming $\eta$-RS on a $\phi$-pseudo Ricci symmetric Kenmotsu manifold $M$ then $f = -(a + \lambda)$, $\eta$ is closed and $1 + (a + \lambda)^2 = 0$.

Also in view of Corollary 2.3.1, we get $\lambda = -(a + b) = (n - 1) > 0$. This leads to the following:
Corollary 2.3.3. If $\xi$ is a torse forming RS on a $\phi$-pseudo Ricci symmetric Kenmotsu manifold $M$ then the RS $(g, \xi, \lambda)$ is always expanding.