Chapter 3
Study of Some Local Bifurcations in the Logistic Map


In this chapter we have studied some local bifurcations shown by the Logistic map. The analytic and graphical techniques which are used for such investigations are already described in chapter 2 of the thesis. We also derived suitable numerical techniques to find out the higher order periodic points and bifurcation points which can not be found analytically. With the help of bifurcation points found through the numerical mechanism, we have established the Feigenbaum universal constant, also known as Feigenbaum delta which gives us the approximate value of the parameter after which chaos creep into the non-linear deterministic systems viz. The Logistic map which otherwise behave quite regularly before reaching the above mentioned critical value of the parameter at which chaos starts.

3.00 Introduction:
The universality discovered by the elementary particle theorist, Mitchell J. Feigenbaum in 1975 in one-dimensional iterations with the logistic map, \( x_{n+1} = \mu x_n(1-x_n) \), where \( \mu \) is a parameter has successfully led to discover that
large classes of non-linear systems exhibit transitions to chaos which are universal and quantitatively measurable [66, 67, ].

One of his fascinating discoveries is that if a family $f$ presents period doubling bifurcations then there is an infinite sequence $\{\mu_n\}$ of bifurcation values such that

$$\lim_{n \to \infty} \frac{\mu_n - \mu_{n+1}}{\mu_{n+1} - \mu_n} = \delta$$

where $\delta$ is a universal number which is now termed as Feigenbaum constant.

Chaos and order have long been viewed as antagonistic states in science. One of the surprises, revealed through the studies of the nonlinear maps is that both the antagonistic states can be ruled by a single law. An even bigger surprise is the discovery that there is a very well defined ‘route’ which leads from one state – ‘order’ – into the other state – ‘chaos’. Furthermore, it is recognised that this ‘route’ is ‘universal’. Here ‘route’ implies that there are abrupt qualitative changes-called ‘bifurcations’- which mark the transition from ‘order’ into ‘chaos’ like a schedule. ‘Universal’ implies that these bifurcations can be found in many natural systems both qualitatively and quantitatively.

To discuss the dynamics of any system, it becomes very much essential to determine its fixed/periodic points of every order and to discuss their stability as the values of the controlling parameters are varied. But because of the presence of some higher degree terms in the function through iteration of which the dynamical system is produced, it becomes a very tedious job and sometimes impossible to determine analytically the periodic points of higher order. In that case we are forced to adopt some numerical techniques to determine those points.
One generally thinks that a chaotic system needs a complicated formula for its mathematical representation. But, there are very simple functions which can lead not only to chaos, but can make us understand how this chaotic situation gets developed from ‘ordered’ behaviour. The logistic function is one of such functions about which we discussed in chapter 2 and explained why it can be considered as a problem of dynamical systems. The logistic function is

\[ f(x) = \mu x(1 - x) \]  

(1)

Where \( \mu \) indicates the ‘fertility’ or ‘growth rate’ of a population with limited resources. Here, \( 0 \leq \mu \leq 4 \) and \( x \in [0,1] \), the reason of which were discussed in chapter 2.

We can see that this function represents an inverted parabola (Fig.2.1), intersecting the \( x \) axis in (0,0) and (1,0).

![Logistic Map function](image)
For values $0 \leq \mu \leq 4$, the height of the parabola will be in the interval $[0, 1]$. If we make the difference equation

$$x_{n+1} = \mu x_n (1-x_n)$$

With the help of this map and iterate it for a large number of times, we will observe the discrete dynamics of the population that the function models.

Over the last more than twenty five years, the logistic map has served as an example to understand nonlinear dynamics and chaos. As R. M. May stressed some 36 years ago, the patterns formed by iterates of the logistic map are simple to compute but illustrate the complexities possible in nonlinear dynamics [64]. The bifurcation of the logistic map, which summaries the long-time dynamics as a function of the control parameter $\mu$ in equation 1, is one of the most commonly reproduced images of dynamical systems. Also, using this map, M.J. Feigenbaum derived his famous renormalization-group theory of scaling exponents [51, 66].

The universality discovered by Feigenbaum [66] with nonlinear models has successfully led to observe that large classes of nonlinear systems exhibit transition to chaos through period doubling route.

3.2. SchwarzianDerivative:

The Schwarzian derivative of a function $f(x)$ which is defined in the interval $(a, b)$ having higher order derivatives is given by

$$S(f(x)) = \frac{f''''(x)}{f'(x)} - \frac{3}{2} \left( \frac{f''(x)}{f'(x)} \right)^2$$
We have calculated the Schwarzian derivative for the Logistic model $f(x) = \mu x(1 - x)$ and found it to be equal to

$$-\frac{6\mu^2}{(\mu(1-x)-\mu x)^2}$$

which is negative for our considered range of the parameter and for all $x_n \in [0, 1]$. This verifies that our considered map is having a chaotic nature for certain range of the parameter. But, the question is how the so simple looking model which should be deterministic as our common sense say can turn to be chaotic for certain values of the parameter. It must have certain ‘route’ through which it passes to become a chaotic one from being regular. The following investigations are aimed in this line.

### 3.3. Transcritical bifurcation in the Logistic map:

We have already mentioned that bifurcation events are those where some change in the qualitative behaviour of system takes place. These qualitative change may take place regarding the stability nature of fixed points and periodic points of a discrete dynamical system produced by a map. There are certain situations where a fixed point exists for all values of a parameter of the system and can never be destroyed. However, such a fixed point may change its nature of stability as the parameter is varied i.e. a stable fixed point may become unstable or an unstable one can turn out to be a stable one. This type of bifurcation is termed as Transcritical bifurcation.

#### 3.3.1 Stability analysis of the fixed points of the Logistic map:

The logistic map $f(x) = \mu x(1 - x)$ has two fixed points 0 and $1 - \frac{1}{\mu}$ which are solutions of the equation $f(x) = x$. Geometrically, they can be visualised by the points of intersection of the map and the line $y = x$ which are shown below.
To determine the stability nature of these fixed points we employ the analytical ‘linear stability analysis’ technique described in chapter 2 of the thesis.

Since $\frac{df}{dx} = \mu(1 - 2x)$, hence the derivative of $f(x)$ at these fixed points are:

$$\left. \frac{df}{dx} \right|_{x=0} = \mu < 1 \text{ for } \mu < 1.$$  

This means that the fixed point 0 is an attracting or stable fixed point for $\mu < 1$. Also, as

$$\left. \frac{df}{dx} \right|_{x=1-\frac{1}{\mu}} = 2 - \mu > 1 \text{ for } \mu < 1.$$  

The other fixed point $1 - \frac{1}{\mu}$ is a repelling or unstable fixed point for $\mu < 1$. So, it is seen that at $\mu = 1$, a Transcritical bifurcation takes place because for $\mu > 1$ the two fixed points exchange their stability i.e., the fixed point 0 which was stable for
\( \mu < 1 \) becomes an unstable fixed point for \( \mu > 1 \) and the fixed point \( 1 - \frac{1}{\mu} \) which was unstable for \( \mu < 1 \) becomes a stable fixed point for \( \mu > 1 \).

\[ \text{Figure 3.3. Cobweb diagram of Logistic Map for } \mu = 0.4 \text{ and } x_0 = 0.5. \]

The above Cobweb diagram is drawn for \( \mu = 0.4 \) and \( x_0 = 0.5 \) where the iterates are seen converging towards the fixed point \( x = 0 \). Note that for this parameter value i.e. \( \mu = 0.4 \) the other fixed point \( x = 1 - \frac{1}{\mu} \) is found to be negative and hence not seen in the picture. Thus, the figure authenticates that for the parameter value \( \mu = 0.4 \) the fixed point \( x = 0 \) is a stable one whereas the other fixed point \( x = 1 - \frac{1}{\mu} \) is unstable. In the following figure we have drawn the Cobweb diagram for \( \mu = 1.4 > 1 \) with same initial value \( x_0 = 0.5 \). This time the iterates converge to
the other fixed point \( x = 1 - \frac{1}{\mu} \) verifying the fact that in this case the fixed point

\( x = 1 - \frac{1}{\mu} \) is a stable one whereas the other fixed point \( x = 0 \) is an unstable one.

These diagrams visually make it clear that when the parameter crosses the value \( \mu = 1 \), the two fixed points exchange their nature of stability i.e. the fixed point which was stable earlier becomes unstable and vice-versa.

![Cobweb diagram of Logistic Map for \( \mu = 1.4 \) and \( x_0 = 0.5 \).](image)

**Figure 3.4.** Cobweb diagram of Logistic Map for \( \mu = 1.4 \) and \( x_0 = 0.5 \).

As further verification of the fact we have shown the following bifurcation diagram also. The figure clearly shows that after the parameter value \( \mu = 1 \), there is a qualitative change in the bifurcation diagram.
3.4 Period doubling bifurcation and period doubling route to chaos in the Logistic map:

In a period-doubling bifurcation, the previously stable fixed/periodic points become unstable after attaining some value of the control parameter, and stable periodic trajectories of period, doubled to the previous one appears near it. The original fixed/periodic points continues to exist as unstable fixed/periodic points, and all the remaining points are attracted towards the new stable periodic trajectories of doubled period. This process repeats itself up to certain critical value of the parameter where chaos creeps into the previously deterministic system.

3.4.1 Stability analysis of the periodic points and detection of bifurcation points of the Logistic map:

From our discussion in section 3.3.1, we can show that the fixed points 0 and $1 - \frac{1}{\mu}$ of the logistic map $f(x) = \mu x(1-x)$ become unstable as soon as the parameter $\mu$
crosses the value $\mu = 3$. So, it is clear that $\mu = 3$ is a bifurcation value. Moreover, it is to be noted that at the parameter value $\mu = 3$, $\frac{df}{dx}\bigg|_{x=\frac{1}{\mu}} = -1$.

The question that arises is what happens to the system when parameter $\mu$ crosses the value $= 3$? To get a glimpse of it we take help of graphical techniques termed as ‘time series plot’ and ‘cobweb diagram’ which are shown below for the logistic map for the parameter value $\mu = 3.1$. The utility of ‘time series plot’ and ‘cobweb diagram’ were discussed already in chapter 2 of the thesis.

\[ \text{Figure 3.6. Cobweb diagram (top) and the Time series plot (bottom) of Logistic Map verifying that at } \mu = 3.1 \text{ it shows a period 2 behaviour.} \]
The pictures show that the states of the system starts oscillating between two discrete values (instead of going towards a fixed point) and hence we can have an idea that periodic points of period 2 come into play their roles in the dynamics generated by the iteration of the map.

We already know how to find these periodic points both mathematically and geometrically. Mathematically, they are the solutions of the equation $f^2(x) = x$ and geometrically, these are the points of intersection of the map $f^2(x)$ and the line $y = x$ which are shown below:

![Figure 3.7. Fixed points of Second iterate of Logistic Map i.e. $f^2$ geometrically.](image)

The fixed points of the second iterated map $f^2$ or the periodic points of period 2 for the map $f$ are given by

$$f^2(x) = x$$

*i.e.* $-\mu^3x^4 + 2\mu^3x^3 - (\mu^2 + \mu^3)x^2 + (\mu^2 - 1)x = 0$

Solving the above equation we get four solutions, viz.

$$0, \quad 1 - \frac{1}{\mu}, \quad \frac{\mu + 1 + \sqrt{\mu^2 - 2\mu - 3}}{2\mu}, \quad \frac{\mu + 1 - \sqrt{\mu^2 - 2\mu - 3}}{2\mu}$$
It is not easy to find the solutions of the above 4th degree polynomial analytically. But some information helps us in doing so.

Since, for a fixed point the criterion to be followed is $f(x) = x$, hence we have

$$f(f(x)) = x \text{ i.e. } f(x) = x \text{ or } f^2(x) = x.$$  This suggests us that the solutions of $f(x) = x$ are also solutions of $f^2(x) = x$. Thus, out of the 4 solutions of $f^2(x) = x$, two are already known solutions of $f(x) = x$ which were found to be 0 and $1 - \frac{1}{\mu}$. So, we divide the 4th degree polynomial repeatedly by $(x - 0)$ and \{x - \left(1 - \frac{1}{\mu}\right)\} to arrive at a quadratic equation the solutions of which could be found analytically. Those were found to be

$$\frac{\mu + 1 + \sqrt{\mu^2 - 2\mu - 3}}{2\mu} \text{ and } \frac{\mu + 1 - \sqrt{\mu^2 - 2\mu - 3}}{2\mu}$$

Thus, out of the above four solutions of $f^2(x) = x$, the first two are fixed points of $f$, about which we already discussed and saw that those became unstable after the control parameter $\mu$ attains the value $\mu = 3$.

In fact, the other two solutions, the fixed points of $f^2(x)$, are the periodic points of period 2 for the logistic map $f$.

At this point, it is to be noted that as we are considering only the real solutions, for the two fixed points

$$x_1 = \frac{\mu + 1 + \sqrt{\mu^2 - 2\mu - 3}}{2\mu}, \quad x_2 = \frac{\mu + 1 - \sqrt{\mu^2 - 2\mu - 3}}{2\mu}$$
We must have $\mu \geq 3$ which clearly signifies that those points were non-existent for $\mu < 3$. Hence, for $\mu$ values between 0 and 4, these solutions are defined only for $\mu \geq 3$. Moreover at $\mu = 3$, we set $x_1 = x_2 = \frac{\mu - 1}{\mu}$ i.e., the two solutions bifurcate from the fixed point $1 - \frac{1}{\mu}$. Figure 3.2 shows a bifurcation diagram at $\mu = 3$.

Thus, we say that at $\mu = 3$, the logistic map trajectories undergo a period doubling bifurcations. Just below $\mu = 3$, the trajectories converge to a single value of $x$ but just above $\mu = 3$, the trajectories tend to alter between two values of $x$.

Let us see how the derivatives of the map function and the second iterate function change at the bifurcation value.

Equation $\frac{df}{dx}\big|_{x=\frac{\mu - 1}{\mu}} = 2 - \mu$ tells us that $\frac{df}{dx}$ passes through the value $-1$ as $\mu$ increases through 3. Next we can evaluate the derivative of the second iterate
function by using the chain-rule of differentiation which is given by the following rule.

\[
\frac{df^2(x)}{dx} = \frac{df(f(x))}{dx} = \frac{df}{dx} \frac{df}{dx(x)}
\]

If we now evaluate the derivative at one of the fixed points say, \(x_1\), we find

\[
\left.\frac{df^2(x)}{dx}\right|_{x_1} = \left.\frac{df}{dx}\right|_{f(x_1)=x_2} \left.\frac{df}{dx}\right|_{x_1} = \left.\frac{df^2(x)}{dx}\right|_{x_2}
\]

Equation (2) states a rather surprising and important result-

The derivative of \(f^2\) are the same at both the fixed points (periodic points) that are part of the two cycles. This result tells us that both these fixed points (periodic points) are stable or both are unstable and that they have the same ‘degree’ of stability or instability. Again, since the derivative of \(f(x)\) is equal to \(-1\) for \(\mu = 3\), equation (2) tells us that the derivative of \(f^2\) is equal to \(+1\) for \(\mu = 3\). As \(\mu\) increases further, the derivative of \(f^2\) decreases and the fixed points (periodic points) become stable. [50].

For \(\mu\) just greater than 3, we see that the slope of \(f^2\) at those two fixed points is less than 1 and hence they are stable fixed points of \(f^2\). Besides this, the unstable fixed point of \(f(x)\) located at \(1 - \frac{1}{\mu}\) is also an unstable fixed point of \(f^{(2)}(x)\). The two 2-cycle fixed points of \(f^{(2)}\) continue to be stable fixed points until \(\mu = 1 + \sqrt{6}\).
This can be found analytically by calculating \( \frac{d f^2(x)}{dx} \) and checking till when

\[
\left| \frac{d f^2(x)}{dx} \right| < 1.
\]

This value of \( \mu \), which is denoted by \( \mu_2 \), the derivative of \( f^2 \) evaluated at the two cycle fixed points is equal to \(-1\) and for values of \( \mu \) larger than \( \mu_2 \), the derivative is more negative than \(-1\). Hence for \( \mu \) values greater than \( \mu_2 \), the 2-cycle points are unstable fixed points. Thus, analytically we could found out that there were two bifurcation points one given by \( \mu_1(say) = 3 \) and the other given by \( \mu_2(say) = 1+\sqrt{6} \).

Employing the graphical techniques mentioned already we get an idea that for \( \mu \) values just greater than \( \mu_2 = 1+\sqrt{6} \), the trajectories settle into a 4 cycle, i.e. the trajectory cycles among 4 values which we can label as \( x_1^* \), \( x_2^* \), \( x_3^* \) and \( x_4^* \). The figures are as shown below.
Figure 3.9. Cobweb diagram (top) and the Time series plot (bottom) of Logistic Map verifying that at $\mu > 1 + \sqrt{6}$ it shows a period 4 behaviour.

To determine these periodic points we need to solve an eight degree equation viz. $f^4(x) = x$ which is impossible analytically and hence we have to adopt numerical methods to find out. The following numerical method helps us in finding out the next and higher order periodic points and bifurcation points.

### 3.5 Numerical scheme to find Periodic Points, Derivatives of Different Iterates of the Map and the Bifurcation Points

(i) To find a fixed point or a periodic point of the Logistic map $f$, we applied the Newton-Recurrence formula given by:

$$x_{n+1} = x_n - \frac{g(x_n)}{\frac{d}{dx} g(x_n)}$$

where $n = 1, 2, 3, ...$

Later on, we will see that the map $g$ is equal to $f^{(k)} - I$, where $I$ is the identity function and $f$ is defined by the map $f(x, \mu) = \mu x (1 - x)$. 
We know that the Newton Raphson [89] formula gives the zero(es) of a map, and to apply this numerical tool one needs a number of recurrence formulae which are given below:

Let the initial value of \( x \) be \( x_0 \).

Then

\[
f(x_0) = \mu x_0(1-x_0) = x_1 \quad \text{(say)} \quad \text{[for convenience we omit the parameter \( \mu \)]}
\]

\[
f^2(x_0) = f(x_1) = \mu x_1(1-x_1) = x_2 \quad \text{(say)}.
\]

Proceeding in this manner, the following recurrence formula can be established:

\[
x_n = \mu x_{n-1}(1-x_{n-1}), \quad n = 1, 2, 3, \ldots
\]

(ii) Again, the derivative of \( f^k \) can be obtained as follows:

\[
\frac{df}{dx}\bigg|_{x=x_0} = \mu(1-2x_0).
\]

By the chain rule of differentiation we get

\[
\frac{df^2}{dx}\bigg|_{x=x_0} = \frac{df}{dx}\bigg|_{f(x_0)} \frac{df}{dx}\bigg|_{x=x_0} = (\mu(1-2x_1))(\mu(1-2x_0))
\]

where \( x_1 = f(x_0) \).

Proceeding in this way we can obtain

\[
\frac{df^k}{dx}\bigg|_{x=x_0} = (\mu(1-2x_{k-1}))(\mu(1-2x_{k-2}))\ldots(\mu(1-2x_0)).
\]

We recall that the value of \( m \) will be the bifurcation value for the map \( f^k \) when its derivative \( \frac{df^k}{dx} \) at a periodic point equals \( -1 \).

We notice that if we put

\[
I = \frac{df^k}{dx} + 1
\]
then \( I \) turns out to be a function of the parameter \( \mu \). The bifurcation value of the parameter \( \mu \) of the period \( k \) occurs when \( I(\mu) \) equals zero. This means, in order to find a bifurcation value of period \( k \), one needs the zero(es) of the function \( I(\mu) \), which is given by the Secant method \([89]\), applied to the function \( I(\mu) \). The rule for the secant method is given by

\[
\mu_{k+1} = \mu_k - \frac{I(\mu_k)(\mu_k - \mu_{k-1})}{I(\mu_k) - I(\mu_{k-1})}.
\]

This method depends very sensitively on the initial condition. Unless the initial approximations are good, we cannot find the bifurcation points correctly. So, we have to find the first two bifurcation points up to a high precision which we found out analytically in the preceding section. Then these two values together with the relation \( \mu_3 \approx \frac{\mu_2 - \mu_1}{\delta} + \mu_2 \) give us a good initial approximation for the 3rd bifurcation value. After finding the third bifurcation value, same process can be repeated to find the fourth and higher bifurcation values. So, it is of prime importance that we find the first two bifurcation values up to a high precision.

With the help of computer programs (given in the appendix) generated by the algorithm discussed in detail above, we obtained the bifurcation points which are furnished below:
Table 3.5

<table>
<thead>
<tr>
<th>Bifurcation points</th>
<th>Periods</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_1 = 3$</td>
<td>Start of Period 2 points</td>
</tr>
<tr>
<td>$\mu_2 = 3.44948974278317…$</td>
<td>Start of Period 4 points</td>
</tr>
<tr>
<td>$\mu_3 = 3.54409035955192…$</td>
<td>Start of Period 8 points</td>
</tr>
<tr>
<td>$\mu_4 = 3.56440726609543…$</td>
<td>Start of Period 16 points</td>
</tr>
<tr>
<td>$\mu_5 = 3.56875941954382…$</td>
<td>Start of Period 32 points</td>
</tr>
<tr>
<td>$\mu_6 = 3.56969160980139…$</td>
<td>Start of Period 64 points</td>
</tr>
<tr>
<td>$\mu_7 = 3.56989125937812…$</td>
<td>Start of Period 128 points</td>
</tr>
<tr>
<td>$\mu_8 = 3.56993401837397…$</td>
<td>Start of Period 256 points</td>
</tr>
<tr>
<td>$\mu_9 = 3.5699431760484…$</td>
<td>Start of Period 512 points</td>
</tr>
<tr>
<td>$\mu_{10} = 3.56994513734217…$</td>
<td>Start of Period 1024 points</td>
</tr>
</tbody>
</table>

Based on these bifurcation values, we computed

\[
\delta_1 = \frac{\mu_2 - \mu_1}{\mu_3 - \mu_2} = 4.751446218177…,
\]

\[
\delta_2 = \frac{\mu_3 - \mu_2}{\mu_4 - \mu_3} = 4.656251017651…,
\]

\[
\delta_3 = \frac{\mu_4 - \mu_1}{\mu_5 - \mu_4} = 4.668242235582…,
\]

\[
\delta_4 = \frac{\mu_5 - \mu_4}{\mu_6 - \mu_5} = 4.668739469275…,
\]

\[
\delta_5 = \frac{\mu_6 - \mu_7}{\mu_8 - \mu_9} = 4.669132150630…
\]

and so on.

Then the Feigenbaum delta is evaluated as

\[
\delta = \lim_{k\to\infty} \delta_k = 4.669201609…
\]

The nature of $\delta$ is universal i.e. it is the same for a wide range of different iterations.
3.6 Using $\delta$ to Make Predictions:

Many new universal properties have been discovered by Feigenbaum [35, 36], for families of maps which depend on a parameter $\mu$. One of his fascinating discoveries is that if a family $f^\mu$ represents period-doubling bifurcation then there is an infinite sequence $\{\mu_n\}$ of bifurcation values such that

$$\lim_{n \to \infty} \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n} = \delta$$

where $\delta$ is a universal number known as the Feigenbaum constant, which does not depend at all on the form of the specific family of maps. The value of $\delta$ is $4.6692016091029...$ in the dissipative case and $8.721097200...$ in the conservative case.

At a practical level, the existence of a universal number such as $\delta$ allows us to make quantitative predictions about the behaviour of a nonlinear system, even if we can not solve the equations describing the system. More importantly, this is true even if we do not know what the fundamental equations for the systems are, as is often the case. For example, if we observe that a particular system undergoes a period doubling bifurcation from period 1 to period 2 at a parameter value $\mu_1$ (say), and from period 2 to period 4 at a parameter value $\mu_2$ (say), then we can use $\delta$ to predict that the system will make a transition from period 4 to period 8 at $\mu_3$ given by

$$\mu_3 \approx \frac{\mu_2 - \mu_1}{\delta} + \mu_2.$$  

(3.6.1)
The above expression gives us a reasonable prediction of the parameter value near which we should expect to see the transitions, when the first two bifurcations are already known.

We can also use $\delta$ to predict the parameter value to which the period-doubling sequence converges and at which point chaos begins. To see how this works, we first write an expression for $\mu_4$ in terms of $\mu_3$ and $\mu_2$, in analogy with (3.6.1). Here, we are, of course, assuming that the same $\delta$ value describe each ratio. This is not exact, in general, but it does allow us to make a reasonable prediction.

$$\mu_4 \approx \frac{\mu_3 - \mu_2}{\delta} + \mu_3 \quad (3.6.2)$$

We now use (3.6.1) in (3.6.2) to obtain

$$\mu_4 \approx \left( \mu_2 - \mu_1 \right) \left( \frac{1}{\delta} + \frac{1}{\delta^2} \right) + \mu_2 \quad (3.6.3)$$

If we continue to use this procedure to calculate $\mu_5, \mu_6$ and so on, we just get more terms in the sum involving powers of \left( \frac{1}{\delta} \right). We recognize this sum as a geometric series. We can sum the series to obtain the result,

$$\mu_\infty \approx \mu_2 + \frac{\left( \mu_2 - \mu_1 \right)}{\delta - 1}.$$

### 3.7 Accumulation Point:

Let $\{\mu_n\}$ be the sequence of bifurcation points. In the preceding section we described how we can arrive at the approximate result [50] -

$$\mu_\infty \approx \mu_2 + \frac{\left( \mu_2 - \mu_1 \right)}{\delta - 1}.$$
However, this expression is exact when \( \delta_n = \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n} \), the bifurcation ratio, is equal for all values of \( n \). In fact \( \{\delta_n\} \) converges to \( \delta \) as \( n \to \infty \),

\[
i.e. \lim_{n \to \infty} \delta_n = \delta
\]

So, we consider the sequence \( \{\mu_{\infty, n}\} \), the points of which are given by

\[
\mu_{\infty, n} = \frac{\mu_n - \mu_{n-1}}{\delta - 1} + \mu_n
\]

Where \( \mu_n \) are the bifurcation points found out numerically as described earlier. Clearly, we have \( \lim_{n \to \infty} \mu_{\infty, n} = \mu_\infty \), which is the accumulation point after which chaos creep in to the dynamics.

Using the bifurcation points found by numerical methods, the sequence of accumulation points \( \{\mu_{\infty, n}\} \) are calculated for some values of \( n \). The values are as follows:

\[
\begin{align*}
\mu_{\infty, 1} &= 3.571993161965258 \ldots \\
\mu_{\infty, 2} &= 3.569872703191509 \ldots \\
\mu_{\infty, 3} &= 3.5699444125125472 \ldots \\
\mu_{\infty, 4} &= 3.5699455504551323 \ldots \\
\mu_{\infty, 5} &= 3.5699456678654817 \ldots \\
\mu_{\infty, 6} &= 3.5699456716448745 \ldots \\
\mu_{\infty, 7} &= 3.569945671861678 \ldots \\
\mu_{\infty, 8} &= 3.5699456718704905 \ldots \\
\mu_{\infty, 9} &= 3.5699456718709253 \ldots 
\end{align*}
\]

The above sequence converges to the value 3.569945671870 ... , which is the required accumulation point.
3.8 Bifurcation Diagram:

The bifurcation diagram has been one of the frequently used tools to study the chaotic or periodic behaviour of one dimensional maps. We can summarize the behaviour of the logistic map with the help of bifurcation diagram.

Horizontal axis represents the bifurcation parameter
Vertical axis represents the final state of the Logistic Map

Fig. 3.10 Bifurcation diagram for the parameter $\mu$ for $1 \leq \mu \leq 4$

From our previous discussion it was seen that for $1 < \mu < 3$, $1 - \frac{1}{\mu}$ is the stable point attractor and its value increases as the value of $\mu$ increases. For $\mu > 3$, the attractor is a period-2 cycle, as indicated by the two branches. As $\mu$ increases, both branches split simultaneously, yielding a period 4 cycle. This splitting is the period doubling bifurcation. A cascade of further period-doublings occurs as $\mu$ increases, yielding period-8, period-16, and so on, until at $\mu = \mu_\infty \approx 3.56994567 \ldots$, the map becomes chaotic (this is marked by the shaded portion) and the attractor changes from a finite to an infinite set of points [7]. From the bifurcation diagram it becomes clear that the distance between bifurcation points gets shorter and shorter. Period 2, period 4 are well visible and period 8 can be seen on close
observation only whereas higher order periods seem to disappear into the black mass behind it. This black mess is where chaos enters the system after the system parameter attains the accumulation point.

A close look into the bifurcation diagram reveals that within the chaotic region also there are some white patches which are termed as periodic windows. In these windows, the dynamics produced by the iteration of the map behaves regularly or in a deterministic way. The interesting point to be noted here is that there are deterministic regions within the chaotic region itself.

Another important fact is that if we zoom into this chaotic region more and more small periodic windows come into picture. This is possible due to the fractal nature of the attractor set after the parameter value attains the accumulation point. In fact the attractor system becomes a Cantor like set, many properties of which can be derived from the analytic study of the Cantor set.

*Fig. 3.11 Bifurcation diagram for the parameter $\mu$ for $3.4 \leq \mu \leq 4$*
3.9 Lyapunov Exponent:

The Lyapunov exponent ‘$\lambda$’ is considered for the verification to find out how intensely perfect the accumulation point is. It is found to be positive at the parameter values greater than the accumulation point and negative when the
parameter value is less than the accumulation point and at the accumulation point Lyapunov exponent is equal to zero.

It can be started by considering an attractor point $x_0$ and calculate the Lyapunov exponent, which is the average of the sum of logarithm of the derivatives of the function at the iteration points. The formula may be constructed as follows [50]:

$$\lambda = \frac{1}{n} \left[ \log |f'(x_0)| + \log |f'(x_1)| + \log |f'(x_2)| + \log |f'(x_3)| + \ldots + \log |f'(x_n)| \right]$$

From the diagram of Lyapunov exponent, we see that some fraction lie in the negative side of the parameter axis indicating regular actions (periodic orbits) and the fraction lying on the positive side of the parameter axis confirms the existence of chaos for our model.

*Horizontal axis represents the parameter values*

*Vertical axis represents the Lyapunov exponents for the Logistic Map*

*Fig. 3.14: Lyapunov exponent of the Logistic map.*
Horizontal axis represents the parameter values
Vertical axis represents the Lyapunov exponents and final states for the Logistic Map

Fig. 3.15: Combined diagram of Lyapunov exponent and Bifurcation diagram for the Logistic map.

Above, we have shown a combined diagram of Bifurcation diagram and the Lyapunov exponents for the Logistic map for the whole parameter range and they are seen to be in perfect conformity with each other. At each bifurcation point and the accumulation point the Lyapunov exponents are seen to be zero whereas the Lyapunov exponents are found to be negative where the Logistic map shows regular periodic behaviour and after the accumulation point where chaos creeps into the system Lyapunov exponents are found to be positive. The noteworthy fact is that within the chaotic region also, the Lyapunov exponents are sometimes seen to be negative. In fact, those are the regions where regular behaviour is shown (periodic windows) inside the chaotic region.
3.10. Tangent Bifurcation (Saddle node Bifurcation):

The logistic map \( f(x) = \mu x (1-x) \) undergoes another important type of bifurcation called saddle node or tangent bifurcation. One marked difference in this case is that this type of bifurcation occurs within the chaotic region only. In fact, due to this type of bifurcation, the periodic windows marked by regular behaviour occur within the chaotic region. If we look carefully in to the bifurcation diagram (figure 3.10), we observe that within the largest periodic window, the iterated values oscillate between three values i.e. the window is marked by period 3 points.

The third-iterate map \( f^3(x) \) of the logistic map \( f(x) = \mu x (1-x) \) is the key in understanding the birth of the period-3 cycle. Any point \( x \) in a period-3 cycle repeats every three iterates, by definition. So, such points satisfy \( x = f^3(x) \) and are therefore fixed points of the third-iterate map. Since \( f^3(x) \) is an eighth-degree polynomial in case of the Logistic map, we get eight solutions of it i.e. there are 8 such points which are shown graphically below for the parameter value 3.835.

![Fixed points of the third iterate of the Logistic map i.e. of \( f^3 \).](image)

The solutions of the equation \( f^3(x) = x \) is given by the intersections between the graph of \( f^3(x) \) and the diagonal line \( y = x \). There are eight solutions, six solutions...
are marked with $s_i, i = 1, 2, 3$ and $u_i, i = 1, 2, 3$ and the other two solutions are not the genuine period-3 solutions; they are actually fixed points or period-1 points for which $f(x) = x$. In figure 3.16, the points marked with $s_i, i = 1, 2, 3$ correspond to a stable period-3 cycle as $\left| \frac{df^3}{dx} \right| < 1$ at these points. Again the points marked with $u_i, i = 1, 2, 3$ corresponds to an unstable period-3 cycle as $\left| \frac{df^3}{dx} \right| > 1$ at these points which are shown in the following table.

<table>
<thead>
<tr>
<th>Fixed Point</th>
<th>Value of the derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1 = 0.152074…$</td>
<td>-0.394972…</td>
</tr>
<tr>
<td>$u_1 = 0.167205…$</td>
<td>2.32052…</td>
</tr>
<tr>
<td>$s_2 = 0.494514…$</td>
<td>-0.394972…</td>
</tr>
<tr>
<td>$u_2 = 0.534015…$</td>
<td>2.32052…</td>
</tr>
<tr>
<td>$u_3 = 0.954313…$</td>
<td>2.32052…</td>
</tr>
<tr>
<td>$S_3 = 0.958635…$</td>
<td>-0.394972…</td>
</tr>
</tbody>
</table>

Period 3 orbit at the tangent bifurcation can be obtained from the following two equations:

\[ f^3(x) = x, \]
\[ \frac{d}{dx}f^3(x) = 1 \]

Out of the above two conditions, the first one comes directly from the definition of period 3 and the second condition comes from the fact that the tangent bifurcation (saddle node bifurcation also for one dimensional maps) occurs when the period three orbit becomes tangent to the line $y = x$. 

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Below, we have shown an analytical technique following [36] how to find the parameter value where the tangent bifurcation occurs.

### 3.11. Estimation of Parameter value analytically where period 3 starts in the Logistic map:

We have to find the smallest value of the parameter \( \mu \) for which the logistic map has a non trivial 3-periodic orbit. To find it out, we use the fact that every three periodic sequence \( \{x_n\} \) can be written in the form

\[
x_n = m + \beta \omega + \overline{\beta \omega}
\]

Where \( m, \beta \) are constants, \( \omega \) is the complex cube root of unity and the over-bars indicate the complex conjugation.

Substituting the above in to \( x_{n+1} - f(x_n) = 0 \), where \( f(x) = \mu x(1 - x) \), we have

\[
m + \beta \omega^{n+1} + \overline{\beta \omega^{n+1}} - \mu(m + \beta \omega^n + \overline{\beta \omega^n})(1 - m - \beta \omega^n - \overline{\beta \omega^n}) = 0
\]

Using the identities \( \omega^2 = \overline{\omega} \) and \( \overline{\omega}^2 = \omega \), we can express the left hand side of the above equation as a linear combination of the three functions \( \{1, \omega^n, \overline{\omega^n}\} \).

Now, equating the coefficients of \( 1, \omega^n, \overline{\omega^n} \) to zero, from the above equation, we get

\[
system{2 \mu \beta \overline{\beta} = \mu m - m - \mu m^2 \beta \omega = \mu \beta - \mu \overline{\beta}^2 - 2 \mu m \beta \overline{\beta} \overline{\omega} = \mu \beta - \mu \beta^2 - 2 \mu m \beta}
\]

Solving the above three equations, we obtain a quadratic equation in \( m \) as
9m^2 \mu^2 - (9\mu^2 + 3\mu)m + (2\mu^2 + 2\mu + 2) = 0

Solutions of this quadratic equation are

\[ m = \frac{9\mu^2 + 3\mu \pm \sqrt{(9\mu^2 + 3\mu)^2 - 36\mu^2(2\mu^2 + 2\mu + 2)}}{18\mu^2} \]

The smallest positive value of \( \mu \in [1, 4] \) for which 3-periodic orbits are possible is therefore the positive root of

\[(9\mu^2 + 3\mu)^2 - 36\mu^2(2\mu^2 + 2\mu + 2) = 0 \text{ i.e. } \mu = 1 + 2\sqrt{2}.\]

Below, we have verified it with the help of already described geometrical techniques.[36,74].

We have shown the Cob-web diagram for the logistic map with \( m = 1 + 2\sqrt{2} \) and \( x_0 = 0.4 \) which verifies the period three behaviour.

Fig 3.17: Cob-web diagram of the logistic map for \( \mu = 1 + 2\sqrt{2}, \ x_0 = 0.4 \)
At this point it is further to be noticed that from Li and Yorke’s result ‘Period Three implies Chaos’ it is established that our considered map exhibits chaotic behaviour for some values of the parameter $\mu$.

Below, we have shown the Time Series plot and the plot of the third iterate of the logistic map for $\mu = 1 + 2\sqrt{2} = 3.828427124 \ldots$ which also authenticates our result found analytically.

Further, in the plot of the third iterate it is seen that the function becomes tangent to the line $y = x$.

*Horizontal axis represents the number of iterations
Vertical axis represents the iterated values of the logistic map

Fig 3.18: Time series plot of the logistic map for $\mu = 1 + 2\sqrt{2}$

*Fig 3.19: Plot of the third iterate of the logistic map for $\mu = 1 + 2\sqrt{2}$
In 1975, the article: ‘Period three implies chaos’, was published in the American Mathematical Monthly by Li and Yorke [103]. (Period three means that there is a point $x_0$ such that $f^3(x_0) = f(f(f(x_0))) = x_0$. $f^k(x_0) \neq x_0$ for $k = 1,2$; in other words, the image of $x_0$ comes back to $x_0$ after three iterations.)

In that article, Li and Yorke announced that a new theorem for continuous functions of a single variable was discovered. The theorem states that if a continuous function has period three, it must have period $n$ for every positive integer $n$. Of course, soon afterwards, it was found that Li and Yorke’s theorem is only a special case of a remarkable theorem published a decade earlier by Soviet mathematician A.N. Sarkovskii, in a Ukrainian journal. However, it was in Li-Yorke’s article that the new concept related to chaos was first introduced. Researchers were surprised to notice that iterations of even a very simple continuous function of a single variable can display extremely complicated chaotic behaviour.

Before formally stating Sarkovskii’s Theorem, it is necessary to define Sarkovskii’s Ordering. This ordering of the natural numbers begins with all odd numbers, written in increasing order. These are followed by 2 times the odds, $2^2$ times the odds, $2^3$ times the odds, and so on. The powers of 2 come last, in decreasing order. This ordering can be written as follows:

$$
3\triangleright 5\triangleright 7\cdots \triangleright 2 \cdot 3 \triangleright 2 \cdot 5 \triangleright 2 \cdot 7 \cdots \triangleright 2^2 \cdot 3 \triangleright 2^2 \cdot 5 \triangleright 2^2 \cdot 7 \cdots \triangleright 2^3 \cdot 3 \triangleright 2^3 \cdot 5 \triangleright 2^3 \cdot 7 \cdots \cdots \triangleright 2^3 \triangleright 2^2 \triangleright 2^2 \triangleright 2 \triangleright 1.
$$
Sarkovskii re-ordered the natural numbers and proved that if $l \triangleright m$ (which means $l$ is ‘less than’ $m$ in Sarkovskii’s ordering) and if a function has period $l$ then it must have period $m$. The number 3 is the ‘smallest’ in Sarkovskii’s ordering. So, obviously, period 3 implies all the other periods, and Li-Yorke’s theorem was not a new one. However, it was in Li-Yorke’s article that the new concept related to chaos was first introduced.

### 3.12 Conclusions:

1. We verified the **chaotic nature of the map**

   \[ f(x) = \mu x(1-x) \]

   through the techniques of Schwarzian derivative, Linear stability analysis, Time series plot, Cobweb diagram, Bifurcation diagram and the Lyapunov exponents some of which are analytic, some are geometric and the rest are numerical methods.

2. We showed analytically that there occurs a **Transcritical bifurcation** in the Logistic map at parameter value $\mu = 1$ and substantiated it with the help of Cobweb diagram and Bifurcation diagram.

3. We detected the **period three point** analytically and verified it with the help of Cob-web diagram, Time series plot and Orbit diagram.

4. For the Logistic map $f(x) = \mu x(1-x)$ we calculated several bifurcation points with our suitable and appropriately developed numerical techniques and established the **Feigenbaum delta** which is a universal quantity for Period Doubling bifurcation route to chaos to justify the claim that the simple looking deterministic system viz. Logistic map which generally
shows regular or deterministic behaviour ultimately shows chaotic behaviour via *Period Doubling* route.