Chapter – III

Propagation of Diverging Spherical Shock Wave in a Self-Gravitating Gas

Introduction :

The study of shock waves is an important research field for the safety assessments and predictions of disasters due to explosions. Shock waves are produced due to the sudden release of enormous amount of energy from sources such as a nuclear explosion. Shock waves produced due to the explosion or implosion in presence of magnetic field has received much attention in the past decades and the mainstay of theoretical descriptions is still the Rankine – Hugoniot MHD boundary conditions derived by taking into account the equation of state for an ideal gas. Many theoretical and experimental studies were reported by various investigators on planer, cylindrical and spherical shock waves. A number of approaches, including the similarity method (Krehl [7]), power series solution method (Sakurai [12]), CCW method (Chester [3], Chisnell [4] and

Whitham [13]) have been used for the theoretical investigation of MHD shock waves in homogeneous and unhomogeneous media. The Rankine – Hugoniot MHD shock condition play a fundamental role in all the above well known methods (viz. Boyd and Sunderson [2], Gardiner and Stone [5], Hosseini and Takayama [6] and Nath [11]).

CCW [1 – 3] method has been used to study the propagation of a diverging spherical shock wave in a self-gravitating gas, having an initial density distribution $\rho_0 = \rho' e^{-\lambda r}$, where ρ' is the density at the origin, λ is a non-dimensional constant and r the propagation distance, for the two cases (i) when the shock is weak and (ii) when it is strong. Analytical relation for shock velocity have been derived and their numerical estimates have been computed. Using initial density distribution, Kumar *et al*. [8] and Kumar and Saxena [9, 10] have investigated the propagation of shock wave through self-gravitating gas for plane, cylindrical and spherical symmetries of the shock. Simultaneously for both the cases, using CCW method and taking $\rho_0 = \rho' e^{-\lambda r}$ as the initial density distribution, the propagation of a spherical shock wave through a self-gravitating gas has been studied simultaneously for both weak and strong cases, in this chapter.

Basic Equations :

The equations governing the flow of the gas enclosed by the shock front under the influence of its own gravitation are

$$\left.\begin{aligned}
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial t} + \frac{1}{\rho} + \frac{\partial p}{\partial r} + \frac{Gm}{r^2} = 0, \\
&\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \rho\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) = 0, \\
&\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial r} + \gamma p\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) = 0, \\
&\frac{\partial m}{\partial r} - 4\pi\rho\, r^2 = 0,
\end{aligned}\right\} \qquad (3.1)$$

where, m (r, t), u(r, t), p(r, t) and ρ(r, t) denote respectively the mass, velocity, pressure and density inside the sphere of radius r at time t.

Let p_0 and ρ_0 denote the undisturbed values of the pressure and density in front of the shock wave. u, p and ρ be the values of respective quantities at any point immediately after the passage of the shock, then the well known Rankine – Hugoniot conditions permit us to express u, p and ρ, in terms of the undisturbed values of these quantities by means of the following equations :

$$\left.\begin{aligned} p &= \rho_0\ a_0^{\ 2}\left\{\frac{2}{\gamma+1}M^2 - \frac{(\gamma-1)}{y(\gamma+1)}\right\}; \\ \rho &= \rho_0\ \frac{(\gamma+1)M^2}{(\gamma-1)M^2+2}; \\ u &= \frac{2_0^{\ a}}{(\gamma+1)}\left(M - \frac{1}{M}\right); \\ U &= a_0\,M, \end{aligned}\right\} \tag{3.2}$$

where U is the shock velocity and a_0 is sound velocity in undisturbed medium and M the Mach number.

Case – I For weak shock M is taken as

$$M = 1 + \in, \tag{3.3}$$

where $\in$ is the parameter which is negligible in comparison to unity.

Using this relation, the boundary conditions (3.2) become

$$\left.\begin{aligned} p &= \frac{\delta\ p_0}{\delta+1}\left\{\frac{(\delta+1)}{\delta} + 4\in\right\}, \\ \rho &= \rho_0\left\{1 + \frac{4\in}{(\delta+1)}\right\}, \\ u &= \frac{4a_0\in}{(\delta+1)} \quad \text{and} \quad U = a_0\,(1+\in). \end{aligned}\right\} \tag{3.4}$$

Case – II

For strong shock, U >> a_0, then the boundary conditions (3.2) reduce to

$$\left.\begin{aligned} p &= \frac{2\rho_0 \; U^2}{(\delta+1)}, \\ \rho &= \rho_0 \frac{(\delta+1)}{(\delta-1)} \\ \text{and} \quad u &= \frac{2U}{(\delta+1)}. \end{aligned}\right\} \tag{3.5}$$

For diverging shocks the characteristic form of system of equations (1) is easily obtained by forming a linear combination of first and third equation of the system of equations (1) in only one direction in (r, t) plane is

$$dp + \ell c\, du + 2\rho c^2 \frac{u}{u+c}\frac{dr}{r} + \rho c \frac{Gm}{u+c}\frac{dr}{r^2} = 0. \tag{3.6}$$

The final step is to substitute the shock conditions (3.4) and (3.5) into this relation. A first order differential equation in $\in$(r) and U^2 is obtained which determines the shock. For weak shock, substituting equation (3.4) into equation (3.6), we get

$$2\,d\in + \left(\frac{dp_0}{p_0} + \frac{da_0}{a_0} + 2\frac{dr}{r}\right)\in + \frac{\delta - 1}{4}$$

$$\frac{1}{a_0^{\ 2}}\left(\frac{dp_0}{\rho_0} + Gm\frac{dr}{r^2}\right) = 1. \qquad (3.7)$$

Now assuming the initial density distribution $\rho_0 = \rho' \; e^{-\lambda r}$, the mass m can be written as

$$m = \int dm = \int 4\pi e' \; e^{-\lambda_r} \; r^2 \; dr$$

$$= K - 4\pi\rho' \; e^{-\lambda_r} \left(\frac{r^2}{\lambda} + \frac{2r}{\lambda^2} + \frac{2}{\lambda^3}\right), \qquad (3.8)$$

where K is a integration constant and is equal to (N + 8π) ρ′ for m = Nρ′, say (at r = 0 and λ = 1) where N is a dimensional constant.

Also, the condition of hydrostatic equilibrium prevailing in front of shock is written by substituting $u = 0 = \frac{\partial}{\partial f}$ in first equation of (3.1), as

$$\frac{1}{\rho}\frac{dp_0}{dr} + \frac{Gm}{r^3} = 0. \qquad (3.9)$$

Integration of equation (3.9) yields :

$$\frac{P_0}{G\rho'} = -K\int \frac{e^{-\lambda r}}{r^2}\,dr + 4\pi\rho'$$

$$\left(\frac{1}{\lambda}\int e^{-2\lambda r}\,dr + \frac{2}{\lambda^2}\int \frac{e^{-2\lambda r}}{r}\,dr + \frac{2}{\lambda^3}\int \frac{e^{-2\lambda r}}{r^2}\,dr\right). \tag{3.10}$$

Now, for the two case viz. (i) when shock is weak, i.e. r >> 1 the expression for the pressure in unperturbed state can be written as a series of increasing powers of 1/r. Therefore, for this situation which is true at distances, it can be written as

$$\frac{P_0}{G\rho'} = K\left(\frac{e^{-\lambda r}}{\lambda r^2} - \frac{2e^{-\lambda r}}{\lambda^2 r^3} + \frac{6e^{-\lambda r}}{\lambda^3 r^4} - \frac{24e^{\lambda r}}{\lambda^4 r^5} + \frac{120e^{\lambda r}}{\lambda^5 r^6}\right.$$

$$\left. + \frac{720}{\lambda^5}\int \frac{e^{-\lambda r}}{r^7}\,dr\right) - 4\pi\rho'\left\{\frac{e^{-2\lambda r}}{2\lambda^2} + \frac{e^{-2\lambda r}}{\lambda^3 r}\right.$$

$$+\left(\frac{e^{-2\lambda r}}{2\lambda^4 r^2} + \frac{2e^{-2\lambda r}}{4\lambda^5 r^3} - \frac{6e^{-2\lambda r}}{8\lambda^6 r^4} + \frac{24e^{-2\lambda}}{16\lambda^7 r^5}\right.$$

$$\left.\left. - \frac{120e^{-2\lambda r}}{32\lambda^8 r^6} - \frac{720}{32\lambda^8}\int \frac{e^{-2\lambda r}}{r^7}\,dr\right)\right\}. \tag{3.11}$$

However, for strong case i.e., for smaller values of r (r << 1) expression (3.10) can be expanded, as

$$\frac{P_0}{G\rho'} = K\left\{\frac{e^{-\lambda r}}{r} + \lambda\left(\log r - \lambda r + \frac{\lambda^2 r^2}{4} - \frac{\lambda^3 r^3}{18} + \frac{\lambda^4 r^4}{96} + \ldots\right)\right.$$

$$+ \; 4\pi\rho' \left\{ -\frac{e^{-2\lambda r}}{2\lambda} - \frac{2e^{-2\lambda r}}{\lambda^3 r} - \frac{2}{\lambda^2}\left(\log r - 2\lambda r + \lambda^2 r^2\right) \right.$$

$$\left. \left. -\frac{4}{9}\lambda^3 r^3 + \ldots \ldots \ldots \right) \right\}. \qquad (3.12)$$

Thus, the expression for the pressure in unperturbed state for weak and strong cases will be given by (3.11) and (3.12), respectively.

Consequently equation (3.7) reduces to

$$\frac{d\in}{\in} = -\frac{1}{2}\left(\frac{dp_0}{p_0} + \frac{da_0}{a_0} + \frac{2dr}{r} \right). \qquad (3.13)$$

On integrating, it yields

$$\in = K_1 \, p_0^{-1/2} \; a_0^{-1/2} \; r^{-1}, \qquad (3.14)$$

where K_1 is a constant of integration.

And, a_0 can be written as

$$a_0 = \sqrt{8G} \left\{ \frac{K}{\lambda}\frac{1}{r^2} - \frac{4\pi\rho'}{\lambda^2} e^{-\lambda_r} \left(\frac{1}{2} + \frac{1}{\lambda r} + \frac{1}{2\lambda^2 r^2} \right) \right\}^{1/2}. \qquad (3.15)$$

Substituting p_0, a_0 in the equation (3.15), we get

$$\in = K_1 \left(\frac{1}{8}\right)^{1/4} r^{1/2} e^{\lambda r/2} \left\{ 1 - \frac{3\pi\rho'}{\lambda K} r^2 e^{-\lambda r} \left(\frac{1}{2} + \frac{1}{\lambda r} + \frac{1}{2\lambda^2 r^2} \right) \right\}, \qquad (3.16)$$

where $K_1 = \left(\frac{KG}{\lambda}\right)^{-3/4} \rho^{-1/2}$, and having retained terms upto $\frac{1}{r^2}$.

Consequently U and $\frac{U}{a_0}$ can be written as

$$U = \sqrt{8G} \left\{ \frac{K}{\lambda}\frac{1}{r^2} - \frac{4\pi\rho'}{\lambda^2} e^{-\lambda r}\left(\frac{1}{2}+\frac{1}{\lambda r}+\frac{1}{2\lambda^2 r^2}\right)\right\}^{1/2}$$

$$\left[1+K_1\left(\frac{1}{8}\right)^{1/4} r^{1/2}\, e^{\lambda r/2}\left\{1-\frac{3\pi\rho'}{\lambda K} r^2\right.\right.$$

$$\left.e^{-\lambda_r}\left(\frac{1}{2}+\frac{1}{\lambda r}\frac{1}{2\lambda^2 r^2}\right)\right] \tag{3.17}$$

$$\frac{U}{a_0} = 1+K_1\left(\frac{1}{8}\right)^{1/4} r^{1/2}\, e^{r/2}$$

$$\left\{-\frac{3\pi\rho'}{\lambda K} r^2\, e^{-\lambda r}\left(\frac{1}{2}+\frac{1}{\lambda r}+\frac{1}{2\lambda^2 r^2}\right)\right\}. \tag{3.18}$$

For strong shock, substituting equation (3.5) into equation (3.6), we get

$$\frac{dU}{dr}+\frac{A}{r}U^2 = -B\frac{Gm}{r^2} \tag{3.19}$$

where A = 8δ [{2 + √ 2δ (δ – 1)} $\left\{\frac{2\delta}{\delta-1}+\sqrt{\frac{2\delta}{\delta-1}}\right\}\Big]^{-1}$

and

$$B = \{(\delta + 1)^2 \sqrt{2\delta}\,(\delta - 1)\}\,[(\delta - 1)\,\{2 + \sqrt{2\delta}\,(\delta - 1)\}$$

$$\left\{\frac{2\delta}{(\delta - 1)} + \sqrt{\frac{2\delta}{\delta - 1}}\right\}\Bigg]^{-1}. \qquad (3.20)$$

Integration of equation (3.20) yields

$$U^2 = r^{-A}\left\{K' - BG\int m\, r^{A-2}\, dr\right\},$$

where K′ is a constant of integration.

Using equation (3.8), we get

$$U^2 = r^{-A} K' + BG\left[\frac{K r^{-1}}{1 - A} + 4\pi\rho' e^{-\lambda r}\right.$$

$$\left\{\frac{r^{-3}}{\lambda(A - 3)} + \frac{r^{-2}}{(A - 2)(A - 3)} + \frac{2r^{-2}}{\lambda^2 (A - 2)}\right\}$$

$$+ 4\pi\rho'\left\{\frac{\lambda}{(A - 2)(A - 3)} + \frac{2}{\lambda(A - 2)} + \frac{2}{\lambda^3}\right\}$$

(3.21)

and $$\left[\frac{U}{a_0}\right]^2 = r^{-A} K' + BG\left[\frac{K r^{-1}}{1 - A} + 4\pi\rho' e^{-\lambda r}\right.$$

$$\left\{\frac{r^{-3}}{\lambda(A - 3)} + \frac{r^{-2}}{(A - 2)(A - 3)} + \frac{2r^{-2}}{\lambda^2 (A - 2)}\right\}$$

$$+4\pi\rho'\left\{\frac{\lambda}{(A-2)(A-3)}+\frac{2}{\lambda(A-2)}+\frac{2}{\lambda^3}\right\}$$

$$\left\{\frac{r^{-1}}{A-1}-\frac{\lambda}{A}+\frac{\lambda^2 r}{2(A+1)}-\frac{\lambda^3 r^2}{6(A+2)}\right\}\Bigg] \Bigg/$$

$$\delta G\left[\frac{K}{r}+\lambda K e^{\lambda r}\left(\log r-\lambda r\right)-\frac{4\pi\rho'}{\lambda^2}\right.$$

$$\left.\left\{\frac{e^{-\lambda r}}{2}+\frac{2e^{-\lambda r}}{\lambda r}+2e^{\lambda r}\left(\log r-2\lambda r\right)\right\}\right]. \quad (3.22)$$

Results and Discussion :

Expressions (3.18) and (3.22) represent, respectively, the shock velocity of a spherical shock wave propagating through an exponential atmosphere due to its own gravitation for weak and strong cases.

Weak Shock –

It is observed that for the present initial condition, the shock velocity increases with propagation distance r.

Strong Shock –

It is observed that for the present initial condition, the shock velocity decreases with propagation distance r.

References

1. Anand, R. K. (2012) : “Jump Relations Across a Shock in Non-Ideal Gas Flow”. Astrophys. Space Sci., Vol. 342, pp 377 – 388.

2. Boyd, T. and Sanderson, J. (2003) : “The Physics of Plasmas”. Cambridge University Press, Cambridge.

3. Chester, W. (1954) : “The Quasi-Cylindrical Shock Tube”. Phil. Mag., Vol. 45, pp 1293 – 1301.

4. Chisnell, R. F. (1955) : “The Normal Motion of a Shock Wave through a Non-Uniform One Dimensional Medium”. Proc. Roy. Soc. London, Ser A, Vol. 232, pp 350 – 370.

5. Gardiner, T. A. and Stone, J. M. (2005) : “An Unsplit Godunov Method for Ideal MHD Via Constrained Transport”. J. Comp. Phys., Vol. 205, pp 509 – 539.

6. Hosseini, S. H. R. and Takayama, K. (2005) : “Implosion from a spherical Shock Wave Reflected from a Spherical Wall”. J. Fluid Mech., Vol. 530, pp 223 – 239.

7. Krehl, Peter, O. K. (2011) : “Shock Wave Physics and Detonation Physics … … … a Stimulus for the Emergence of Numerous New Branches in Science and Engineering”. Physical Journal H 3 36 : 85, Bibcode : 2011 EPJH … … … 36 … … … 85K.

8. Kumar, S.; Saxena, D. K. and Chaturani, P. (1981) : Proc. Nat. Soc. Fluid Mech. Fluid Power, Vol. 10, p 93.

9. Kumar, S. and Saxena, D. K. (1982) : II Nuovo Cimento, Vol. 72B, p 149.

10. Kumar, S. and Saxena, D. K. (1982) : Proc. Nat. Soc. Fluid Mech. Fluid Power, Vol. 13, pp 487 – 490.

11. Nath, G. (2011) : "MHD Shock Wave Generated by a Moving Piston in a Rotational Axisymmetric isothermal flow of Perfect Gas with Variable Density". Adv. Space Res., Vol. 47, pp 1463 – 1471.

12. Sakurai, A. (1956) : "Propagation of Spherical Shock Waves in Stars". J. Fluid Mech., Vol. 1, pp 436 – 453.

13. Whitham, G. B. (1958) : "On the Propagation of Shock Waves through Regions of Non-Uniform Area of Flow". J. Fluid Mech., Vol. 4, pp 337 – 360.

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