Chapter – II

Propagation of Cylindrical Imploding Shock Waves in Magnetogasdynamics

Introduction :

Assuming a single, imploding, strong cylindrical shock till collapse along axis of symmetry and moving in a continuum medium, Fujimoto and Mishkin [8] have obtained a solution of gasdynamic equations in cylindrical symmetry and have analyzed the problem of implosive shock in detail. They have considered the motion to be self-similar and thus have assumed that the system possess no characteristic length. The self-similarity exponent has been obtained analytically. But in overall analysis, they have ignored the interaction of magnetic field with other gasdynamic variables.

A shock wave (also called shock front or simply “shock”) is a type of propagating disturbance. Like an ordinary wave, it carries energy and can propagate through a medium (solid, liquid, gas or plasma) or in some cases in the absence of a material medium, through a field such as the electromagnetic field. Shock waves are characterized by an abrupt, nearly discontinuous, change in the characteristics of the medium Anderson [1]. Analytic solutions to the wave equations for steady vertical compression waves in an isothermal hydrostatic atmosphere with uniform horizontal magnetic field have been presented by Arora [2] and Arora *et al*. [3]. Bharadwaz [5], Blythe [6], Maurya and Kumar [11], Stanyukevich [16] and Taylor [17] have studied shock waves characteristics in a magnetic field. Baty *et al*. [4], Gretler and Wehle [9], Landau and Lifshitz [10], Nath [12, 13], Rosenau and Frankenthal [14], Singh and Nath [15], Vishwakarma and Nath [18], Vishwakarma and Pandey [19] and Zel'dovitch and Raizer [21] studied the non-standard analysis and shock wave in jump condition, propagation of MHD shock in a thermally conducting medium, propagation of MHD shock waves in gaseous media and MHD spherical shock waves in a non-ideal gas with radiation.

In this chapter, the propagation of a cylindrical imploding shock wave in magneto gasdynamics has been considered. It has been found that for a particular value of γ, role of magnetic field is unimportant with respect to other flows variables in gasdynamics.

Basic Equations and Boundary Conditions :

Following Whitham [20], magnetohydrodynamic equations for axially symmetric cylindrical shock wave is

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \rho\frac{\partial u}{\partial r} + \rho\frac{u}{r} = 0, \tag{2.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{h}{\rho}\frac{\partial h}{\partial r} = 0, \tag{2.2}$$

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial r} + h\frac{\partial u}{\partial r} + \frac{\partial u}{\partial r} = 0, \tag{2.3}$$

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho\gamma}\right) + u\frac{\partial}{\partial r}\left(\frac{p}{\rho\gamma}\right) = 0, \tag{2.4}$$

where, u, ρ, p and h represent flow velocity, density, pressure and component of magnetic field in axial symmetry, respectively. 'r' represents position of the fluid (radial distance from axis of symmetry) at any time 't'.

The shock surface moves into gas of density ρ_0, field h_0 with position depicted by R(t) and its velocity by $\dot{R}(t)$. The shock surface behaves as one of the boundaries for integration of the differential equations representing motion.

The jump conditions for the flow variables across the shock boundary is given by Courant and Friedrichs [7], Whitham [20] and Zel'dovich and Raizer [21]

$$p = \frac{2}{\gamma+1}\,\rho_0\,\dot{R}^2, \tag{2.5}$$

$$\rho = \left(\frac{\gamma+1}{\gamma-1}\right)\rho_0, \tag{2.6}$$

$$h = \left(\frac{\gamma+1}{\gamma-1}\right)h_0, \tag{2.7}$$

$$u = \left(\frac{2}{\gamma+1}\right)\dot{R}, \tag{2.8}$$

where, γ is then adiabatic index.

These are strong shock conditions derived in Whitham [20].

Solutions :

Taking non-dimensional parameter,

$$\xi = \frac{r}{R(t)}, \tag{2.9}$$

we consider the following self-similar solution of equations (2.1) to (2.4),

$$p = \left(\frac{2}{\gamma+1}\right)\rho_0\,\dot{R}^2\,P(\xi), \tag{2.10}$$

$$\rho = \left(\frac{\gamma+1}{\gamma-1}\right) \rho_0 \, R(\xi), \tag{2.11}$$

$$h = \left(\frac{2\rho_0}{\gamma+1}\right)^{1/2} \dot{R}\, H(\xi), \tag{2.12}$$

$$u = \left(\frac{2}{\gamma+1}\right) \dot{R}\, U_1(\xi). \tag{2.13}$$

At the shock front $\xi = 1$ and

$$P(1) = 1,$$

$$R(1) = 1,$$

$$U_1(1) = 1, \tag{2.14}$$

$$H(1) = \frac{(\gamma+1)^{3/2}}{\sqrt{2(\gamma-1)\, M_A}},$$

where, M_A is Alfven Mach number at shock front defined as

$$M_A^2 = \frac{\dot{R}^2}{\left(\dfrac{h^2{}_0}{\rho_0}\right)}. \tag{2.15}$$

The derivatives of the product function

$$f(r,\ t) = \sum (\xi)\, T(t), \tag{2.16}$$

are given by

$$\frac{\partial f}{\partial r} = \frac{1}{R}\ T\,\Sigma'. \tag{2.17}$$

$$\frac{df}{dt} = \Sigma\,\dot{R} + \frac{U\dot{R}}{R}\ T\ \Sigma', \tag{2.18}$$

where, the transformation

$$U_1 \rightarrow \frac{\gamma+1}{2}\left(U + \xi\right), \tag{2.19}$$

has been used.

It makes

$$u = \dot{R}\left(U+\xi\right). \tag{2.20}$$

Using self-similar solutions (2.10) – (2.13), the partial differential equations (2.1) – (2.4) may be reduced to a set of ordinary differential equations as

$$-\frac{dR}{R} = \frac{dU}{U} + \frac{d\xi}{\xi} + 2\frac{d\xi}{U}, \tag{2.21}$$

$$-\frac{2(\gamma-1)}{(\gamma+1)^2}\ \frac{dP}{R} - \frac{4(\gamma-1)}{(\gamma+1)^2}\,\frac{H\,dH}{R}$$

$$= U\,dU + \lambda\xi\ d\xi + (\lambda+1)\ U\,d\xi, \tag{2.22}$$

$$-\frac{dH}{H} = (\lambda+2)\frac{d\xi}{\xi} + \frac{dU}{U} + \frac{d\xi}{U}, \tag{2.23}$$

$$-\frac{dP}{P} = \gamma\left(\frac{dU}{U}+\frac{d\xi}{\xi}\right) + 2(\lambda+\gamma)\frac{d\xi}{U}, \qquad (2.24)$$

where

$$\lambda = \frac{d\ 1n\ \dot{R}}{d\ 1n\ R} = \frac{R}{\dot{R}}\frac{d\dot{R}}{dR} = \frac{R\ \ddot{R}}{R^2}. \qquad (2.25)$$

The variables t and ξ are separable if

$$\lambda = \text{constant}, \qquad (2.26)$$

so that

$$R(t) = R_0\left(1 \pm \frac{t}{t_C}\right)^{\alpha}, \quad \alpha = \frac{1}{I - \lambda}, \qquad (2.27)$$

where positive sign stands for explosion and negative sign for implosion models.

Making assumption (Fujimoto and Mishkin [8])

$$\sigma = \sigma(\xi) = \exp\left[-\int_1^{\xi}\frac{d\xi'}{U(\xi)}\right],$$

$$\sigma(1) = 1,$$

$$\frac{d\sigma}{d\xi} = -\frac{\alpha}{U},$$

$$\sigma(\infty) = \exp\left[-\int_0^{\infty}\frac{d\xi}{U}\right]. \qquad (2.28)$$

We can integrate (2.21), (2.23) and (2.24) from the shock boundary ξ = 1 to some point behind the shock front, say, at ξ, and get

$$R(\xi) = -\left(\frac{\gamma-1}{\gamma+1}\right) \cdot \frac{\sigma^2}{\xi U}, \tag{2.29}$$

$$P(\xi) = \left[\left(\frac{1-\gamma}{1+\gamma}\right)\frac{1}{U}\right]^{\gamma} \sigma^2(\lambda+\gamma), \tag{2.30}$$

$$H(\xi) = -\left[\frac{(\gamma+1)^{1/2}}{2M_A}\right]\frac{\sigma^{\lambda+2}}{\xi U}. \tag{2.31}$$

Explosion Case –

Assumption that energy E, contained in shock wave remains constant i.e. time-independent, leads to the result

$$E = \int_0^R \left[\frac{P}{\gamma-1} + \frac{1}{2}\rho u^2 + \frac{h^2}{2}\right] 2\pi\gamma \; d\xi$$

$$= \frac{4\pi\alpha^2}{(\gamma+1)t_C{}^2}\rho_0 R_0{}^4\left(1+\frac{t}{t_C}\right)^{4\alpha-2}$$

$$\int_0^1 \left[\frac{P}{\gamma-1} + \frac{RU^2{}_1}{\gamma-1} + H^2\right]\xi \; d\xi \tag{2.32}$$

and so, E is time independent if,

$$\alpha = 0.5. \tag{2.33}$$

Thus self-similarity exponent remains intact even if interaction with magnetic field is taken into account.

Implosion Case –

At the shock front $\xi = 1$, from (2.19) we have

$$U(1) = -\left(\frac{\gamma-1}{\gamma+1}\right). \qquad (2.34)$$

Using (2.14) and (2.34), the slope of non-dimensional variables are given by

$$1 + \left(\frac{\gamma+1}{\gamma-1}\right)^2 \frac{1}{M_A{}^2},$$

$$\frac{dU(1)}{d\xi} = \left(\frac{\gamma+1}{\gamma-1}\right)^2 \frac{1}{M_A{}^2}\left[\lambda + \left(\frac{\gamma+3}{\gamma-1}\right)\right]$$

$$- \frac{\left[6\lambda(\gamma+1) + \gamma^2 + 6\gamma + 1\right]}{(\gamma+1)^2}, \qquad (2.35)$$

$$\left[1 + \left(\frac{\gamma+1}{\gamma-1}\right)^2 \frac{1}{M_A{}^2}\right]\frac{dP(1)}{d\xi}$$

$$= \left(\frac{\gamma+1}{\gamma-1}\right)^3 \frac{1}{M_A{}^2}\lambda(2-\gamma) - \frac{2(2\gamma-1)}{(\gamma-1)}$$

$$\left[\lambda + \frac{\gamma(\gamma-1)}{(\gamma+1)(2\gamma-1)}\right], \qquad (2.36)$$

$$\left[1 + \left(\frac{\gamma+1}{\gamma-1}\right)^2 \frac{1}{M_A{}^2}\right] \frac{dH(1)}{d\xi}$$

$$= \frac{(\gamma+1)^{3/2}}{2(\gamma-1)^2} \frac{(\gamma-5)}{M_A}\left[\lambda - \frac{2(\gamma-1)}{(\gamma+1)(\gamma-5)}\right]. \qquad (2.37)$$

As U_1 is monotonically decreasing function of

$$\frac{dU_1(1)}{d\xi} < 0. \qquad (2.38)$$

So that from (2.19)

$$\frac{dU(1)}{d\xi} < -1. \qquad (2.39)$$

In the case of the shock, the gas is at rest, so that

$$U_1(\infty) = 0 \Rightarrow \underset{\xi\to\infty}{U(\xi)} = -\xi\frac{dU(\infty)}{d\xi} = -1. \qquad (2.40)$$

Using equations (2.22), (2.23), (2.24), (2.29), (2.30) and (2.31) we can obtain the differential equation for U as under

$$-\frac{2}{(1+\gamma)^{(1+\gamma)}}\left[\frac{1-\gamma}{\xi U}\right]^{\gamma} 2(\lambda+\gamma-1)$$

$$\left[\gamma\left(\xi\frac{dU}{d\xi}+U\right)+2(\lambda+\gamma)\xi\right]$$

$$-\left[\xi\frac{dU}{d\xi}+(\lambda+2)+U\right]\left(\frac{1}{\xi U}\right)^2\frac{1}{M^2{}_A}\sigma^2(\lambda+1)$$

$$= U\frac{dU}{d\xi} + \lambda\xi + (\lambda+1)U. \qquad (2.41)$$

The unknown function σ can be eliminated for a particular value of $\gamma = 2$ by differentiating (2.41) logarithmically. Thus (2.41) for $\gamma = 2$ becomes

$$\sigma^{2(\lambda+1)} = -\frac{U\frac{dU}{d\xi}+\lambda\xi+(\lambda+1)U}{\left(\frac{4}{27}+\frac{1}{M^2{}_A}\right)\left[\frac{dU}{d\xi}+U+(\lambda+2)\right]\left(\frac{1}{\xi U}\right)^2}$$

and its logarithmic differentiation discloses

$$\frac{2(\lambda+1)}{U}\frac{\frac{d^2U}{d\xi^2}+2\frac{dU}{d\xi}+\lambda+2}{\frac{dU}{d\xi}+U+(\lambda+2)\xi} - \frac{2}{U\xi}\left(\frac{dU}{d\xi}+U\right)$$

$$-\frac{U\frac{d^2U}{d\xi^2}+\left(\frac{dU}{d\xi}\right)^2+\lambda+(\lambda+1)\frac{dU}{d\xi}}{U\frac{dU}{d\xi}+\lambda\xi+(\lambda+1)U}. \qquad (2.42)$$

Using transformations

$$x = \frac{dU}{d\xi}, \quad y = \frac{U}{\xi}, \quad \frac{x}{y} = \frac{d\ln U}{d\ln \xi} \tag{2.43}$$

we may write

$$\frac{dx}{d\xi} = \frac{d^2U}{d\xi^2},$$

$$\frac{dy}{d\xi} = \frac{x-y}{\xi}, \tag{2.44}$$

$$\frac{d^2U}{d\xi^2} = \frac{x - y}{\left(\frac{dy}{dx}\right)}$$

and the equation (2.42) is transformed into

$$\frac{dy}{dx} = \frac{y(y-x)\ (y^2+y-\lambda)}{G(x, y;\ \lambda)}, \tag{2.45}$$

where, $G(x,y;\lambda) = 2(\lambda+1)\left[xy+\lambda+(\lambda+1)y\right]\left[x+y+\lambda+2\right]$

$$- y\left[xy+\lambda+(\lambda+1)y\right]\left[2x+\lambda+2\right]$$

$$+ \ 2(x+y)(x+y+\lambda+2)\ (xy+\lambda+(\lambda+1)y)$$

$$+ \ y(x+y+\lambda+2)\ \left[x^2+(\lambda+1)x+\lambda\right]. \tag{2.46}$$

At the front of the shock

$$\xi = 1,$$

$$y(1) = U(1) = -\left(\frac{\gamma - 1}{\gamma + 1}\right) = -\frac{1}{3} < 0 \quad \text{for} \quad \gamma = \frac{5}{3}$$

$$x(1) = \frac{dU(1)}{d\xi} < -1. \tag{2.47}$$

In the case of the shock

$$x(\infty) = -1, \quad y(\infty) = -1. \tag{2.48}$$

Analytic Determination of Self Similarity Coefficient λ –

The pressure vanishes in the case of the shock wave

$$\lim_{\xi \to \infty} P(\xi) = 0. \tag{2.49}$$

Pressure rises at the shock front so that

$$\frac{dP(1)}{d\xi} > 0. \tag{2.50}$$

Thus P(ξ) has maximum at some value $1 < \xi < \infty$ behind shock

for maxima $\quad \dfrac{dP}{d\xi} = 0,$

from (2.36) and (2.37) we have for $\gamma = 2$

$$\left(1 + \frac{9}{M_A^{\,2}}\right) \frac{dP(1)}{d\xi} = -\sigma\left[\lambda + \frac{2}{9}\right], \tag{2.51}$$

$$\left(1 + \frac{9}{M_A^{\ 2}}\right) \frac{dH(1)}{d\xi} = -\frac{3^{6/2}}{2M_A}\left[\lambda + \frac{2}{9}\right]. \qquad (2.52)$$

From (2.51) and (2.52) it is obvious that

$$\frac{dP(1)}{d\xi} > 0, \quad \frac{dH(1)}{d\xi} > 0, \quad \text{if} \quad \lambda < -\frac{2}{9}. \qquad (2.53)$$

Thus, for $\lambda < -2/9$, pressure and magnetic field both show rising trend at the shock front.

Following transformation ξ, U $\rightarrow$ x, y as in (2.43) equations (2.22), (2.23) and (2.24) take the form

$$-\frac{2}{9}\frac{1}{R}\frac{dP}{d\xi} - \frac{4}{9}\frac{H}{R}\frac{dH}{d\xi} = \left[xy + \lambda + (A+1)y\right]\xi, \qquad (2.54)$$

$$-\frac{1}{H}\frac{dH}{d\xi} = \left[x + y + \lambda + 2\right], \qquad (2.55)$$

$$-\frac{1}{P}\frac{dP}{d\xi} = \frac{2}{U}\left[x + y + \lambda + 2\right]. \qquad (2.56)$$

From equations (2.55) and (2.56) it is clear that P and H assume their maximum value at the same point.

Thus, $\frac{dP}{d\xi} = 0$ implies

$$x + y + \lambda + 2 = 0, \tag{2.57}$$

showing $\frac{dH}{d\xi} = 0$.

Then from equation (2.54)

$$xy + \lambda + (\lambda + 1)y = 0. \tag{2.58}$$

From equations (2.57) and (2.58) : after eliminating x, we have

$$y^2 + y - \lambda = 0 \tag{2.59}$$

or $$y = \frac{-1 \pm \sqrt{(1+4\lambda)}}{2}.$$

The reduced pressure will have single maximum $P_m(\xi_m)$ at $\xi = \xi_m$ when the discriminant of the quadratic equation (2.59) is zero i.e.

$$1 + 4y = 0 \tag{2.60}$$

or $$\lambda = \lambda_m = -\frac{1}{4}.$$

The maximum pressure P_m occurs at (x_m, y_m) given by

$$x_m = -\frac{5}{4}, \qquad y_m = -\frac{1}{2}. \tag{2.61}$$

The value of maximum pressure $P_m(\xi_m)$ may be obtained from (2.30).

Thus,

$$P_m(\xi_m) = \frac{4}{9}\frac{1}{4}\sigma^{3/2}(\xi_m), \tag{2.62}$$

$$\sigma(\xi_m) = \exp\left[-\int_1^{\xi_m}\frac{d}{U}\right] = \exp\left[-\int_{y(1)}^{y_m}\frac{dy}{y(x-y)}\right]$$

$$= \exp\left[-\int_{-1/3}^{1/2}\frac{dy}{y(x-y)}\right]. \tag{2.63}$$

In view of equations (2.57) and (2.58) it is obvious from equation (2.46) that

$$G(x, y, \lambda) = 0,$$

when (i) pressure is maximum

and (ii) when $x(\infty) = y(\infty) = -1$ i.e. at the tail of the shock,

the numerator of equation (2.45) also vanishes at these points.

Conclusion :

By considering axially symmetric implosion model, the effect of MHD implosive shock on flow pattern has been discussed. We have seen that for a particular value of $\gamma(=2)$ the magnetic field is maximum behind shock only at the point where gas dynamic pressure attains its maximum. The value of the self-similarity exponent remains unaltered and for this value of γ, interaction of magnetic field with other flow variables has no impact on the position of maximum pressure and gasdynamic variables behavior as if no magnetic field were present.

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