Chapter – V

Study of the Plane Magnetohydro-dynamics Shock Waves Propagation in a Self Gravitating Gas

Introduction :

For arbitrary choices of initial density distributions of the forms, say, $\rho_0 = \dfrac{\rho' r^{-w}}{\rho' \exp(-\lambda r)}$, Rana [8] has used CCW method to investigate the propagation of plane hydro magnetic shock waves in an ideal gas under its own gravitation. The magnetic field has often been assumed to have constant axial and constant azimuthal, constant axial and variable azimuthal, constant axial, variable azimuthal and radial, components. These studies have three limitations viz. (i) it take care of state just behind the shock, (ii) computation of numerical estimates of flow variables only at PSFL (Permissible Shock Front Location) and (iii) the disturbances behind the flow which overtake the shock and hence influence its motion. Many authors have modified CCW approach by including the EOD behind the flow on the motion of shock waves through uniform and non-uniform media and improved the accuracy of results and noted that the EOD behind the flow overtake the shocks and hence influence their motion considerably, quantitatively. However, the results remain unchanged, qualitatively.

In this chapter, the propagation of plane shock waves in a self gravitating gas in presence of a magnetic field having only constant axial $\left(H_{z_0}\right)$ and azimuthal components $\left(H_{\theta_0}\right)$, simultaneously for the two situations viz., (i) when the shock is weak and (ii) when it is strong, represented by CCW approach, has been better described by taking into account the EOD behind the flow on their motion. The density in the unperturbed state has arbitrary been assumed to vary, as $\rho'_0 = \rho' r^{-w}$, where ρ' is the density at the plane of symmetry and w is a constant. Case of weak shock is explored under two conditions viz., when the applied magnetic field is weak and (ii) when it is strong. Also, the expressions for flow variables have also been obtained for strong shocks. PSFL have been evaluated from the fulfillment of initial density distribution condition. The dependence of flow variables upon governing parameters r, β^2, β_2^2, D, w and ξ has been arrived and their numerical estimates have been computed with the initial strength of the shock at some instant at the initial condition and also compared with the numerical results obtained here describing free propagation. It has been found that the results remain unchanged, qualitatively. However, the trends of variation of (i) the shock strength, the shock velocity, the particle velocity, the density and the pressure in case of WSWMF (Weak Shock Weak Magnetic Field), (ii) the shock velocity, the particle velocity, the density and the presume in case of WSSMF (Weak Shock Strong Magnetic Field) and (iii) the shock strength, the shock velocity and the particle velocity in case of SS (Strong Shock), have reversed.

In the field of propagation of hydromagnetic Shock waves several authors as like Carrus *et al*. [1] and Singh *et al*, [14] have discussed the propagation of shock waves in a Stellar Model, Nath *et al*. [4] find the propagation of shock wave in a rotating interplanetory atmosphere with increasing energy. MHD Shock Wave Generated by a Moving Piston in a Rotational Axisymmetric Isothermal Flow of Perfect Gas (Nath [5]), Propagation of Shock Waves in an Exponential Medium with Heat Conduction and Radiation Heat Flux (Vishwakarma and Nath [16]). Kumar and Singh [2], Kumar and Kishor [3], Purohit [6], Ramchandran and Smith [7], Rogers [9], Rosenau and Frankenthal [10], Sakurai [11], Sedov [12], Singh and Vishwakarma [13], Singh and Nath [15] and Whitham [17] have used different method to investigate the many problems.

Basic Equations :

The equation governing the flow of the gas enclosed by plane shock front under the influence of its own gravitation and magnetic, having only constant axial and azimuthal components are

$$
\left.
\begin{aligned}
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\mu}{2\rho}\frac{\partial}{\partial r}\left(H_\theta^2 + H_z^2\right) \\
&\qquad\qquad\qquad\qquad + \frac{\mu}{\rho}\frac{H_\theta^2}{r} + \frac{Gm}{r^2} = 0, \\
&\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \rho\frac{\partial u}{\partial r} = 0, \\
&\frac{\partial P}{\partial t} + u\frac{\partial p}{\partial r} + p\gamma\frac{\partial u}{\partial r} = 0, \\
&\frac{\partial H_\theta}{\partial t} + u\frac{\partial H_\theta}{\partial r} + H_\theta\frac{\partial u}{\partial r} = 0, \\
&\frac{\partial H_z}{\partial t} + u\frac{\partial H_z}{\partial r} + H_z\frac{\partial u}{\partial r} = 0, \\
&\frac{\partial m}{\partial r} - \rho = 0,
\end{aligned}
\right\} \quad (5.1)
$$

where, r is the radial co-ordinate u, p, ρ, H_θ, H_z, μ and m are respectively, the particle velocity, the pressure, the density, azimuthal components of magnetic field, axial component of magnetic field, permeability of the gas, mass inside a cylinder of unit cross-section and of length r and $a^2 = \gamma p/\rho$ (γ being the adiabatic index of the gas).

Boundary Conditions :

The magneto hydrodynamic conditions can be written in terms of single parameter $\xi = \rho/\rho_0$ as

$$\rho = \rho_0 \xi, \quad H = H_0 \xi, \quad u = \frac{\xi - 1}{\xi} U,$$

$$U^2 = \frac{2\xi}{(\gamma+1) - (\gamma-1)\xi}\left[a_0^2 + \frac{b_0^2}{2}\{(2-\gamma)\xi + \gamma\} \right], \quad (5.2)$$

and $$p = p_0 + \frac{2\rho_0(\xi-1)}{(\gamma+1) - (\gamma-1)\xi}\left[a_0^2 + \frac{\gamma-1}{4} b_0^2 (\xi-1)^2 \right],$$

where, the superscript '0' stands for the states immediately ahead of the shock front, U is the shock velocity, a_0 is the sound speed $\sqrt{\gamma p_0/\rho_0}$ and b_0 is the Alfven speed $\sqrt{\mu H_0^2/\rho_0}$.

Weak Shock – For every weal shock, the parameter ξ is written as

$$\rho/\rho_0 = \xi = 1 + \in(r), \quad (5.3)$$

where, $\in(r)$ is the another parameter which is negligible in comparison with unit, i.e. $\in(r) << 1$. Now, we consider the two cases of weak and strong magnetic fields.

Case – I : For weak magnetic field at $b_0^2 << a_0^2$ i.e. $\mu H_0^2 / \gamma p_0 << 1$, under this condition the boundary conditions (5.2) for very weak shock reduce to

$$\left.\begin{aligned} &\rho = \rho_0(1+\in), \quad H_z = H_{z_0}(1+\in), \\ &U = \left[1+\frac{\gamma+1}{4}\in\right]a_0, \\ &p = p_0(1+\gamma\in) \\ \text{and} \quad &u = \in a_0. \end{aligned}\right\} \tag{5.4}$$

Case – II : For strong magnetic field $b_0^2 >> a_0^2$ i.e. $\mu H_0^2 / \gamma p_0 >> 1$, by using this condition and equation (5.3), the boundary conditions (5.2) become

$$\left.\begin{aligned} &\rho = \rho_0(1 + \in), \\ &H_z = H_{z_0}(1+\in), \\ &U = \left(1 + \frac{3}{4}\in\right)b_0, \\ &p = p_0(1+\gamma\in) \qquad \text{and} \quad u = \in b_0. \end{aligned}\right\} \tag{5.5}$$

Strong Shock – In the limiting case of a strong shock, p/p_0, is large. Now consider the two cases of weak and strong magnetic field.

Case – I : The purely non-magnetic way when $\xi - \left\{\frac{(\gamma+1)}{(\gamma-1)}\right\}$ is small.

Case – II : When $b_0^2 >> a_0^2$, i.e. $\mu H_0^2 >> \gamma p_0$ i.e., the ambient magnetic pressure is large compared with the fluid pressure. In terms of ξ, the shock may be strong (i) for ξ values just greater than one and β^2 should be large and (ii) for ξ values closed to $(\gamma + 1)/(\gamma - 1)$, the shock will also be strong for smaller values of β^2. And $\xi = \left\{\frac{(\gamma+1)}{(\gamma-1)}\right\}$, corresponds to the case of shock of infinite shock strength. In terms of ξ, the boundary conditions (5.2) become

$$\rho = \rho_0\,\xi, \quad H_z = H_{z_0}\,\xi,$$

$$U^2 = \frac{2\xi a_0^2}{(\gamma+1)-(\gamma-1)\xi}\left[1+\frac{1}{2}\{(2-\gamma)\xi\}\frac{b_0^2}{a_0^2}\right],$$

$$\frac{p}{p_0} = \chi(\xi)\frac{U^2}{a_0^2} + L \qquad \text{and} \qquad u = \frac{\xi-1}{\xi}U, \tag{5.6}$$

where, $$\chi = \chi(\xi) = \frac{\gamma(\gamma-1)\,(\xi-1)^3}{2\xi\{(2-\gamma)(\xi+\gamma)\}}$$

and $$L = \frac{(\gamma+1)\xi - (\gamma-1)}{(\gamma+1) - (\gamma-1)\xi}.$$

Characteristic Equation :

For diverging shock the characteristic form of the system of equations (5.1) is easily obtained by forming a linear combination of first and third equations of the system of equations (5.1) in only one direction in (r, t) plane. Equations first and third of the system of equations (5.1) can be written as

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial r} + \frac{\partial p_t}{\partial r} + \rho\frac{Gm}{r^2} = 0 \qquad (5.7)$$

and $$\frac{\partial p_t}{\partial t} + u\frac{\partial p_t}{\partial r} + \rho c^2\frac{\partial u}{\partial r} = 0 \qquad (5.8)$$

where $$p_t = p + \frac{\mu}{2}\left(H_\theta^2 + H_z^2\right) + \int \mu H_\theta^2 \frac{dr}{r} \qquad (5.9)$$

is the total pressure including magnetic pressure

and $$c^2 = a^2 + b^2 = \frac{\gamma p}{\rho} + \frac{\mu\left(H_\theta^2 + H_r^2\right)}{\rho}. \qquad (5.10)$$

Linear combination of two equations (5.7) and (5.8) can be written as

$$\frac{\partial p_t}{\partial t} + \rho h\frac{\partial u}{\partial t} + \left(u + h\right)\frac{\partial p_t}{\partial r}$$

$$+ \rho\left(uh + c^2\right)\frac{\partial u}{\partial r} + h\rho\frac{Gm}{r^2} = 0. \qquad (5.11)$$

The conditions that this combination involves the derivatives in only one direction are given by

$$\frac{\partial p_t}{\partial t} = (u + h)\frac{\partial p_t}{\partial r} \qquad \text{or} \qquad \frac{\partial r}{\partial t} = u + h \tag{5.12}$$

$$\text{and} \qquad h\frac{\partial u}{\partial t} = (uh + c^2)\frac{\partial u}{\partial r} \qquad \text{or} \qquad h\frac{\partial r}{\partial t} = c^2 + hu. \tag{5.13}$$

Equations (5.12) and (5.13) give h = ±c, i.e. $\frac{\partial r}{\partial t} = u \pm c.$ (5.14)

It expresses the fact that the characteristic curves in (r, t) plane represent the motion of possible disturbances whose velocity differs from particle velocity u, by the value ±c (local sound speed), respectively, for diverging and converging shocks. Now putting h = ±c, into equation (5.11), we get

$$\frac{\partial p_t}{\partial t} + \rho c\frac{\partial u}{\partial t} + (u + c)\frac{\partial p_t}{\partial r}$$

$$+ \rho c(u + c)\frac{\partial u}{\partial r} + \rho c\frac{Gm}{r^2} = 0, \tag{5.15}$$

substituting

$$\frac{\partial p_r}{\partial t} = \frac{\partial p}{\partial t} + \frac{\partial}{\partial t}\left\{\frac{\mu}{2}\left(H_\theta^2 + H_z^2\right) + \int \mu H_\theta^2 \frac{dr}{r}\right\} \tag{5.16}$$

and $$\frac{\partial p_t}{\partial r} = \frac{\partial p}{\partial t} + \frac{\partial}{\partial r}\left\{\frac{\mu}{2}\left(H_\theta^2 + H_z^2\right) + \int \mu H_\theta^2 \frac{dr}{r}\right\} \tag{5.17}$$

into equation (5.15), we get

$$\frac{\partial p}{\partial t} + (u+c)\frac{\partial p}{\partial r} + \frac{\partial}{\partial t}\left\{\frac{\mu}{2}\left(H_\theta^2 + H_z^2\right) + \int \mu H_\theta^2 \frac{dr}{r}\right\}$$

$$+ (u+c)\frac{\partial}{\partial r}\left\{\frac{\mu}{2}\left(H_\theta^2 + H_z^2\right) + \int \mu H_\theta^2 \frac{dr}{r}\right\}$$

$$+ \rho c\left\{\frac{\partial u}{\partial t} + (u+c)\frac{\partial u}{\partial r}\right\} + \rho c \frac{Gm}{r^2} = 0. \tag{5.18}$$

Writing total derivatives for p, H_θ, H_z and u, the equation (5.18) becomes

$$dp + \mu H_\theta dH_\theta + \mu H_z dH_z + \rho c\, du + \mu H_\theta^2 \frac{dr}{r}$$

$$+ \frac{\rho c}{u+c}\frac{Gm}{r^2} dr = 0. \tag{5.19}$$

The equation (5.19) represents the characteristic form of the system of equations (5.1) for diverging shock, i.e., the form in which equation contains derivations in only one direction in (r, t) plane. In order to estimate the strength of overtaking disturbances an independent C_+ characteristic is considered. The differential relation valid across a C_+

disturbance is obtained by replacing c by c_- in equation (5.19) and written as :

$$dp + \mu H_\theta dH_\theta + \mu H_z dH_z - \rho c\, du + \mu H_\theta^2 \frac{dr}{r}$$

$$- \frac{\rho c}{u - c} \frac{Gm}{r^2} dr = 0. \qquad (5.20)$$

Equation (5.20) represents the characteristic form of the system of equation (5.1) for converging shock.

Analytical Relations for Flow Variables :

The final step is to substitute the shock conditions (5.4) or (5.5) or (5.6) into equations (5.19) and (5.20). A first order differential equation in $\in(r)$ or U^2 is obtained which determines the shock. The equilibrium state of the gas assumed to be specified by the conditions

$$\frac{\partial}{\partial r} = 0 = u \text{ and } H_{z_0} = \text{constant} = H_{\theta_0}. \qquad (5.21)$$

The initial density distribution in unperturbed state is assumed to be of the form

$$\rho_0 = \rho' r^{-w} \qquad (5.22)$$

where, ρ' is the density at the plane of symmetry and w is a constant.

Using equation (5.21) and first equation of the system of equations (5.1), the hydrostatic equilibrium prevailing in front of the shock can be written as

$$\frac{1}{\rho_0}\frac{\partial p_0}{\partial r} + \frac{\mu}{\rho_0}\frac{H_\theta^2}{r} + \frac{G m}{r^2} = 0 . \tag{5.23}$$

From the sixth equation of the system of equations (5.1) can be written as :

$$m = \int \rho' r^{1-w} \, dr \quad \text{and} \quad m = \frac{\rho'}{(1-w)} r^{1-w} . \tag{5.23a}$$

From equations (5.22), (5.23) and (5.23a), we get

$$\frac{p_0}{p'} = K + K_1 r^{-2w} - \gamma \beta_2{}^2 (\log r), \tag{4.24}$$

a_0 and b_0 can be written as :

$$\frac{a_0}{a'} = \sqrt{r^w \left\{K + K_1 r^{-2w} - \gamma \beta_2{}^2 (\log r)\right\}} \tag{5.25}$$

$$\text{and} \quad b_0 = a_0 \sqrt{\frac{\beta^2}{\left\{K + K_1 r^{-2w} - \gamma \beta_2{}^2 (\log r)\right\}}} , \tag{5.26}$$

where K is the constant of integration, p′ is the pressure at the plane of symmetry in the unperturbed state and G is the universal gravitational constant and $$K_1 = \frac{\gamma}{2Dw(1-w)},$$

$$D = \frac{a'^2}{G\rho'},$$

$$\beta_1^{\,2} = \frac{\mu H_{z_0}^2}{\gamma p'}, \qquad \beta_2^{\,2} = \frac{\mu H_{\theta_0}^2}{\gamma p'},$$

$$\beta^2 = \beta_1^{\,2} + \beta_2^{\,2}.$$

The pressure curves in the unperturbed state given by equation (5.24) with propagation distance, equations (5.24), (5.25) and (5.26) can be written as

$$\frac{dp_0}{p_0} = -\left[\frac{\gamma\beta_2^{\,2}}{K}\frac{dr}{r} + \frac{2w\gamma K_1}{K^2} r^{-4w-1}\,dr + \frac{2\gamma K_1}{K}\right.$$

$$\left.\left\{w - \frac{\beta_2^{\,2}}{2k} + \frac{w\gamma\beta_2^{\,2}}{K}(\log r)\, r^{-2w-1}\,dr\right\}\right], \qquad (5.27)$$

$$\frac{da_0}{a_0} = -\left[\frac{1}{2}\left\{w - \frac{\gamma\beta_2^{\,2}}{K}\right\}\frac{dr}{r} + \frac{w\gamma K_1}{K^2} r^{-4w-1}\,dr\right.$$

$$\left. + \frac{\gamma K_1}{K}\left\{w - \frac{\beta_2^{\,2}}{2K} + \frac{w\gamma\beta_2^{\,2}}{K}(\log r)\, r^{-2w-1}\,dr\right\}\right], \qquad (5.28)$$

and $$\frac{db_0}{b_0} = \frac{w}{2}\frac{dr}{r}. \qquad (4.29)$$

Weak Shock with Weak Magnetic Field (WSWMF) :

Substituting the shock conditions (5.4) into equation (5.19) and using equation (5.23) and neglecting the second and higher order terms of $\in(r)$, since $\in(r) << 1$, we get

$$\frac{d\in}{\in} + \frac{1}{2}\left(1 - \frac{\mu H_0^2}{2\gamma p_0}\right)\left(\frac{dp_0}{p_0} + \frac{da_0}{a_0} + \frac{2\mu H_{\theta_0}^2}{\gamma P_0}\right) = 0. \tag{4.30}$$

The validity of above relation is $\frac{\mu H_0^2}{2\gamma p_0} << 1$ and this equation can further be rewritten as :

$$\frac{d\in}{\in} + \frac{1}{2}\left[1 - \frac{\beta^2}{2}\frac{p'}{p_0}\right]\left[\frac{dp_0}{p_0} + \frac{da_0}{a_0} + \frac{2\beta_2{}^2 p'}{p_0}\frac{dr}{r}\right] = 0. \tag{5.31}$$

On substituting the values of $\frac{p_0}{p'}$, $\frac{da_0}{a_0}$ and $\frac{dp_0}{p_0}$ in equation (5.31) and on integration, we get

$$\in(r) = K' r^{R_1} \exp \Delta_1, \tag{5.32}$$

where k' is the constant of integration,

and $\Delta_1 = -R_2 r^{-2w} - R_3(\log r) r^{-2w} + R_4 r^{-4w} - R_5(\log r) r^{-4w}$

$$+ R_6 r^{-6w} + R_7 r(\log r)^2,$$

$$R_1 = \frac{1}{4}\left\{ -w - \frac{4\beta_2^{\ 2}}{K} + \frac{\gamma}{k}\left(3\beta_2^{\ 2} + \frac{w}{2}\beta^2\right) + \frac{(4-3\gamma)\beta^2\beta_2^{\ 2}}{2K^2}\right\},$$

$$R_2 = \frac{\beta_2^{\ 2}K_1}{16wK^2}\left\{2-(4-3\gamma) - \frac{w\beta^2}{\beta_2^{\ 2}}(1+6\gamma) - \frac{2\beta^2}{K}(2-3\gamma)\right\},$$

$$R_3 = \frac{3(4K-\beta^2)\gamma^2 K_1\beta_2^{\ 2}}{4K^3},$$

$$R_4 = \frac{K_1}{32wK^3}\Big[48\gamma K - 6\gamma(1+KK_1)\beta^2$$

$$- \frac{(4-3\gamma)K\beta^2\beta_2^{\ 2}}{wK} - \frac{3(1+K_1)\gamma^2\beta_2^{\ 2}}{2K}\Big],$$

$$R_5 = \frac{3(1+K_1)\gamma^2 K_1\beta^2\beta_2^{\ 2}}{16K^4},$$

$$R_6 = \frac{\gamma K_1^{\ 2}\beta^2}{8K^4},$$

$$R_7 = \frac{w\gamma\beta^2\beta_2^{\ 2}}{16K^2}.$$

Remember that equation (5.32) describes free propagation.

Using equation (5.30) into (5.4), we get

$$du_- = \in da_0 + a_0\left[-\frac{1}{2}\frac{dp_0}{p_0} - \frac{1}{2}\frac{da_0}{a_0} - \frac{\mu H_{\theta_0}^2}{\gamma P_0}\frac{dr}{r}\right]\in. \tag{5.33}$$

On substituting the shock conditions (5.4) into equation (5.20) and using equation (5.23), we get

$$\frac{d\in}{\in} - \left(\frac{\gamma p_0}{\mu H_\theta^2}\right)\left[\left(1-\frac{2}{\gamma}\right)\frac{dp_0}{p_0} - \frac{da_0}{a_0}\right] = 0. \qquad (5.34)$$

Using equation (5.34) into equation (5.4), we get

$$du_+ = \in da_0 + a_0 \left[\left\{-\frac{\gamma p_0}{\mu H_\theta^2}\right\}\right.$$

$$\left.\left\{\left(1-\frac{2}{\gamma}\right)\frac{dp_0}{p_0} - \frac{da_0}{a_0}\right\}\in\right]. \qquad (5.35)$$

Now, in presence of both C_+ and C_- disturbances, the fluid velocity increment behind the shock will be related as

$$du_+ + du_- = \in da_0 + a_0 d\in. \qquad (5.36)$$

Substituting equations (5.33) and (5.35) into equation (5.36), we get

$$\frac{d\in}{\in} - \left\{1+\frac{2(2-\gamma)}{\gamma\beta^2}\frac{p'}{p_0}\right\}\frac{dp_0}{p_0} - \frac{3}{2}\frac{da_0}{a_0}$$

$$+\beta_2{}^2\frac{p_0}{p'}\frac{dr}{r} = 0 \qquad (5.37)$$

On substituting the values of $\frac{p_0}{p'}$, $\frac{da_0}{a_0}$ and $\frac{dp_0}{p_0}$ in equation (5.37) and on integration, we get

$$\in\left(r\right) = K'' r^{R_1'} \exp \Delta_1' , \tag{5.38}$$

where K″ is the constant of integration.

$$\Delta_1' = R_2' r^{-2w} - R_3' (\log r) r^{-2w} - R_4' r^{-4w} - R_5' (\log r) r^{-4w} + R_6' r^{-6w}$$

$$R_1' = = \left[\frac{3}{4} w - \frac{(\gamma + 4)\, \beta_2{}^2}{4K} + \frac{(\gamma - 2)\, \beta_2{}^2}{\beta^2} \right]$$

$$R_2' = \frac{K_1}{4Kw} \left[w\gamma - \frac{\gamma^2\, \beta_2{}^2}{2K} - \frac{4(\gamma - 2)\, wK}{\beta^2} - \frac{(\gamma + 4)\beta_2{}^2}{2K} \right.$$

$$\left. - \frac{\gamma(\gamma - 2)\beta_2{}^2}{K\beta^2} \right]$$

$$R_3' = \frac{\gamma K_1 \beta_2{}^2}{K^2} \left\{ \frac{(\gamma - 2)}{\beta^2} - \frac{\gamma}{4} \right\},$$

$$R_4' = \frac{K_1}{4wK^2} \left[\frac{w\gamma}{2} - \frac{2w(\gamma - 2)(K - K_1)}{\beta^2} - \frac{(\gamma - 2) K_2 \beta_2{}^2}{\beta^2} \right.$$

$$\left. - \frac{(\gamma - 2)^2 K_1 \beta_2{}^2}{2\beta^2} \right],$$

$$R_5' = \frac{\gamma(\gamma - 2) K_1{}^2 \beta_2{}^2}{2K^2 \beta^2},$$

$$R_6' = \frac{(\gamma - 2) K_1{}^2}{3K^2 \beta^2}.$$

Weak Shock with Strong Magnetic Field (WSSMF) :

Substituting the shock conditions (5.5) into equations (5.19) and (5.23), we get

$$\frac{d\in}{\in}+\frac{1}{2}\left\{1-\frac{\gamma p_0}{2\mu H_0^2}\right\}$$

$$\left\{\frac{\gamma dp_0}{\mu H_{0_0}^2}+\frac{db_0}{b_0}+\frac{db_0}{b_0}+\frac{2\mu H_{\theta_0}^2}{dr}\frac{dr}{r}\right\} = 0. \qquad (5.39)$$

The validity condition of above relation is $\frac{\gamma p_0}{2\mu H_0^2} << 1$ and this equation can further be re-written as

$$\frac{d\in}{\in}+\frac{1}{2}\left\{1-\frac{1}{2\beta^2}\left(\frac{p_0}{p'}\right)\right\}$$

$$\left\{\frac{dp_0}{\beta^2 p_0}\left(\frac{p_0}{p'}\right)+\frac{db_0}{b_0}+\frac{2\beta_2^{\ 2}}{\beta^2}\frac{dr}{r}\right\} = 0. \qquad (5.40)$$

By substituting the values of $\frac{p_0}{p'}$, $\frac{dp_0}{p_0}$ and $\frac{db_0}{b_0}$ into equation (5.40) and on integration, we get

$$\in(r) = P' r^{P_1} \exp\Delta_2, \qquad (5.41)$$

where P′ is the constant of integration.

and $\quad \Delta_2 = -P_2 r^{-2w} + P_3 r^{-4w} - P_4(\log r)r^{-4w} + P_5 r^{-6w} - P_6(\log r)^2$

and $$P_1 = \frac{1}{8\beta^2}\left[w(2\beta^2 - K) + \frac{4(2-\gamma)\beta_2^{\,2}}{\gamma} \right],$$

$$P_2 = \frac{(\gamma+8)}{16}\frac{K_1}{\beta^2},$$

$$P_3 = \frac{\gamma K_1^{\,2}}{4w\beta^2 K}\left[\frac{(1-K_1)w}{K_1} + \frac{(1-\gamma)\beta_2^{\,2}}{4K} \right],$$

$$P_4 = \frac{(1+\gamma)\gamma K_1^{\,2}\beta_2^{\,2}}{4K^2\beta^2},$$

$$P_5 = \frac{\gamma K_1^{\,2}}{6K^2\beta^2},$$

$$P_6 = \frac{w\gamma\beta_2^{\,2}}{16\beta^2}.$$

Remember that equation (5.41) describes free propagation.

Using the equation (5.39) into equation (5.5), we get

$$du_- = \in db_0 + b_0\left[-\frac{1}{2}\left(1 - \frac{\gamma P_0}{2\mu H_0^2}\right) \left(\frac{\gamma dp_0}{\mu H_0^2} + \frac{db_0}{b_0} + \frac{2\mu H_{\theta_0}^2}{\mu H_0^2}\frac{dr}{r} \right) \right] \in. \qquad (5.42)$$

On substituting the shock condition (5.5) into equation (5.20) and using equation (5.23), we get

$$\frac{d\in}{\in}+\left(\frac{\mu H_0^2}{\gamma P_0}\right)\left\{\frac{(\gamma-2)\,dP_0}{\mu H_0^2}-\frac{db_0}{b_0}-\frac{\mu H_{\theta_0}^2}{\mu H_0^2}\frac{dr}{r}\right\} = 0. \tag{5.43}$$

Using equation (5.43) into equation (5.5), we get

$$du_+ = \in db_0 + b_0\left[\left(-\frac{\mu H_0^2}{\gamma P_0}\right)\left\{\frac{(\gamma-2)dp_0}{\mu H_0^2}-\frac{db_0}{b_0}-\frac{\mu H_{\theta_0}^2}{\mu H_0^2}\frac{dr}{r}\right\}\right]\in. \tag{5.44}$$

Now, in presence of both C_+ and C_- disturbances, the fluid velocity behind the shock will be related as

$$du_+ + du_- = \in db_0 + b_0\, d\in. \tag{5.45}$$

Substituting equations (5.42) and (5.44) into equation (5.45), we get

$$\frac{d\in}{\in}=\left[-(\gamma-2)-\frac{1}{2\beta_2{}^2}\left(\frac{p'}{p_0}\right)\right]\frac{dp_0}{p_0}+\left[\frac{1}{2}+\frac{1}{4\beta^2}\frac{p'}{p_0}+\beta^2\frac{p_0}{p'}\right]\frac{db_0}{b_0}+\left[\beta_2{}^2\frac{p_0}{p'}-\frac{\beta_2{}^2}{\beta^2}\right]\frac{dr}{r}. \tag{5.46}$$

On substituting the values of $\frac{p_0}{p'}$, $\frac{dp_0}{p_0}$ and $\frac{db_0}{b_0}$ into equation (5.46) and on integration, we get

$$\in(r) = P'' r^{p_1'} \exp\Delta_2', \tag{5.47}$$

where P″ is the constant of integration.

and $$\Delta_2' = -P_2' r^{-2w} + P_3'(\log r) r^{-2w} + P_4' r^{-4w} + P_5'(\log r) r^{-4w} + P_6' r^{-6w} + P_7'(\log r) r^{-4w}$$

$$P_1' = \left[\frac{w}{8}\left(2 + 4K\beta^2 + \frac{K}{\beta^2}\right) + \frac{(\gamma-2)\beta_2^{\,2}}{2\beta^2} + \frac{(\gamma-1)\beta_2^{\,2}}{K}\right],$$

$$P_2' = \frac{K_1}{2w\gamma}\left[2w(\gamma-2) - \frac{w}{2K} + \frac{\gamma\beta_2^{\,2}}{2} + \frac{wK(\gamma+8)}{8\beta^2} - \frac{\gamma(1-\gamma)\beta_2^{\,2}}{2} + \frac{(\gamma^2-3\gamma+3)\beta_2^{\,2}}{K}\right],$$

$$P_3' = \frac{\gamma\beta_2^{\,2}K_1}{K^2}\left\{(\gamma-2) + \frac{\gamma K}{2\beta^2}\right\},$$

$$P_4' = \frac{K_1}{4wK}\left[\frac{2w(\gamma-2)}{K} + \frac{w\gamma(1-K_1)}{\beta^2} + \frac{\beta_2^{\,2}\gamma}{2K\beta^2}\left\{K_1 + \frac{1}{2}(K_1 - \gamma + K\gamma)\right\}\right],$$

$$P_5' = \frac{(\gamma K - \gamma + K_1)\gamma K_1 \beta_2^{\,2}}{4K^2\beta^2},$$

$$P_6' = \frac{\gamma K_1^2}{6K^2\beta^2},$$

$$P_7' = \frac{w\gamma\beta_2^{\ 2}}{4}\left(\frac{\beta_2^{\ 2}}{K^2} - \frac{1}{4\beta^2}\right).$$

Equation (5.47) describes the propagation parameter $\in(r)$, which includes the effect of disturbances behind the flow on the motion of shock.

Strong Shock (SS) :

Substituting the shock conditions (5.6) into equation (5.19) and using equation (5.23), we get

$$A\frac{dU^2}{dr} - B\frac{U^2}{r} + C\frac{a'^2}{D(1-w)}r^{-w-1} + E\beta_2^{\ 2}a'^2\, r^{w-1} = 0, \tag{5.48}$$

where, $A = \left[\frac{\chi(\xi)}{\gamma} + \frac{1}{2}(\xi-1)\sqrt{\frac{\chi(\xi)}{\xi}}\right],$

$$B = \frac{w\chi(\xi)}{\gamma},$$

$$C = \left[\frac{\xi\sqrt{\xi\chi(\xi)}}{(\xi-1) + \sqrt{\xi\chi(\xi)}} - L\right]$$

and $E = \left(\xi^2 - L\right).$

The equation (5.48) can be written as

$$\frac{dU^2}{dr} - B_1\frac{U^2}{r} + B_2\frac{a'^2}{D(1-w)}r^{-w-1} - B_3\, a'^2\, r^{w-1} = 0, \tag{5.49}$$

where $B_1 = \dfrac{B}{A}$, $B_2 = \dfrac{C}{A}$ and $B_3 = \dfrac{E}{A}$.

On integration (5.49), we get

$$U^2 = T r^{B_1} + B_2 \frac{a'^2}{D(1-w)(w+B_1)} r^{-w} - B_3 \frac{a'^2 \beta_2{}^2}{(w-B_1)} r^{w}, \tag{5.50}$$

where T is the constant of integration.

Remember that equation (5.50) describes free propagation.

For the C_- disturbances generated by the shock, the fluid velocity increment using equation (5.49) into equation (5.6) may be expressed as :

$$du_- = \frac{\xi-1}{\xi} \frac{1}{2U} \left[B_1 \frac{U^2}{r} - B_2 \frac{a'^2}{D(1-w)} r^{-w-1} - B_3 a'^2 \beta_2{}^2 r^{w-1} \right] dr. \tag{5.51}$$

On substituting the shock condition (5.6) into equation (5.20) and using equation (5.23), we get

$$A^* \frac{dU^2}{dr} - B' \frac{U^2}{r} + c' \frac{a'^2}{D(1-w)} r^{-w-1} + E a'^2 \beta_2{}^2 r^{w-1} = 0, \tag{5.52}$$

where, $A^* = \left[\dfrac{\chi(\xi)}{\gamma} - \dfrac{1}{2}(\xi-1)\sqrt{\dfrac{\chi(\xi)}{\xi}} \right],$

$$B' = \frac{w\chi(\xi)}{\gamma},$$

$$C' = \left[\frac{\xi\sqrt{\xi\chi(\xi)}}{(\xi-1) - \sqrt{\xi\chi(\xi)}} + L \right]$$

and $$E' = \left(\xi^2 - L\right).$$

Equation (5.52) can be written as :

$$\frac{dU^2}{dr} - B_1 * \frac{U^2}{r} + B_2 * \frac{a'^2}{D(1-w)} r^{-w-1} + B_2 * a'^2 \beta_2{}^2 r^{w-1} = 0, \tag{5.53}$$

where $B_1 * = \frac{B}{A*}$, $B_2 * = \frac{C}{A*}$ and $B_3 * = \frac{E'}{A*}$.

For the C_+ disturbances generated by the shock, the fluid velocity increment using equation (5.53) into equation (5.6) may be expressed as :

$$du_+ = \frac{\xi-1}{\xi}\frac{1}{2U}\left[B_1 * \frac{U^2}{r} + B_2 * \frac{a'^2}{D(1-w)} r^{-w-1} - B_3 * a'^2 \beta_2{}^2\right] dr. \tag{5.54}$$

Now, in presence of both C_+ and C_- disturbances, the fluid velocity behind the shock will be related as :

$$du_- = du_+ = \frac{\xi-1}{\xi} dU. \tag{5.55}$$

Substituting equations (5.51) and (5.54) into equation (5.55), we get

$$\frac{dU^2}{dr} - \left(B_1 + B_1 *\right)\frac{U^2}{r} + \left(B_2 - B_2 *\right)\frac{a'^2}{D(1-w)} r^{-w-1}$$

$$+ \left(B_3 + B_3 *\right) a'^2 \beta_2{}^2 r^{w-1} = 0 . \qquad (5.56)$$

On integration equation (5.56), we get

$$U^2 = T * r\left(B_1 + B_1 *\right) + \frac{\left(B_2 - B_2 *\right) a'^2}{D(1-w)\,(w + B_1 + B_1 *)} r^{-w}$$

$$- \frac{\left(B_3 + B_3 *\right)\beta_2{}^2 \, a'^2}{\left(w - B_1 - B_1 *\right)} r^{w} , \qquad (5.57)$$

where T* is the constant of integration.

Equation (5.57) describes the propagation parameter U^2, which includes the effect of disturbances behind the flow on the motion of shock.

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