Chapter 0

Introduction

As the title suggests, this thesis deals with some problems in spectral sets, Duggal transformations, and Aluthge transformations of bounded linear operators on Hilbert spaces.

The notion of spectral sets was introduced by J. von Neumann in 1951. If $T$ is a bounded linear operator on a Hilbert space, a closed proper subset $X$ of the complex plane $\mathbb{C}$ is called a spectral set for $T$ if $X$ contains the spectrum of $T$ and $\| f(T) \| \leq \| f \|_{\infty}$, for all rational functions $f$ having poles off $\bar{X}$, where $\bar{X}$ denotes the closure of $X$ in the Reimann sphere $\mathbb{S}$, and $\| f \|_{\infty}$ the norm of $f$ in the $C^*$-algebra $C(\partial \bar{X})$.

If $T = U |T|$ is the polar decomposition of a bounded linear operator $T$ on a Hilbert space, where $U$ is a partial isometry such that $T$ and $U$ have the same kernel, then the operator $\tilde{T} = |T|^{1/2} U |T|^{1/2}$ is known as the Aluthge transformation of $T$. Aluthge transformation was first studied by A. Aluthge in 1990 in the paper [1] in relation with the $p$–hyponormal and $\log$–hyponormal operators. Aluthge transformation has received considerable attention in recent years. One reason is the connection of Aluthge transformation with the invariant subspace problem. Il Bong Jung, Eungil Ko, and Carl Pearcy proved in [22] that
$T$ has a nontrivial invariant subspace if and only if $\tilde{T}$ does. Another reason is related with the iterated Aluthge transformation. In [22], Jung, Ko, and Pearcy defined the $n^{th}$ Aluthge transform $\tilde{T}^{(n)}$ for each non-negative integer $n$, as $\tilde{T}^{(n)} = (\tilde{T}^{(n-1)})$ and $\tilde{T}^{(0)} = T$. They conjectured that for every bounded linear operator on a Hilbert space, the Aluthge sequence $\{\tilde{T}^{(n)}\}_{n=0}^{\infty}$ is norm convergent to a quasinormal operator. M. Chô, I. B. Jung, and W. Y. Lee in [10], showed that the conjecture is not true in the case of infinite dimensional Hilbert spaces. The finite dimensional case is under study and yet to be resolved completely. Aluthge transformation is very useful in the study of non-normal operators.

For a bounded linear operator $T$ on a Hilbert space with polar decomposition $T = U |T|$, the operator $\hat{T} = |T| U$ is the Duggal transformation of $T$. For each non-negative integer $n$, the $n^{th}$ Duggal transformation $\hat{T}^{(n)}$ can be defined as $\hat{T}^{(n)} = (\hat{T}^{(n-1)})$ and $\hat{T}^{(0)} = T$. In 2003, Ciprian Foias, Il Bong Jung, Eungil Ko, and Carl Pearcy initiated the study of Duggal transformations in [16], and proved several analogous results for Aluthge transformations and Duggal transformations. They named the transformation after B. P. Duggal who inspired them to study this transformation. The volume of work done on Duggal transformations, is considerably less, compared to that on Aluthge transformations.

Though the scope of this study embraces Duggal transformations and Aluthge transformations, it devotes more attention to Duggal transformations. Spectral sets of operators on Hilbert spaces are studied, and some relation between spectral sets and the above mentioned transformations are established. The main results of this work are on the convergence of the norms of the Duggal iterates, the convergence of the Duggal iterates and spectral sets, contractivity and positivity of the maps between the Riesz Dunford algebras determined by an operator and its Aluthge transformation and Duggal transformation, the polar decomposition of Duggal transformations and Aluthge transformations, the minimal spectral sets, and $n$–level spectral sets.
Apart from the introduction, the thesis contains four chapters. Chapter 1 is devoted to the basic definitions and results that are necessary for the study. This chapter covers some topics on bounded linear operators on Hilbert spaces, rational and holomorphic functional calculi, $C^*$-algebras, and spectral sets.

Aluthge and Duggal transformations form the object of study in chapter 2. T. Yamazaki in [35] proved that for every bounded linear operator $T$ on a Hilbert space, the sequence of the norms of the Aluthge iterates of $T$ converges to the spectral radius $r(T)$. Derming Wang in [33] gave another proof of this result. In an attempt to prove, the analogue of the result of Yamazaki, that for every bounded linear operator $T$ on a Hilbert space, the sequence of the norms of the Duggal iterates of $T$ converges to the spectral radius $r(T)$, it is proved that for certain classes of operators, the sequence of the norms of the Duggal iterates converges to the spectral radius. This result is given as theorem 2.2.2. Further investigation leads to an example of a finite dimensional operator showing that there exist bounded linear operators on Hilbert spaces such that the sequence of the norms of the Duggal iterates does not converge to the spectral radius, and this is a striking difference of Duggal transformation compared to Aluthge transformation. This example is exhibited in section 2.2.3.

After Yamazaki’s result in 2002 on the convergence of the sequence of the norms of the Aluthge iterates, in 2003, T. Ando and T. Yamazaki in [3], proved that in the case of a $2 \times 2$ matrix, the sequence of the iterated Aluthge transformations itself converges. In 2006, Jorge Antezana, Enrique R. Pujals and Demetrio Stojanoff [4], proved the convergence of iterated Aluthge transformation sequence for diagonalizable matrices. In section 2.2.3, an example is constructed showing that the analogous results for these three results fail to hold in the case of Duggal transformations. In 2004, T. Ando proved in [2] that if $A$ is an $n \times n$ matrix over $\mathbb{C}$, then the convex hull of the spectrum $\sigma(A)$ equals the numerical range $W(A)$ if and only if $A$ and $\tilde{A}$ have the same numerical range. In section 2.3.1,
the study argues that this result does not hold if the Aluthge transformation is replaced by the Duggal transformation.

If $T$ is a bounded linear operator on a Hilbert space, there is an algebra of bounded linear operators determined by the Riesz Dunford functional calculus associated with the operator $T$, known as the Riesz Dunford algebra determined by $T$. In section 2.3.2, the thesis analyzes the Aluthge and the Duggal transformations of the operator $T$ when the partial isometry $U$ in the polar decomposition of $T$ happens to be a coisometry. The homomorphisms between the Riesz Dunford algebras determined by $T$, $\tilde{T}$, and $\hat{T}$ are studied. The notion of $n$–level spectral set is introduced between spectral sets and complete spectral sets, for every positive integer $n$. The thesis shows that if $T = U|T|$ is the polar decomposition of $T$, and $U$ a coisometry, then $T$ and $\hat{T}$ have the same collection of spectral sets, have the same collection of complete spectral sets, and have the same collection of $n$–level spectral sets. Further the thesis proves that if $T$ is an invertible operator, and if for some $n$, the $n$th Duggal iterate $\hat{T}^{(n)}$ is normal, then $T$ is normaloid; in fact, in such cases, $f(T)$ is normaloid for every rational function $f$ having poles off $\sigma(T)$.

In [14], Ken Dykema and Hanne Schultz proved that if $\mathcal{H}$ is any Hilbert space, then the Aluthge transformation map $T \to \tilde{T}$ is continuous on the space $\mathcal{L}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$. In section 2.3.3, the thesis examines the continuity of the Duggal transformation map $T \to \hat{T}$. Using the continuity of the Duggal transformation map $T \to \hat{T}$ on the set of invertible operators, results regarding the relation between the spectral sets of an operator and the spectral sets of the limit of the sequence of the Duggal iterates, are obtained. The following theorem is one of these results. If $T$ is an invertible operator on a Hilbert space, then the sequence of Duggal iterates $\{\hat{T}^{(n)}\}_{n=0}^{\infty}$ can converge to an invertible operator if and only if $T$ is quasinormal. The study proceeds to prove the result: Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{L}(\mathcal{H})$ be an invertible operator.
Suppose that $\hat{T}^{(n)} \to S$ in $\mathcal{L}(\mathcal{H})$, and $X$ is a closed proper subset of $\mathbb{C}$ such that $X$ contains a neighborhood of $\sigma(S)$. Further, assume that $f(\hat{T}^{(n)}) \to f(S)$ for all rational functions $f$ having poles off $\tilde{X}$. Then, $X$ is a spectral set for $T$ if and only if $X$ is a spectral set for $S$.

In section 2.4, the contractivity and positivity of the map $f(T) \to f(\tilde{T})$ between the Riesz Dunford algebras determined by $T$ and $\tilde{T}$, and of the map $f(T) \to f(\hat{T})$ between the Riesz Dunford algebras determined by $T$ and $\hat{T}$, are used to prove some of the consequences including the following theorem. If $T$ is a bounded linear operator on a Hilbert space $\mathcal{H}$ such that the Riesz Dunford algebra determined by $T$ is closed in $\mathcal{L}(\mathcal{H})$ and $f, g$ are holomorphic functions on neighborhoods of $\sigma(T)$ satisfying $(f(T))^* = g(T)$, then $(f(\tilde{T}))^* = g(\hat{T})$, $g(\hat{T})).$

The polar decomposition of Aluthge transformation and Duggal transformation of operators is the focus of study of chapter 3. In [20], Masatoshi Ito, Takeaki Yamazaki, and Masahiro Yanagida obtained several results on the polar decomposition of Aluthge transformation. In [21], Ito, Yamazaki, and Yanagida showed results on the polar decomposition of the product of two operators and of Aluthge transformation. They also showed properties and characterizations of binormal and centered operators from the viewpoint of the polar decomposition and Aluthge transformation. In [20], Ito, Yamazaki, and Yanagida gave an example of a binormal, invertible operator $T$ such that the Aluthge transformation $\tilde{T}$ is not binormal. In chapter 3, it is shown that if $T$ is a binormal, invertible operator, then the Duggal transformation $\hat{T}$ is binormal. Some of the consequences of applying Aluthge transformation and Duggal transformation successively on an invertible operator $T$ are discussed. As a result, theorem 3.2.10 shows that if $T$ is invertible and binormal, then $\hat{(\tilde{T})} = \tilde{(\hat{T})}$. The thesis further extends this result to iterated Aluthge transformations and Duggal transformations. The study proceeds to show that if $T$ is an invertible operator with polar decomposition
$T = U |T|$, then the polar decomposition of $\hat{T}$ is $\hat{T} = U |\hat{T}|$.

Let $T = U|T|$ be the polar decomposition of an operator $T$. A theorem in [21] says that $T$ is binormal if and only if $\tilde{T} = \tilde{U} |\tilde{T}|$ is the polar decomposition of the Aluthge transformation $\tilde{T}$. The thesis discusses a similar situation for Duggal transformations. Necessary and sufficient condition for $\hat{T}$ to have the polar decomposition $\hat{T} = \hat{U} |\hat{T}|$ is obtained. The following theorem is an important consequence. If $T$ is binormal, then $\hat{T} = \hat{U} |\hat{T}|$ is the polar decomposition of $\hat{T}$.

In section 3.2.3, the discussion on polar decomposition of Aluthge transformations and Duggal transformations is concluded by giving a modification of the proof of a theorem in [21], using the semigroup properties of factors in the polar decomposition of a bounded linear operator on a Hilbert space.

In [29], M. Schreiber characterized by means of normal dilations, those operators, the closure of whose numerical range is a spectral set. He obtained results on the equality of the convex hull of the spectrum with the closure of the numerical range in relation to the spectrality of the numerical range. In section 3.3, Aluthge and Duggal transformations are discussed in the context of these results.

Minimal spectral sets and $n$–level spectral sets are the subjects of study of chapter 4. If $\mathcal{H}$ is a Hilbert space and $T \in \mathcal{L}(\mathcal{H})$, then a closed subset $X$ of $\mathbb{C}$ is called a minimal spectral set for $T$, if $X$ is a spectral set for $T$ such that $X$ contains no other spectral set for $T$. As observed by J. P. Williams in [34], for every operator $T$ on a Hilbert space $\mathcal{H}$, there is a minimal spectral set, and in fact, every spectral set contains a minimal spectral set. A theorem in [7] says that $\sigma(T)$ is a spectral set for $T$ if and only if $f(T)$ is normaloid for every rational function $f$ having poles off the spectrum $\sigma(T)$. In chapter 4, it is observed that these statements are equivalent to the uniqueness of the minimal spectral set. Situations when the minimal spectral set of an operator is unique are discussed.

If $X$ is a closed proper subset of the complex plane, $\mathcal{R}(X)$ the subalgebra
of the $C^*$-algebra $C(\partial \tilde{X})$ consisting of the rational functions having poles off $\tilde{X}$, $\mathcal{R}(X)$ the set of all complex conjugates of members of $\mathcal{R}(X)$, and $X$ happens to be an $n$–level spectral set for the operator $T$ on a Hilbert space $\mathcal{H}$, then there is a map $\tilde{\rho} : \mathcal{R}(X) + \mathcal{R}(\bar{X}) \to \mathcal{L}(\mathcal{H})$ that extends the natural functional calculus map $\rho : \mathcal{R}(X) \to \mathcal{L}(\mathcal{H})$. In section 4.3, results on the contractivity and positivity of the map $\tilde{\rho}$ are obtained.

Further research that is possible beyond the thesis, and some of the problems and possibilities that are left open, are briefly outlined in the Epilogue.
Chapter 1

Preliminaries

1.1 Introduction

This chapter is devoted to the basic definitions and results that are necessary for the study, and covers some topics on bounded linear operators on Hilbert spaces, rational and holomorphic functional calculi, $C^*$-algebras, and spectral sets.

Notation: In what follows, $\mathbb{C}$ denotes the set of all complex numbers, $\mathbb{R}$ the set of all real numbers, and $\mathbb{R}_+$ the set of all non-negative real numbers. We denote by $\mathcal{H}$ a Hilbert space, $\langle , \rangle$ the inner product, and $\mathcal{L}(\mathcal{H})$ the $C^*$-algebra of all bounded linear operators on $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$, we denote the spectrum of $T$ by $\sigma(T)$, and the adjoint of $T$ by $T^*$. If $\mathcal{M}$ is a closed subspace of $\mathcal{H}$, then we denote by $P_\mathcal{M}$ the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$. All the projections we consider are orthogonal projections. If $T \in \mathcal{L}(\mathcal{H})$, we denote by $\text{ran}T$, the range $\{Tx : x \in \mathcal{H}\}$, and by $\text{ker}T$ the set $\{x \in \mathcal{H} : Tx = 0\}$. $T$ always denotes a bounded linear operator on a Hilbert space $\mathcal{H}$, unless specified otherwise. If $X$ is a subset of a normed linear space, or of $\mathbb{C}$, we denote by $\overline{X}$, the closure of $X$ in the respective space, except in section 4.3. In section 4.3, the overbar is
used to denote complex conjugation. The term ‘operator’ is freely used to mean ‘bounded linear operator on the Hilbert space $\mathcal{H}$’.

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**Definition 1.2.1.** Let $\mathcal{H}$ be a complex Hilbert space, and $T \in \mathcal{L}(\mathcal{H})$. The set of $x \in \mathcal{H}$ such that $Tx = 0$ is a closed subspace of $\mathcal{H}$. Let $\mathcal{Y} = \{x \in \mathcal{H} : Tx = 0\}$. (ie., $\mathcal{Y} = \ker T$). Let $\mathcal{X} = \mathcal{Y}^\perp$. The projection $E = P_\mathcal{X}$ is called the *support* of $T$.

If $E$ is the support of $T$, then $TE = T$. Also $E$ is the smallest of the projections $E_1$ of $\mathcal{L}(\mathcal{H})$ such that $TE_1 = T$. (If $E_1$ and $E_2$ are two projections, we say that $E_1 \leq E_2$ if $\text{ran} E_1 \subset \text{ran} E_2$).

The closure of $T(\mathcal{H})$ is a closed linear subspace $\mathcal{Y}$ of $\mathcal{H}$. Let $F = P_\mathcal{Y}$. Then $F$ is the smallest of the projections $F_1$ of $\mathcal{L}(\mathcal{H})$ such that $T^*F_1 = T^*$. Also, $F$ is the support of $T^*$.

**Definition 1.2.2 (Partial isometry).** Let $U \in \mathcal{L}(\mathcal{H})$, and $E$ its support. We say that $U$ is a *partial isometry* if $U$ is an isometry on $\mathcal{X} = E(\mathcal{H})$.

If $U \in \mathcal{L}(\mathcal{H})$ is a partial isometry, then $U(\mathcal{H}) = U(\mathcal{X})$ is a closed linear subspace $\mathcal{Y}$ of $\mathcal{H}$; and $U$ maps $\mathcal{X}$ isometrically onto $\mathcal{Y}$. Let $F = P_\mathcal{Y}$. We say that $E$ is the *initial projection* of $U$ and that $F$ is the *final projection* of $U$. We say that $\mathcal{X}$ is the *initial space* of $U$ and that $\mathcal{Y}$ is the *final space* of $U$.

Let $x \in \mathcal{X}$, $y = Ux \in \mathcal{Y}$. For every $z \in \mathcal{H}$, we have $\langle x, z \rangle = \langle x, Ez \rangle = \langle Ux, UEz \rangle = \langle y, UEz \rangle = \langle y, Uz \rangle = \langle U^*y, z \rangle$. Hence $x = U^*y$. Thus the mapping $x \rightarrow Ux$ of $\mathcal{X}$ onto $\mathcal{Y}$ has for its inverse (isometric) mapping the mapping $y \rightarrow U^*y$ of $\mathcal{Y}$ onto $\mathcal{X}$.

Since, furthermore, the support of $U^*$ is $F$, we see that $U^*$ is a partial isometry with initial projection $F$ and final projection $E$. We also see that $U^*U = E$.
and $UU^* = F$. Thus $U^*U$ and $UU^*$ are the initial and final projections of $U$.

Conversely, if $V \in \mathcal{L}(\mathcal{H})$ such that $V^*V$ is a projection, then $V$ is a partial isometry. Similarly, if $W \in \mathcal{L}(\mathcal{H})$ and $WW^*$ is a projection, then $W^*$ is a partial isometry, and hence $W$ is a partial isometry.

**Remark 1.2.3.** If $U$ is a nonzero partial isometry, then $\|U\| = 1$. (Proof: $U$ is a nonzero partial isometry $\implies U^*U$ is a nonzero projection $\implies 1 \in \sigma(U^*U) \subset \{0, 1\} \implies \|U\|^2 = r(U^*U) = 1$, where $r(U^*U)$ denotes the spectral radius of the operator $U^*U$.)

**Definition 1.2.4** (Polar decomposition of an operator). Let $T \in \mathcal{L}(\mathcal{H})$, $E$ the support of $T$, $F$ the support of $T^*$, $\mathcal{X} = E(\mathcal{H})$ and $\mathcal{Y} = F(\mathcal{H})$. We put $|T| = (T^*T)^{1/2}$. We have, for every $x \in \mathcal{H}$, $\|Tx\|^2 = \langle T^*Tx, x \rangle = \langle |T|^2x, x \rangle = \langle |T|x, |T|x \rangle = \|T|x\|^2$. Therefore, $\|Tx\| = \|T|x\|$ for all $x \in \mathcal{H}$. Hence $|T|$ has support $E$, and consequently, $|T|(\mathcal{H}) = \mathcal{X}$.

Furthermore, the mapping $|T|x \rightarrow Tx$ is a linear isometry of $|T|(\mathcal{H})$ onto $T(\mathcal{H})$, and therefore extends to a linear isometry $V : \mathcal{X} = |T|(\mathcal{H})$ onto $\mathcal{Y} = T(\mathcal{H})$. Let $U$ be the partial isometry with support $E$ and which coincides with $V$ on $\mathcal{X}$; this partial isometry has $E$ as initial projection and $F$ as final projection.

We have $T = U|T|$ an equality called the polar decomposition of $T$.

Thus any $T \in \mathcal{L}(\mathcal{H})$ can be written as

$$T = U|T|$$

where $U$ is a partial isometry and $|T|$ is a positive operator such that the initial projection of $U$ is $E$ which is the support of $T$, the final projection of $U$ is $F$ which is the support of $T^*$, and such that the support of $|T|$ is also $E$. Note that $TE = T$, $UE = U$ and $|T|E = |T|$.

On the other hand, if we have an equality $T = U_1T_1$ where $T_1$ is positive
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hermitian and where $U_1$ is a partial isometry whose initial projection is support of $T_1$, then we have $T_1 = |T|$ and $U_1 = U$.

The equality

$$T^* = U^*(U|T|U^*)$$

is the polar decomposition of $T^*$.

The following theorem in [27] deals with polar decomposition.

**Theorem 1.2.5.** [27] Let $A$ be a bounded operator on a Hilbert space $\mathcal{H}$. Then there is a partial isometry $U$ such that $A = U|A|$. The partial isometry $U$ is uniquely determined by the condition that $\ker U = \ker A$. Moreover, $\ran U = \overline{\ran A}$.

**Remark 1.2.6.** [12] Let $A$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$. If $T \in A$ and $T = U|T|$ the polar decomposition of $T$, then $U \in A$ and $|T| \in A$. If $\mathcal{M}$ is a left ideal in $A$ and $T \in A$, then $T \in \mathcal{M}$ if and only if $|T| \in \mathcal{M}$.

**Definition 1.2.7** (Shifts). An operator $S_+$ on a Hilbert space $\mathcal{H}$ is a **unilateral shift** if there exists an infinite sequence $\{H_k\}_{k=0}^\infty$ of nonzero pairwise orthogonal subspaces of $\mathcal{H}$ such that $\mathcal{H} = \bigoplus_{k=0}^\infty H_k$ and $S_+$ maps each $H_k$ isometrically onto $H_{k+1}$.

Since $S_+|H_k : H_k \to H_{k+1}$ is unitary (ie., a surjective isometry), it follows that $\dim H_{k+1} = \dim H_k$, for every $k \geq 0$. This constant dimension is the multiplicity of $S_+$. The adjoint $S_+^*$ of $S_+$ lies in $\mathcal{L}(\mathcal{H})$, and is referred to as the **backward unilateral shift**, also denoted by $S_-$. Writing $\bigoplus_{k=0}^\infty x_k$ for $\{x_k\}_{k=0}^\infty$ in $\mathcal{H} = \bigoplus_{k=0}^\infty H_k$, it follows that $S_+$ and $S_-$ are given by the formulas

$$S_+x = 0 \oplus \bigoplus_{k=1}^\infty U_kx_{k-1}$$
and

$$S_+^*x = \bigoplus_{k=0}^\infty U_k^*x_{k+1}$$

for every $x = \bigoplus_{k=0}^\infty x_k$ in $\mathcal{H} = \bigoplus_{k=0}^\infty H_k$, where 0 is the origin of $H_0$ and $U_{k+1}$ is any unitary transformation of the space $H_k$ onto the space $H_{k+1}$ so that $S_+|H_k = U_{k+1}$,
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for each \( k \geq 0 \). These are identified with the infinite matrices

\[
S_+ = \begin{bmatrix}
0 & \cdots & \\
U_1 & 0 & \cdots \\
& U_2 & 0 & \cdots \\
& & U_3 & 0 & \cdots \\
& & & \ddots & \ddots \\
\end{bmatrix}
\]

and

\[
S^*_+ = \begin{bmatrix}
0 & U_1^* & \cdots \\
0 & U_2^* & \cdots \\
& 0 & U_3^* & \cdots \\
& & & \ddots & \ddots \\
\end{bmatrix}
\]

of transformations.

In particular, if \( \mathcal{H} \) is an infinite dimensional separable Hilbert space; that is, if the Hilbert space \( \mathcal{H} \) has a countably infinite orthonormal basis, say \( \{e_k\}_{k=0}^\infty \), then set \( \mathcal{H}_k = \text{span} \{e_k\} \) for \( k \geq 0 \) so that \( \text{dim} \mathcal{H}_k = 1 \) for all \( k \geq 0 \). It can be verified that \( S_+ \) is a unilateral shift of multiplicity 1 on \( \mathcal{H} \) if it shifts the orthonormal basis \( \{e_k\}_{k=0}^\infty \) for \( \mathcal{H} \); that is, if \( S_+ e_k = e_{k+1} \) for every \( k \geq 0 \). Conversely, an operator \( S_+ \) is a unilateral shift of multiplicity 1 on \( \mathcal{H} \) only if it shifts some orthonormal basis \( \{u_k\}_{k=0}^\infty \) for \( \mathcal{H} \); that is, only if \( S_+ u_k = u_{k+1} \) for every \( k \geq 0 \).

If \( x = \bigoplus_{k=0}^\infty x_k \) is any vector in the Hilbert space \( \mathcal{H} = \bigoplus_{k=0}^\infty \mathcal{H}_k \), then

\[
\|S_+ x\|^2 = \|0 \oplus \bigoplus_{k=1}^\infty U_k x_{k-1}\|^2 = \sum_{k=1}^\infty \|U_k x_{k-1}\|^2 = \sum_{k=1}^\infty \|x_{k-1}\|^2 = \sum_{k=0}^\infty \|x_k\|^2 = \|x\|^2.
\]

Thus the unilateral shift \( S_+ \) is an isometry. This can also be verified observing that \( S^*_+ S_+ = I \).

An operator \( S \) on a Hilbert space \( \mathcal{H} \) is a bilateral shift if there exists an infinite sequence \( \{\mathcal{H}_k\}_{k=-\infty}^\infty \) of nonzero pairwise orthogonal subspaces of \( \mathcal{H} \) such that \( \mathcal{H} = \bigoplus_{k=-\infty}^\infty \mathcal{H}_k \) and \( S \) maps each \( \mathcal{H}_k \) isometrically onto \( \mathcal{H}_{k+1} \).

**Definition 1.2.8 (Coisometry).** Let \( \mathcal{H} \) be a Hilbert space, and \( U \in \mathcal{L}(\mathcal{H}) \). We say that \( U \) is a coisometry if \( UU^* = I \), where \( I \) is the identity operator in \( \mathcal{L}(\mathcal{H}) \).

The operator \( U \) is a coisometry if and only if \( U^* \) is an isometry. Obviously every unitary operator is a coisometry. Also an invertible coisometry is always unitary. The backward unilateral shift is an example of a coisometry.
**Definition 1.2.9** (Quasinormal operator). An operator $T$ is called quasinormal if $T$ commutes with $T^*T$.

Every normal operator is quasinormal. Converse is not true. For instance, if $A$ is an isometry, then $A^*A = I$. So $A$ commutes with $A^*A$. If $A$ is not unitary, then $A$ is not normal. For example, the unilateral shift is quasinormal, but not normal.

**Definition 1.2.10** (Dilations). Suppose that $\mathcal{H}$ is a subspace of a Hilbert space $\mathcal{K}$ and let $P$ be the projection from $\mathcal{K}$ onto $\mathcal{H}$. If $B \in \mathcal{L}(\mathcal{K})$, then $B$ induces an operator $A$ on $\mathcal{H}$, defined by

$$Ax = PBx, \ x \in \mathcal{H}.$$ 

We have $AP = PB$.

The operator $A$ is called the *compression* of $B$ to $\mathcal{H}$, and $B$ is called a *dilation* of $A$ to $\mathcal{K}$.

If $\mathcal{H}$ is invariant under $B$ then $A$ is the restriction of $B$ to $\mathcal{H}$ and $B$ is an extension of $A$ to $\mathcal{K}$. The following are some known facts.

- Every operator has a normal dilation.
- If $\|A\| \leq 1$, then $A$ has a unitary dilation.
- If $0 \leq A \leq 1$, then $A$ has a dilation that is a projection.

An operator $B$ is called a *power dilation* (or *strong dilation*) of an operator $A$ if $B^n$ is a dilation of $A^n$ for $n = 1, 2, \ldots$ (i.e., $A^n x = PB^n x, \ x \in \mathcal{H}, \ n = 1, 2, \ldots$).

**Definition 1.2.11** (Subnormal operator). An operator is *subnormal* if it has a normal extension. More precisely, an operator $A$ on a Hilbert space $\mathcal{H}$ is
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subnormal if there exists a normal operator $B$ on a Hilbert space $K$ such that $H$ is a subspace of $K$, $H$ is invariant under the operator $B$, and the restriction of $B$ to $H$ coincides with $A$.

Normal $\Rightarrow$ subnormal.

On finite dimensional Hilbert spaces, the concepts normal and subnormal are the same. Unilateral shift is subnormal, the bilateral shift is the normal extension.

Normal $\Rightarrow$ quasinormal $\Rightarrow$ subnormal.

On finite dimensional Hilbert spaces the three concepts, normal, quasinormal and subnormal, are the same.

Subnormal $\not\Rightarrow$ quasinormal.

For example, let $U$ be the unilateral shift and $0 \neq c$ be a scalar. Let $B$ be the normal extension of $U$. Then $B$ is the bilateral shift on a Hilbert space $K$. Let $I_K$ be the identity operator on $K$. We know that if $S$ and $T$ are normal operators such that $S$ commutes with $T^*$ and $T$ commutes with $S^*$, then $S + T$ is normal. Therefore, the operator $B + cI_K$ is normal. Also, $B + cI_K$ is an extension of $U + cI$. This shows that $U + cI$ is subnormal. But $U + cI$ is not quasinormal, since if $U + cI$ were quasinormal, $U + cI$ would commute with $(U + cI)^*(U + cI)$, i.e., $(U + cI)[(U + cI)^*(U + cI)] = [(U + cI)^*(U + cI)](U + cI)$ which would imply that $UU^* = U^*U$ and would contradict the fact that $U$ is not normal.

**Definition 1.2.12** (Spectral radius). If $T \in \mathcal{L}(\mathcal{H})$, the spectral radius $r(T)$ of $T$ is defined as

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$  

We see that

$$r(T) = \lim_{k \to \infty} \| T^k \|^{1/k}.$$
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**Definition 1.2.13** (Normaloid). An operator $T$ is said to be normaloid if $r(T) = \| T \|$.

**Definition 1.2.14** (Numerical range). If $T \in \mathcal{L}(\mathcal{H})$, then the set $W(T) = \{ \langle Tx, x \rangle : x \in \mathcal{H}, \| x \| = 1 \}$ is called the numerical range of $T$.

**Theorem 1.2.15** (Toeplitz-Hausdorff theorem). The numerical range of an operator is always convex.

In 1918, Toeplitz proved that for any operator $A$, the boundary of $W(A)$ is a convex curve, but left open the possibility that it had interior holes. In 1919, Hausdorff proved that it did not.

**Theorem 1.2.16.** The closure of the numerical range includes the spectrum.

**Theorem 1.2.17.** If $T$ is normal, then $\overline{W(T)}$ is the closed convex hull $C(\sigma(T))$ of the spectrum $\sigma(T)$ of $T$.

If $T$ is not normal, then it can happen that $C(\sigma(T)) \neq \overline{W(T)}$. For example, let $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then $\sigma(A) = \{0\}$ and $W(A) = \{z : |z| \leq 1/2\}$.

**Definition 1.2.18** (Convexoid operator). An operator $T$ is said to be convexoid if $C(\sigma(T)) = \overline{W(T)}$.

**Definition 1.2.19** (Unitarily equivalent operators). Let $\mathcal{H}$ be a Hilbert space, and $S, T \in \mathcal{L}(\mathcal{H})$. We say that $S$ and $T$ are unitarily equivalent if there exists a unitary operator $U$ in $\mathcal{L}(\mathcal{H})$ such that $S = U^*TU$.

Unitarily equivalent operators have the same numerical range. (Proof: Suppose that $S$ and $T$ are unitarily equivalent. Assume that $S = U^*TU$, where $U$ is unitary. For every $x \in \mathcal{H}$, $\langle Sx, x \rangle = \langle U^*TUx, x \rangle = \langle TUx, Ux \rangle = \langle Ty, y \rangle$, where $y = Ux$, and we have $\| y \| = \| Ux \| = \| x \|$. On the other hand, for every $x \in \mathcal{H}$, $\langle Tx, x \rangle = \langle TUU^*x, UU^*x \rangle = \langle SU^*x, U^*x \rangle = \langle Sz, z \rangle$, where $z = U^*x$, and we have $\| z \| = \| U^*x \| = \| x \|$. Thus $W(S) = W(T)$.)
1.3 Rational and holomorphic functional calculi

Let $\mathcal{U}$ be an associative algebra with unity. We shall define $\mathbb{C}[t]$ as the integral domain of all polynomials over $\mathbb{C}$ in the variable $t$.

**Theorem 1.3.1.** [7] Let $x \in \mathcal{U}$. Then there is a unique algebra homomorphism $\phi : \mathbb{C}[t] \to \mathcal{U}$ such that $\phi(1) = 1$ and $\phi(t) = x$.

**Notation:** If $p \in \mathbb{C}[t]$ we define $p(x) = \phi(p)$; thus if $p(t) = \sum_{k=0}^{n} a_k t^k$, then $p(x) = \sum_{k=0}^{n} a_k x^k$.

We shall denote by $\mathbb{C}(t)$ the field of fractions of the integral domain $\mathbb{C}[t]$. Thus $\mathbb{C}(t)$ is the set of all rational forms $f = p/q$ where $p, q \in \mathbb{C}[t]$ and $q \neq 0$. The rational form $f = p/q$ is said to be in the reduced form if $p$ and $q$ are relatively prime in the integral domain $\mathbb{C}[t]$.

**Definition 1.3.2** (The algebra $\mathcal{R}(X)$). [7] Let $X$ be a closed proper subset of $\mathbb{C}$, and let $\bar{X}$ denote the closure of $X$, when we regard $X$ as a subset of the Riemann sphere $\mathcal{S}$. That is, $\bar{X} = X$, when $X$ is compact, and otherwise $\bar{X}$ is $X$ together with the point at $\infty$. We let $\mathcal{R}(X)$ denote the quotients of polynomials with poles off $\bar{X}$, that is, the bounded, rational functions on $X$ with a limit at $\infty$. Clearly, $\mathcal{R}(X)$ is a subalgebra of $\mathbb{C}(t)$.

**Theorem 1.3.3.** [7] Let $\mathcal{U}$ be an associative algebra with unity and $A \in \mathcal{U}$. There is a unique algebra homomorphism $\phi : \mathcal{R}(\sigma(A)) \to \mathcal{U}$ such that $\phi(1) = 1$ and $\phi(t) = A$.

**Notation:** If $f \in \mathcal{R}(\sigma(A))$ we define $f(A) = \phi(f)$; writing $f = p/q$ where $p, q \in \mathbb{C}[t]$ and $q$ has no zeros in $\sigma(A)$, we have $f(A) = q(A)^{-1}p(A) = p(A)q(A)^{-1}$.

Note that if $f = p/q$ belong to $\mathcal{R}(\sigma(A))$, then $q$ has no zeros in $\sigma(A)$. This means $0 \notin q(\sigma(A))$. But by the spectral mapping theorem, $q(\sigma(A)) = \sigma(q(A))$. 


1.3. Rational and holomorphic functional calculi

Therefore, \(0 \notin \sigma(q(A))\), and hence \(q(A)\) is invertible. Thus the definition of \(f(A)\) makes sense in theorem 1.3.3.

Let \(H\) be a Hilbert space and \(T \in \mathcal{L}(H)\). Suppose that \(X\) is a closed proper subset of the complex plane \(\mathbb{C}\) such that \(X \supset \sigma(T)\). Obviously, \(\mathcal{R}(X) \subset \mathcal{R}(\sigma(T))\). Therefore, by applying the functional calculus of theorem 1.3.3, \(f(T)\) is defined and exists in \(\mathcal{L}(H)\), for every \(f \in \mathcal{R}(X)\). The mapping \(f \to f(T) : \mathcal{R}(X) \to \mathcal{L}(H)\) is an algebra homomorphism by theorem 1.3.3.

**Definition 1.3.4** (Integrals of Banach space-valued functions). [23] Let \(U\) be a Banach space and \(f\) be a continuous function of the complex variable \(z\), with \(f(z) \in U\). Let \(C\) be a smooth closed curve in the plane \(\mathbb{C}\), defined by \(t \to z(t) : [a, b] \to \mathbb{C}\) (which is continuously differentiable on \([a, b]\)). We define

\[
\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt
\]

as the norm limit of the “Riemann sums” of the form

\[
\sum_{j=1}^n f(z(t'_j))[z(t_j) - z(t_{j-1})]
\]

where \(a = t_0 < t_1 < \ldots < t_n = b\), \(t_{j-1} \leq t'_j \leq t_j\), the limit being taken as \(\max\{|t_j - t_{j-1}| : j = 1, \ldots n\} \to 0\).

**Definition 1.3.5** (Holomorphic functional calculus). [23] Suppose that \(U\) is a Banach algebra and let \(A \in U\). Let \(\sigma(A)\) be the spectrum of \(A\), let \(C\) be a smooth closed curve whose interior contains \(\sigma(A)\), and let \(n\) be a positive integer. It can be seen that

\[
A^n = \frac{1}{2\pi i} \int_C z^n(zI - A)^{-1}dz
\]  

(1.1)

Hence it follows that

\[
p(A) = \frac{1}{2\pi i} \int_C p(z)(zI - A)^{-1}dz
\]  

(1.2)
1.4. Some topics in $C^*$-algebras

for each polynomial $p$, where $C$ is as described above. If $f$ is a holomorphic function (classical complex valued of complex variable), holomorphic in an open set containing $\sigma(A)$, we define

$$f(A) = \frac{1}{2\pi i} \int_C f(z)(zI - A)^{-1}dz$$

(1.3)

for each $A \in \mathcal{U}$, where $C$ is a smooth closed curve whose interior contains $\sigma(A)$. By 1.3.4, the integral on the right hand side of (1.3) converges in the norm. So it represents an element $f(A)$ in $\mathcal{U}$.

We shall denote by $\text{Hol}(\sigma(A))$ the set of functions holomorphic in some open set containing $\sigma(A)$ (the open set may vary with the function). It can be seen that $\text{Hol}(\sigma(A))$ is an algebra.

**Theorem 1.3.6.** [23] The mapping $f \to f(A)$ is a homomorphism from the algebra $\text{Hol}(\sigma(A))$ into $\mathcal{U}$ for each $A$ in the Banach algebra $\mathcal{U}$. If $f$ is represented by the power series $\sum_{n=0}^{\infty} a_n z^n$ throughout an open set containing $\sigma(A)$, then

$$f(A) = \sum_{n=0}^{\infty} a_n A^n.$$ 

**Remark 1.3.7.** Let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{L}(\mathcal{H})$. Let $X$ be a closed proper subset of $\mathbb{C}$ such that $X \supset \sigma(T)$. If $f$ is a rational function, then we see that $f \in \mathcal{R}(X) \implies f = p/q$, where $p, q \in \mathbb{C}(t)$ such that $q$ has no zeros in $X \implies f$ is holomorphic on an open set containing $X \implies f \in \text{Hol}(\sigma(T))$. Thus $\mathcal{R}(X)$ is a subalgebra of $\text{Hol}(\sigma(T))$.

1.4 Some topics in $C^*$-algebras

**Definition 1.4.1** (Operator system). If $S$ is a subset of a $C^*$-algebra $\mathcal{A}$, we set $S^* = \{a : a^* \in S\}$, and we call $S$ self-adjoint when $S^* = S$. If $\mathcal{A}$ has a unit 1
and $S$ is a self-adjoint subspace of $A$ containing 1, then we call $S$ an operator system.

**Definition 1.4.2** (The $C^*$-algebra $M_n(A)$ and the norm in $M_n(A)$). Let $A$ be a $C^*$-algebra, and let $M_n(A)$ denote the set of all $n \times n$ matrices with entries from $A$. We will denote a typical element of $M_n(A)$ by $(a_{ij})$. There is a natural way to make $M_n(A)$ into a $*$-algebra. For $(a_{ij})$ and $(b_{ij})$ in $M_n(A)$, set

$$(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij}),$$

$$c(a_{ij}) = (ca_{ij}) \text{ if } c \in \mathbb{C},$$

$$(a_{ij}) \cdot (b_{ij}) = (\Sigma_{k=1}^n a_{ik} b_{kj}),$$

and

$$(a_{ij})^* = (a_{ji}^*).$$

Also, there is a unique way to introduce a norm such that $M_n(A)$ becomes a $C^*$-algebra.

One way that $M_n(A)$ can be viewed as a $C^*$-algebra is to first choose a one-to-one $*$-representation of $A$ on some Hilbert space $H$, and then let $M_n(A)$ act on the direct sum of $n$ copies of $H$ in the obvious way. It can be easily verified that this defines a one-to-one representation of $M_n(A)$ for which the above multiplication and $*$ operation become operator composition and operator adjoint. It is straight-forward to verify that the image of $M_n(A)$ under this representation is closed and hence a $C^*$-algebra.

Thus we have a way to turn $M_n(A)$ into a $C^*$-algebra. But since the norm is unique on a $C^*$-algebra we see that the norm on $M_n(A)$ defined in this fashion is independent of the particular representation of $A$ we choose.

We shall use the notation $M_n$ for the $C^*$-algebra of all $n \times n$ complex matrices.
In other words, $\mathcal{M}_n = \mathcal{M}_n(\mathbb{C})$.

**Definition 1.4.3.** Let $\mathcal{A}$ be a unital $C^*$-algebra, and $S \subset \mathcal{A}$ be an operator system. If $\mathcal{B}$ is a unital $C^*$-algebra and $\phi : S \to \mathcal{B}$ is a linear map, we say that $\phi$ is a *unital map* if $\phi(1) = 1$. We say that $\phi$ is *self-adjoint* if $\phi(a^*) = (\phi(a))^*$ for all $a \in S$.

**Definition 1.4.4 (Operator space).** Let $\mathcal{A}$ be a $C^*$-algebra and let $\mathcal{M}$ be a subspace, then we call $\mathcal{M}$ an *operator space*.

Clearly $\mathcal{M}_n(\mathcal{M})$ can be regarded as a subspace of $\mathcal{M}_n(\mathcal{A})$, and we let $\mathcal{M}_n(\mathcal{M})$ have the norm structure that it inherits from the unique norm structure on the $C^*$-algebra $\mathcal{M}_n(\mathcal{A})$. Similarly if $S \subset \mathcal{A}$ is an operator system, then we endow $\mathcal{M}_n(S)$ with the norm and the order structure that it inherits as a subspace of the $C^*$-algebra $\mathcal{M}_n(\mathcal{A})$.

**Definition 1.4.5.** Let $\mathcal{A}$ be a $C^*$-algebra, $\mathcal{M}$ be a subspace, and let $n$ be a positive integer. If $\mathcal{B}$ is a $C^*$-algebra and $\phi : \mathcal{M} \to \mathcal{B}$ is a linear map, we define $\phi_n : \mathcal{M}_n(\mathcal{M}) \to \mathcal{M}_n(\mathcal{B})$ by

$$\phi_n((a_{ij})) = (\phi(a_{ij})),$$

and call $\phi_n$ the *$n$th amplification* of $\phi$.

**Definition 1.4.6.** Let $\mathcal{A}$ be a $C^*$-algebra, $\mathcal{M}$ a subspace of $\mathcal{A}$, and let $n$ be a positive integer. If $\mathcal{B}$ is a $C^*$-algebra and $\phi : \mathcal{M} \to \mathcal{B}$ is a linear map, we say that

i. $\phi$ is *positive* if $\phi$ maps positive elements of $\mathcal{M}$ to positive elements of $\mathcal{B}$.

ii. $\phi$ is *contractive* if $\|\phi\| \leq 1$.

iii. $\phi$ is *$n$-positive* if $\phi_n$ is positive.
iv. $\phi$ is *completely positive* if $\phi$ is $n$-positive for all $n$.

v. $\phi$ is *completely bounded* if $\sup_n \|\phi_n\|$ is finite, and, in this case, we set $\|\phi\|_{cb} = \sup_n \|\phi_n\|$.

vi. $\phi$ is *completely isometric* if each $\phi_n$ is isometric.

vii. $\phi$ is *completely contractive* if $\|\phi\|_{cb} \leq 1$.

viii. $\phi$ is *$n$-contractive* if $\|\phi_n\| \leq 1$.

A positive map need not be completely positive; and a bounded map need not be completely bounded. A contractive map need not be $2$–contractive. In general, $\|\phi_n\| \neq \|\phi\|$.

For example, let $\{E_{ij}\}_{i,j=1}^2$ denote the system of matrix units for $M_2$, that is $E_{ij}$ is the $2 \times 2$ matrix with $1$ in the $ij^{th}$ entry and $0$ elsewhere. Let $\phi : M_2 \to M_2$ be the transpose map. It can be easily seen that $\phi(A)$ is positive if $A$ is positive, and that $\|\phi(A)\| = \|A\|$ for all $A \in M_2$. Thus $\phi$ is positive and $\|\phi\| = 1$.

Now, consider $\phi_2 : M_2(M_2) \to M_2(M_2)$. The matrix of matrix units

\[
\begin{bmatrix}
E_{11} & E_{12} \\
E_{21} & E_{22}
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1
\end{bmatrix}
\]

can be written as $B^*B$ where

\[
B = \begin{bmatrix}
\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}}
\end{bmatrix},
\]
and hence is positive. But

$$\phi_2 \left( \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} \right) = \begin{bmatrix} \phi(E_{11}) & \phi(E_{12}) \\ \phi(E_{21}) & \phi(E_{22}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is not positive. For instance, if $T$ is the operator on $\mathbb{C}^4$ that is represented by the above matrix, and if $x = (0, -1, 1, 0)$, then $x \in \mathbb{C}^4$ and $\langle Tx, x \rangle = -2 < 0$.

Also, if

$$D = \begin{bmatrix} E_{11} & E_{21} \\ E_{12} & E_{22} \end{bmatrix},$$

then $\|D\| = 1$, but $\|\phi_2(D)\| = 2$. Therefore, $\|\phi_2\| \geq 2$. Note that $\phi$ is contractive, but not 2–contractive.

### 1.5 Spectral sets

**Definition 1.5.1** (Spectral set, K-spectral set). Let $X$ be a closed proper subset of $\mathbb{C}$, and let $\bar{X}$ denote the closure of $X$ when we regard $X$ as a subset of the Riemann sphere $\mathcal{S}$. That is, $\bar{X} = X$, when $X$ is compact, and otherwise $\bar{X}$ is $X$ together with the point at $\infty$. We let $\mathcal{R}(X)$ denote the quotients of polynomials with poles off $\bar{X}$, that is, the bounded, rational functions on $X$ with a limit at infinity. We regard $\mathcal{R}(X)$ as a subalgebra of the $C^*$-algebra $C(\partial \bar{X})$, which defines norms on $\mathcal{R}(X)$ and on each $\mathcal{M}_n(\mathcal{R}(X))$.

If $X$ is a closed, proper subset of $\mathbb{C}$, and $T \in \mathcal{L}(\mathcal{H})$, with $\sigma(T) \subset X$, then there is still a functional calculus, i.e., a homomorphism $\rho : \mathcal{R}(X) \to \mathcal{L}(\mathcal{H})$, given
by $\rho(f) = f(T)$, where $f(T) = p(T)q(T)^{-1}$ if $f = p/q$ (see the remarks after 1.3.3).

If $\| \rho \| \leq 1$, then $X$ is called a spectral set for $T$.

If $\| \rho \| \leq K$, then $X$ is called a $K$-spectral set for $T$.

If $\| \rho \|_{cb} \leq 1$, then $X$ is called a complete spectral set for $T$.

If $\| \rho \|_{cb} \leq K$, then $X$ is called a complete $K$-spectral set for $T$.

Of course, $\| \rho \|$ is defined as follows.

$$\| \rho \| = \sup \{ \| \rho(f) \| : f \in \mathcal{R}(X), \| f \|_{\infty} = 1 \}$$

$$= \sup \{ \| f(T) \| : f \in \mathcal{R}(X), \| f \|_{\infty} = 1 \},$$

where $\| f \|_{\infty}$ is the norm of $f$ in the $C^*$-algebra $C(\partial \tilde{X})$.

A well known theorem known as von Neumann’s inequality, says that an operator $T$ is a contraction if and only if the closed unit disk is a spectral set for $T$. Thus for every $T \in \mathcal{L}(H)$, the set $\{ z \in C : |z| \leq \| T \| \}$ is always a spectral set for $T$.

By replacing the algebra $\mathcal{R}(X)$ in the above definition by the algebra $\text{Hol}(X)$ and by defining $f(T)$ according to the Riesz-Dunford functional calculus we obtain more general definitions of spectral sets and $K$-spectral sets.

**Theorem 1.5.2** (Berger-Foias-Lebow). If $S$ is a compact, convex spectral set for the operator $T$ on a Hilbert space $H$, and if $\partial S$ denotes the boundary of $S$, then
there exists a normal operator $N$ defined on a larger Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that

i. $\sigma(N) \subset \partial S$

ii. $T^n x = P N^n x$, $x \in \mathcal{H}, n = 1, 2, \ldots$

where $P$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$.

The theorem says that if $S$ is a compact spectral set for $T$, then there exists a strong normal dilation $N$ of $T$ such that $\sigma(N) \subset \partial S$. 