CHAPTER 3

THERMAL MODELING OF TRANSFORMATION HARDENING

3.1 Introduction

One of the prime objectives of world wide research in the field of laser hardening is to develop analytical model to explain mechanism involving many physical phenomenon, their behaviour and resulting output so as to achieve desired properties.

The procedure for modeling this process is to develop a proper understanding in the appropriate transport phenomena, boundary conditions and solution technique. Modeling provides insight and predicts behavior of the process. It reduces the experimental cost and provides optimization of affecting parameters and enhances the understanding of the physical process involved. The analytical solution provides a better appreciation of the physical relationships between the relevant parameter. Analytical model may be easily implemented for simulation work, for analysis and prediction of process under various conditions. This chapter is devoted to develop an analytical model for the process of laser transformation hardening.

The model developed is a two dimensional model considering semi-infinite solid subject to static heat source of uniform heat flux. This model can be used to estimate the temperature at any location for any instant of time within heat affected zone to predict the microstructure of that location and subsequently microstructure of the whole heat affected zone.

3.2 Mathematical modeling-Basics

The principles of laser hardening are similar to those of conventional through hardening. The time scales involved in the former are typically an order of magnitude shorted. Heating is induced by scanning suitably shaped laser beam over the component. In uncoated ferrous alloys the photon of laser beam interacts with free
electrons in the substrate in a layer. Energy is transferred into the bulk of the substrate by classical conduction.

Hence study of heat transfer through conduction will give better understanding of the process.

3.2.1 Conduction

Conduction is heat transfer by diffusion. Conduction refers to the transport of energy in a medium due to the temperature gradient and the physical mechanism is that of random atomic or molecular activity [70].

3.2.2 The heat diffusion equation [70]

A major objective of conduction analysis is to determine the temperature fields in a medium resulting from condition imposed on its boundaries. That is, we wish to know the temperature distribution, which represents how temperature varies with position in the medium. The knowledge of the temperature distribution could be used to ascertain microstructure and structural integrity through determination of thermal stresses, expansion, and deflection. The temperature distribution could also be used to optimize the process parameters. The temperature distribution is determined by applying the energy conservation requirement. That is we define a differential control volume, identify the relevant energy transfer processes, and introduce the appropriate rate equation. The result is a differential equation whose solution provides the temperature distribution in the medium, for prescribed boundary conditions.

Considering a homogeneous medium in which temperature gradients exist and temperature distribution \(T(x, y, z)\) is expressed in Cartesian co-ordinate.

Assuming \(dx, dy, dz\) is an infinitesimally small control volume, and that there are temperature gradients, the conduction heat rates perpendicular to each other of the control surfaces at the \(x, y, z\) co-ordinate are \(q_x, q_y, q_z\) respectively.
The conduction heat rates at the opposite surfaces are

\[
\begin{align*}
q_{x+dx} &= q_x + \frac{\partial q_x}{\partial x} dx \\
q_{y+dy} &= q_y + \frac{\partial q_y}{\partial y} dy \\
q_{z+dz} &= q_z + \frac{\partial q_z}{\partial z} dz
\end{align*}
\]

within the medium the rate of thermal energy generation is

\[E_g = q dx dy dz\]

where \( q \) is rate at which energy is generated per unite volume of the medium.

In addition there may occur changes in the amount of the internal thermal energy stored by the material in a control volume, this energy storage is

\[E_s = \rho c_p \frac{\partial T}{\partial t} dx dy dz\]

Hence on rate basis, the general form of the conservation of energy requirement is

\[E_{in} + E_g - E_{out} = E_{st}\]

where \( E_{in} \) is the conduction rate of energy inflow

\[E_{out} \] is the conduction rate of energy outflow

\[
q_x + q_y + q_z + q dx dy dz - q_{x+dx} - q_{y+dy} - q_{z+dz} = \rho c_p \frac{\partial T}{\partial t} dx dy dz
\]

which gives

\[
\frac{\partial q_x}{\partial x} dx - \frac{\partial q_y}{\partial y} dy - \frac{\partial q_z}{\partial z} dz + q dx dy dz = \rho c_p \frac{\partial T}{\partial t} dx dy dz
\]

Dividing out the dimension of control volume \((dx dy dz)\) we obtained and evaluating the conduction heat rates from Fourier's law as
By deriving it similarly for cylindrical co-ordinate we obtain

\[ q_x = -kd_ydz \frac{\partial T}{\partial x} \]
\[ q_y = -kd_xdz \frac{\partial T}{\partial y} \]
\[ q_z = -kd_xdy \frac{\partial T}{\partial z} \]

We obtain

\[ \frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial T}{\partial z} \right) + q = \rho c_p \frac{\partial T}{\partial t} \]

If the flow is radial, the temperature ‘T’ is independent of \( \theta \)

\[ \frac{\partial T}{\partial \theta} = 0 \]

and neglecting the effect of internal heat generation on the temperature distribution, the equation reduces to

\[ \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \]

Where

\[ \alpha = \frac{k}{\rho c_p} \]

is thermal diffusivity

3.3 Problem definition

The initial goal of laser heat treatment is selective surface hardening. This processing involves a wide range of power densities, interaction time and transport phenomena, and deals with objects of sizes raging from nanometers to meters.

Because transformation hardening is dominantly a heat transfer process, models have justifiably concentrated on calculating temperature distribution. The
temperature is determined by solving the transient heat conduction equation. This solution is used to create several useful relations for obtaining initial estimates of the transformation hardness.

Hence this work attempts to solve the axis symmetrical cylindrical transient heat conduction equation, so the problem is summarized as finding out the analytical solution for transient conduction equation employing cylindrical co-ordinate neglecting the effect of internal heat generation.

The general governing equation of heat conduction of transient behavior in cylindrical co-ordinate is given as

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( k r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( k \frac{\partial T}{\partial \theta} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial T}{\partial z} \right) + q = \rho c_p \frac{\partial T}{\partial t}
\]

and neglecting the effect of internal heat generation on the temperature distribution, the equation reduces to

\[
\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{\alpha}{\partial t} \quad \ldots \text{(3.1)}
\]

Where \( \alpha = \frac{k}{\rho c_p} \) is thermal diffusivity

An analytical solution to the axis-symmetric heat conduction under a Gaussian energy distribution is presented which can be used to calculate the temperature distribution from a short laser pulse. The method is also used to calculate the size of a transformation hardened spot under the incident energy of a stationary laser.

### 3.4 Physical model and assumptions

Laser transformation hardening is a process of hardening very thin surface layer of the substrate to enhance the mechanical properties, leaving the interior of work piece essentially unaffected.

In the present model, the semi-infinite slab of very large size \( (x \to \infty, y \to \infty, z \to \infty) \) is considered as the work piece.
Semi-infinite solids

If one surface of a solid body with a particular temperature distribution is suddenly exposed to convection condition or it has its surface temperature changed suddenly, conduction will produce a change in temperature distribution along the thickness of the body. If this change does not reach the other side or surface of the solid under the time then the solid is modeled as semi infinite solid.

Figure 3.1 shows the schematic diagram of physical model. It is considered that the laser beam of circular cross-section of finite diameter that remains static throughout the time parameter acts as the heat source. In the present model, the semi-infinite slab of very large size \((x \rightarrow \infty, y \rightarrow \infty, z \rightarrow \infty)\) is considered as the work piece. But the geometry of heating pattern due to rotationally symmetric beam is of cylindrical type, hence considering the heat flowing in cylindrical co-ordinate, the initial and final boundary conditions are:

\[
T_{(r,z,t)} = \text{finite} \quad \text{.........................................................(a)} \]

\[
T_{(r,z,0)} = 0 \quad \text{.........................................................(b)} \]

\[
T_{(\infty,z,t)} = 0 \quad \text{.........................................................(c)} \]

\[
T_{(r,\infty,t)} = 0 \quad \text{.........................................................(d)} \]

\[
T_{(0,z,t)} = T_{\text{max}} \quad \text{.........................................................(e)} \]

\[
-k \left( \frac{\partial T}{\partial z} \right)_{(r,0,z)} = 0 \quad \text{..............for} \, r > a \quad \text{..............(f_1)} \]

\[
-k \left( \frac{\partial T}{\partial z} \right)_{(r,0,z)} = Q \quad \text{..............for} \, r \leq a \quad \text{..............(f_2)} \]

Assumptions

1. The ambient temperature is considered 0 °C.
2. Heat loss due to radiation and in any form is neglected.
Fig. 3.1 Co-ordinate system of three-dimensional solid cylinder

3. Only a single spot per a single pulse is considered.
4. The work piece is homogenous and isentropic.
5. No internal heat generation takes place.
6. No variation of temperature along angular direction takes place i.e.
   \[ T = f(r, z) \]
7. The heat source is a circle of finite diameter that remains constant within the time domain.

3.5 Solution methodology

An analytical solution to the transient heat conduction equation is based on successive integral transformation. It involves use of Laplace transformation with respect to \( t' \), because it removes time dependency. Applying Hankel transformation of order zero with respect to \( r \), removes radial dependency. This reduces equation (3.1) to homogenous partial differential equation of second order of constant co-efficient. Then, by successive inverse transformation, the solution of equation is deduced [69, 70].
3.6 Solution to heat conduction equation

Considering a substrate is subjected to a constant source of heat flux of a laser beam of finite diameter so that the conduction of heat in the layer of substrate is of cylindrical co-ordinate. Laser transformation hardening is a time dependent process hence the general governing equation of heat conduction of transient behavior in cylindrical co-ordinate is given as

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( kr \frac{\partial T}{\partial r} \right) + \frac{1}{\rho c} \left( k \frac{\partial T}{\partial \theta} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial T}{\partial z} \right) + q = 0
\]

As per assumption the general governing equation for axis-symmetrical heat conduction of transient behavior is reduces to

\[
\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}
\]  

----- (3.1)

The Laplace transformation of equation (3.1) with respect to \( t \) it reduces to

\[
\int_0^\infty e^{-\alpha t} \left( \frac{1}{\alpha} \frac{\partial T}{\partial t} \right) dt = \int_0^\infty e^{-\alpha t} \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} \right) dt
\]  

----- (3.2)

Considering L.H.S of equation (3.2)

\[
L.H.S = \left( \frac{1}{\alpha} \frac{\partial T}{\partial t} \right)
\]

\[
\int_0^\infty e^{-\alpha t} \left( \frac{1}{\alpha} \frac{\partial T}{\partial t} \right) dt
\]

\[
if \ ... T \rightarrow f(t)
\]

\[
= \int_0^\infty e^{-\alpha t} \left( \frac{1}{\alpha} \frac{\partial f(t)}{\partial t} \right) dt
\]

Integrating by parts

\[
= \frac{1}{\alpha} \left( e^{-\alpha t} \frac{\partial f(t)}{\partial t} \bigg|_0^\infty - \int_0^\infty e^{-\alpha t} \left( \frac{\partial f(t)}{\partial t} \right) dt \right)
\]

\[
= \frac{1}{\alpha} \left( e^{-\alpha f(t)} - e^{0} f(0) + \int_0^\infty e^{-\alpha f(t)} dt \right)
\]
\[ \frac{1}{\alpha} (0 - f(0) + sL(f(t))) \]
\[ \frac{1}{\alpha} (sL(f(t)) - f(0)) \]

It is only function of \( t' \) and hence according to (M.6) (Appendix-1)

\[ L \left[ \frac{1}{\alpha} \frac{\partial T}{\partial t} \right] = \frac{s}{\alpha} \phi(r, z, s) - \frac{1}{\alpha} T(r, z, 0) \]

----- (3.3)

From boundary condition (b), \( T(r, z, 0) = 0 \)

\[ \left[ \frac{1}{\alpha} \frac{\partial T}{\partial t} \right] = \frac{s}{\alpha} \phi(r, z, s) \]

----- (3.2b)

\[ \phi(r, z, s) = L[T(r, z, t)] \]

Now considering R.H.S of equation (3.2)

\[ = \int_0^\infty e^{-\alpha u} \left( \frac{\partial^2 f(t)}{\partial z^2} + \frac{\partial^2 f(t)}{\partial r^2} + \frac{1}{r} \frac{\partial f(t)}{\partial r} \right) \]
\[ = L \left[ \frac{\partial^2 f(t)}{\partial z^2} \right] + L \left[ \frac{\partial^2 f(t)}{\partial r^2} \right] + \frac{1}{r} L \left[ \frac{\partial f(t)}{\partial r} \right] \]
\[ = \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} \]

----- (3.2c)

Substituting (3.2b) \& (3.2c) in equation (3.2) the equation reduces to

\[ \frac{s}{\alpha} \phi(r, z, s) = \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} \]

----- (3.4)

Referring to (H1) (Appendix-2) and taking Hankel transformation of order zero \((Ho)\) equation (3.4) becomes

\[ H_0 \left( \frac{s}{\alpha} \phi(r, z, s) \right) = H_0 \left( \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} \right) \]
\[
H_0 \left( \frac{s}{\alpha} \phi(r,z,s) \right) = H_0 \left( \frac{\partial^2 \phi}{\partial z^2} \right) + H_0 \left( \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} \right) \quad \text{--- (3.5)}
\]

Considering the \((III)\) part of equation (3.5)

\[
= H_0 \left( \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} \right)
\]

\[
= H_0 \left\{ \frac{\partial^2}{\partial r^2} \left( \phi(r,z) \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( \phi(r,z) \right) \right\}
\]

\[
= \int_0^\infty \frac{\partial^2}{\partial r^2} \left( \phi(r,z) \right) J_0(\lambda r) dr + \int_0^\infty \frac{1}{r} \frac{\partial}{\partial r} \left( \phi(r,z) \right) J_0(\lambda r) dr
\]

\[
= \int_0^\infty \left( \frac{\partial^2}{\partial r^2} \left( \phi(r,z) \right) + \frac{\partial}{\partial r} \left( \phi(r,z) \right) \right) J_0(\lambda r) dr
\]

Integrating by parts it becomes

\[
= r J_0(\lambda r) \frac{\partial}{\partial r} \left( \phi(r,z) \right) |_0^\infty - \int_0^\infty J_0(\lambda r) \left( r \frac{\partial}{\partial r} \left( \phi(r,z) \right) \right) dr
\]

\[
= x J_0(x) \frac{\partial}{\partial r} \left( \phi(x,z) \right) - 0 J_0(0) \frac{\partial}{\partial r} \left( \phi(0,z) \right) - \int_0^\infty J_0(\lambda r) \left( r \frac{\partial}{\partial r} \left( \phi(r,z) \right) \right) dr
\]

\[
= - \int_0^\infty \frac{\partial}{\partial r} J_0(\lambda r) \left( r \frac{\partial}{\partial r} \left( \phi(r,z) \right) \right) dr
\]

Now differentiating with respect to \(\tau\) as formulae (fl) (Appendix-3)

\[
= \int_0^\infty \lambda J_1(\lambda r) \left( r \frac{\partial}{\partial r} \left( \phi(r,z) \right) \right) dr
\]
\[ (3.5a) \]

Integrating equation (3.5a) by parts, it becomes

\[ \frac{d}{dr} (\lambda r J_1(\lambda r) \psi(r, z)) = -\int_0^\infty \frac{d}{dr} (\lambda r J_1(\lambda r) \psi(r, z)) dr \]

as \( \psi(\infty, z) = 0 \) at \( r = \infty \)

\[ (3.5b) \]

Now differentiating \( r J_1(\lambda r) \) with respect to 'r' equation (3.5b) becomes

\[ (3.5c) \]

Hence according to (H1) (Appendix-2) Hankel transformation if \( \phi(r) \) is function, defined for all \( r \geq 0 \) then

\[ \Theta(\lambda) = H_0(\phi(r)) = \int_0^\infty r J_0(\lambda r) \psi(r, z) dr \]

Where \( J_0(x) \) is the Bessel function of order zero and the function \( \Theta(\lambda) \) is the Hankel transform of order zero of the original function \( \phi(r) \) denoted by \( H_0(\phi(r)) \).

Where \( r J_0(\lambda r) \) - Hankel kernel of order zero and \( \lambda \) is Hankel transformed variable of \( r \). The inverse of Hankel transform of order zero is denoted by \( H_0^{-1}(\phi(\lambda)) \) is

\[ H_0^{-1}(\phi(\lambda)) = \int_0^\infty \lambda J_0(\lambda r) \Theta(\lambda) d\lambda. \]

Hence equation (3.5c) becomes

\[ -\lambda^2 \int_0^\infty r J_0(\lambda r) \psi(r, z) dr = -\lambda^2 \Theta(\lambda, z) \]

Therefore (III) becomes i.e.
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\[ = H_0 \left( \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} \right) \]

\[ = H_0 \left( \frac{\partial^2 \phi}{\partial r^2} \left( \phi(r, z) \right) + \frac{1}{r} \frac{\partial \phi}{\partial r} \left( \phi(r, z) \right) \right) \]

\[ = -\lambda^2 \theta(\lambda, z) \]

Hence equation (3.5) reduces to (3.6)

\[ H_0 \left( \frac{s}{\alpha} \phi(r, z, s) \right) = H_0 \left( \frac{\partial^2 \phi}{\partial z^2} \right) + H_0 \left( \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} \right) \]

\[ H_0 \left( \frac{s}{\alpha} \phi(r, z, s) \right) = H_0 \left( \frac{\partial^2 \phi}{\partial z^2} \right) + H_0 \left( \frac{\partial^2 \phi}{\partial r^2} \left( \phi(r, z) \right) + \frac{1}{r} \frac{\partial \phi}{\partial r} \left( \phi(r, z) \right) \right) \]

\[ H_0 \left( \frac{s}{\alpha} \phi(r, z, s) \right) = H_0 \left( \frac{\partial^2 \phi}{\partial z^2} \right) - \lambda^2 \theta(\lambda, z) \]

Under the condition that at

\[ r = \infty, \phi(r, z, s) = 0 \quad \text{and} \quad \theta(\lambda) = H_0(\phi(r)) = \int_0^\infty J_0(\lambda r) \psi(r) dr \]

Equation (3.6) can be written as

\[ \frac{s}{\alpha} \int_0^r \phi(r, z, s) r J_0(\lambda r) dr = \int_0^\infty \left( \frac{\partial^2 \phi}{\partial z^2} \right) r J_0(\lambda r) dr - \lambda^2 \theta(\lambda, z, s) \]

\[ \frac{s}{\alpha} \theta(\lambda, z, s) = \frac{\partial^2 \theta}{\partial z^2} - \lambda^2 \theta(\lambda, z, s) \]

\[ \frac{\partial^2 \theta}{\partial z^2} \left( \lambda^2 + \frac{s}{\alpha} \right) \theta = 0 \]

Equation (3.7) is a homogeneous second order partial differential equation having real constant coefficient which is expressed as

\[ \frac{d^2 y}{dx^2} + a \frac{dy}{dx} + by = 0 \]
where $a$ and $b$ are real constants, hence the general solution to this equation is

$$y = y_1 + y_2 = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

where

$$\lambda_1 = \frac{1}{2} \left( -a + \sqrt{a^2 - 4b} \right)$$

$$\lambda_2 = \frac{1}{2} \left( -a - \sqrt{a^2 - 4b} \right)$$

Hence from equation (3.7)

$$\lambda_1 = \sqrt{\lambda^2 + \frac{s}{\alpha}} \quad \lambda_2 = -\sqrt{\lambda^2 + \frac{s}{\alpha}}$$

$$y = y_1 + y_2$$

$$\Theta(\lambda, z, s) = \Theta_1(\lambda, z, s) + \Theta_2(\lambda, z, s)$$

$$\Theta(\lambda, z, s) = A e^{\lambda_1 z} + B e^{\lambda_2 z}$$

$$\Theta(\lambda, z, s) = A e^{\sqrt{\lambda^2 + \frac{s}{\alpha}}} + B e^{-\sqrt{\lambda^2 + \frac{s}{\alpha}}} \quad \text{---- (3.8)}$$

Unknown $A$ and $B$ can be determined by applying initial and boundary conditions.

Therefore, considering boundary condition

$$T(r, \infty, t) = 0 \quad \text{................................................................. (d)}$$

Taking Laplace transformation of equation (d) with respect to $t$ it becomes

$$L(T(r, \infty, t)) = \phi(r, \infty, s)$$

$$\phi(r, \infty, s) = \int_0^\infty e^{-st} T(r, \infty, t)$$

$$\phi(r, \infty, s) = 0 \quad \text{.................................................................}$$

Taking Hankel transformation of order zero of $\phi(r, \infty, s)$ with respect to $'r'$ it becomes
\[ H_0(\phi(r, \infty, s)) = \int_0^\infty \phi(r, \infty, s) r J_0(\lambda r) dr \]

\[ H_0(\phi(r, \infty, s)) = 0 (\lambda, \infty, s) \]

\[ H_0(\phi(r, \infty, s)) = 0 \]

As \( \theta(\lambda, \infty, s) = 0 \), the equation (3.8) reduces to

\[ \theta(\lambda, z, s) = Be^{-\frac{z}{\sqrt{\lambda^2 + s^2}}} \]

Taking the inverse of Hankel transformation of equation (3.9)

\[ H_0^{-1}(\theta(\lambda, z, s)) = \phi(r, z, s) \]

\[ \therefore \phi(r, z, s) = B \int_0^\infty \lambda J_0(\lambda r) e^{\frac{-z}{\sqrt{\lambda^2 + s^2}}} d\lambda \]

Integrating part of equation (3.10) as

\[ \int_0^\infty \lambda J_0(\lambda r) e^{\frac{-z}{\sqrt{\lambda^2 + s^2}}} d\lambda \]

Let \( \lambda^2 + \frac{s}{\alpha} = t^2 \)

if \( \lambda = 0 \) then \( t = \frac{s}{\sqrt{\alpha}} \) and

if \( \lambda = \infty \) then \( t = \infty \)

Differentiating (a) with respect to \( t \) it becomes

\[ d\lambda = \frac{t}{\lambda} dt \]

Hence equation (3.10a) becomes

\[ = \int_0^\infty \lambda J_0(\lambda r) e^{\frac{-z}{\sqrt{\lambda^2 + s^2}}} d\lambda \]
\[
\frac{e^{-\alpha t}}{\sqrt{t^2 - \frac{s}{\alpha}}} \int_{\frac{s}{\alpha}}^{\infty} \frac{1}{r(y)} \, dy = \int_{\frac{s}{\alpha}}^{\infty} e^{-\alpha t} J_0(r(y)) \, dt
\]

Referring Formulae (f.0) (Appendix-3) above integration becomes

\[
\frac{e^{-\alpha t}}{\sqrt{t^2 - \frac{s}{\alpha}}} \int_{\frac{s}{\alpha}}^{\infty} \frac{1}{r(y)} \, dy = \int_{\frac{s}{\alpha}}^{\infty} e^{-\alpha t} J_0(r(y)) \, dt
\]

\[
\frac{e^{-\alpha t}}{\sqrt{t^2 - \frac{s}{\alpha}}} \int_{\frac{s}{\alpha}}^{\infty} \frac{1}{r(y)} \, dy = \frac{z}{(z^2 + r^2)} \left( \left( \frac{s}{\sqrt{\alpha}} \right) \sqrt{\frac{z^2 + r^2}{\alpha}} \right) + \frac{1}{\sqrt{z^2 + r^2}} e^{-\alpha t} J_0 \left( \sqrt{z^2 + r^2} \right)
\]

Equation (3.10) reduces to.

\[
\phi(r, z, s) = B \frac{z}{(z^2 + r^2)} \left( \left( \frac{s}{\sqrt{\alpha}} \right) \sqrt{\frac{z^2 + r^2}{\alpha}} \right) + \frac{1}{\sqrt{z^2 + r^2}} e^{-\alpha t} J_0 \left( \sqrt{z^2 + r^2} \right)
\]

Taking inverse of Laplace transformation of equation (3.11)

\[
L^{-1}(\phi(r, z, s)) = T(r, z, t)
\]

\[
T(r, z, t) = L^{-1} \left( \frac{z}{(z^2 + r^2)} \left( \left( \frac{s}{\sqrt{\alpha}} \right) \sqrt{\frac{z^2 + r^2}{\alpha}} \right) \right)
\]

\[
T(r, z, t) = L^{-1} \left( \frac{z}{(z^2 + r^2)} \left( \left( \frac{s}{\sqrt{\alpha}} \right) \sqrt{\frac{z^2 + r^2}{\alpha}} \right) \right) + \frac{1}{\sqrt{z^2 + r^2}} e^{-\alpha t} J_0 \left( \sqrt{z^2 + r^2} \right)
\]

\[
T(r, z, t) = B \frac{z}{(z^2 + r^2)} \left( L^{-1} \left( \left( \frac{s}{\sqrt{\alpha}} \right) \sqrt{\frac{z^2 + r^2}{\alpha}} \right) \right) + \frac{1}{\sqrt{z^2 + r^2}} L^{-1} \left( e^{-\alpha t} J_0 \left( \sqrt{z^2 + r^2} \right) \right)
\]

Considering term (I) of the above equation
Referring Formulae (f.2) (Appendix-3)

\[
L^{-1} \left( \frac{1}{\sqrt{\alpha}} \left( s^2 e^{-z^2/\alpha} \right) \sqrt{s} \right) = \frac{1}{\sqrt{\alpha}} \left( s^2 \alpha \right) e \left( \frac{s^2 r^2}{4 \alpha^3} \right) \left( z^2 + r^2 \right)^{1/2} H_2 \left( \frac{z^2 + r^2}{4 \alpha} \right)^{1/2}
\]

Now considering the \((II)\) term of the equation (3.12)

\[
L^{-1} \left( e^{-\frac{z^2}{\alpha} \sqrt{z^2 + r^2}} \right) = L^{-1} \left( e^{-p^2} \right) \quad \left( p = \frac{s}{\sqrt{\alpha}} \ldots \text{and} \ldots x = \sqrt{z^2 + r^2} \right)
\]

Referring Formulae (f.3) (Appendix-3)

\[
L^{-1} \left( e^{-p} \right) = \frac{1}{2} \frac{x}{\sqrt{\pi \alpha} \sqrt{3}} e^{-\frac{x^2}{4 \alpha t}}
\]

Using (a) and (b) values in equation (3.12)
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Equation (3.13) contains unknown constant 'B' that can be evaluated using the boundary condition as at \( r = 0 \) equation (3.13) reduces to

\[
T(0, z, t) = \frac{B}{2} \left( \frac{z}{2a_t} \right) \left( \frac{1}{\sqrt{\pi a_t^3}} \right) e^{\left( \frac{z^2}{4a_t} \right)}
\]  

(3.14)
The equation (3.14) reduces to

\[ T(0, z, t) = T_{\text{max}} \]

\[ \Rightarrow T(0, z, t) = T_{\text{max}} = \frac{B}{2} \left( \frac{z}{2\alpha t} \right) \left[ \frac{1}{\sqrt{\pi \alpha t^3}} \right] e^{-\left( \frac{z}{4\alpha t} \right)^2} \]

\[ \Rightarrow B = \frac{\left( \frac{4\alpha t}{z} \right) \left( \frac{\sqrt{\pi \alpha t^3}}{} \right) T_{\text{max}}}{e^{-\left( \frac{z}{4\alpha t} \right)^2}} \] \quad \text{(3.15)}

Hence equation (3.13) becomes

\[ T(r, z, t) = T_{\text{max}} \left( \frac{4\alpha t}{z} \right) \left( \frac{\sqrt{\pi \alpha t^3}}{} \right) \left[ 1 \left( \frac{z}{2\alpha t} \right) \left( \frac{1}{\sqrt{\pi \alpha t^3}} \right) e^{-\left( \frac{z}{4\alpha t} \right)^2} \right] \]

\[ T(r, z, t) = T_{\text{max}} e^{-\left( \frac{z^2}{4\alpha t} \right)} \] \quad \text{(3.16)}

The value of \( T_{\text{max}} \) in equation (3.16) is dependent on the actual nature and geometry of the heat applied source. The value of \( T_{\text{max}} \) can be determined considering the boundary condition as given in equation (f) i.e.

\[ -k \left( \frac{\partial T}{\partial z} \right)_{(r,0,t)} = 0 \quad \text{for } r > a \]

\[ -k \left( \frac{\partial T}{\partial z} \right)_{(r,0,t)} = Q_f \quad \text{for } r \leq a \]

\text{(f)}

Now considering

\[ -k \left( \frac{\partial T}{\partial z} \right)_{(r,0,t)} = Q_f \]

\[ \left( \frac{\partial T}{\partial z} \right) = -\frac{Q_f}{k} \]
Taking Laplace transformation of \( \frac{\partial T}{\partial z} \) with respect to 'z' it becomes

\[
L \left( \frac{\partial T}{\partial z} \right) = \frac{\partial \phi}{\partial z}
\]

\[
\frac{\partial \phi}{\partial z} = L \left( \frac{\partial T}{\partial z} \right) = \int_{0}^{s} \left( -\frac{Q_{f}}{k} \right) e^{-st} dt
\]

Integrating, we get

\[
\frac{\partial \phi}{\partial z} = \left( -\frac{Q_{f}}{k} \right) e^{-st} \bigg|_{0}^{s}
\]

\[
\frac{\partial \phi}{\partial z} = \left( -\frac{Q_{f}}{ks} \right)
\]

Now taking Hankel transformation for \( \frac{\partial \phi}{\partial z} \) as

\[
H_{0} \left( \frac{\partial \phi}{\partial z} \right) = \int_{0}^{\infty} \left( \frac{\partial \phi}{\partial z} \right) rJ_{0}(\lambda r) dr
\]

\[
\left( \frac{\partial \theta}{\partial z} \right) = \int_{0}^{\infty} \left( \frac{\partial \phi}{\partial z} \right) rJ_{0}(\lambda r) dr + \int_{0}^{\infty} \left( \frac{\partial \phi}{\partial z} \right) rJ_{0}(\lambda r) dr
\]

\[
\left( \frac{\partial \theta}{\partial z} \right) = \int_{0}^{\infty} \left( -\frac{Q_{f}}{ks} \right) rJ_{0}(\lambda r) dr + \int_{0}^{\infty} \left( 0 \right) rJ_{0}(\lambda r) dr
\]

\[
\left( \frac{\partial \theta}{\partial z} \right) = \int_{0}^{\infty} \left( -\frac{Q_{f}}{ks} \right) rJ_{0}(\lambda r) dr
\]

Referring formulae (f4) (Appendix-3)

\[
\int_{0}^{\gamma} J_{\gamma-1}(r) dr = X^{\gamma} J_{\gamma}(x) \quad \text{for} \ (\gamma > 0)
\]

\[
\int_{0}^{\gamma} J_{\gamma}(r) dr = XJ_{\gamma}(x) \quad \text{when} \ (\gamma = 1)
\]

Comparing equation (3.17) and (3.17a)
\[
\left( -\frac{Q_f}{k_s} \right) \int_0^a r J_0 (\lambda r) dr = \left( -\frac{Q_f}{k_s} \right) a J_1 (\lambda a) = \left( -\frac{Q_f}{k_s} \right) \frac{a}{\lambda} J_1 (\lambda a)
\]

\[
\frac{\partial \theta}{\partial z} = \left( -\frac{Q_f}{k_s} \right) \frac{a}{\lambda} J_1 (\lambda a)
\]

\[
\frac{\partial \theta}{\partial z} (\lambda, 0, s) = \left( -\frac{Q_f}{k_s} \right) \frac{a}{\lambda} J_1 (\lambda a) \tag{3.18}
\]

Considering equation (3.9) and differentiating with respect to \(z\)

\[
\frac{\partial}{\partial z} (\theta (\lambda, z, s)) = B \frac{\partial}{\partial z} \left( e^{-\sqrt{\frac{\lambda^2 + s}{\alpha}}} \right)
\]

\[
\frac{\partial}{\partial z} (\theta (\lambda, z, s)) = B \left( -\sqrt{\frac{\lambda^2 + s}{\alpha}} \right) e^{-\sqrt{\frac{\lambda^2 + s}{\alpha}}} \tag{3.9a}
\]

\[
\frac{\partial}{\partial z} (\theta (\lambda, 0, s)) = B \left( -\sqrt{\frac{\lambda^2 + s}{\alpha}} \right) \text{ (when } z = 0) \tag{3.9b}
\]

Equating equation (3.18) and (3.9b)

\[
B \left( -\sqrt{\frac{\lambda^2 + s}{\alpha}} \right) = \left( -\frac{Q_f}{k_s} \right) a \frac{J_1 (\lambda a)}{\lambda}
\]

\[
B = \left( \frac{Q_f}{k_s} \frac{a}{\lambda} J_1 (\lambda a) \right) \frac{1}{\sqrt{\lambda^2 + s}}
\]

\[
B = \left( \frac{Q_f a J_1 (\lambda a)}{k_s \lambda \sqrt{\lambda^2 + s}} \right)
\]

Substituting the value of constant \(B\) in equation (3.9)

\[
\theta (\lambda, z, s) = B e^{-\sqrt{\frac{\lambda^2 + s}{\alpha}}}
\]
\[ \theta(\lambda, z, s) = \frac{Q_f a J_1(\lambda a)}{k \lambda \sqrt{\lambda^2 + \frac{s}{a}}} e^{-\frac{\lambda z + s}{a}} \]

\[ \theta(\lambda, z, s) = Q_f a J_1(\lambda a) \left( \frac{e^{-\frac{\lambda z + s}{a}}}{k \lambda \sqrt{\lambda^2 + \frac{s}{a}}} \right) \] ---- (3.19)

Taking the inverse of Hankel transformation of equation (3.19) it becomes

\[ H_0(\theta(\lambda, z, s)) = \phi(r, z, s) = \frac{Q_f a}{k s} \int_0^\infty J_1(\lambda a) \frac{e^{-\frac{\lambda^2 z}{a}}}{\sqrt{\lambda^2 + \frac{s}{a}}} J_0(\lambda r) d\lambda \]

\[ H_0(\theta(\lambda, z, s)) = \phi(r, z, s) = \frac{Q_f a}{k s} \int_0^\infty \frac{e^{-\frac{\lambda^2 z}{a}}}{\sqrt{\lambda^2 + \frac{s}{a}}} J_1(\lambda a) J_0(\lambda r) d\lambda \]

\[ \phi(0, z, s) = \frac{Q_f a}{k s} \int_0^\infty J_1(\lambda a) d\lambda \]

Solving part of equation (3.20) as

\[ \int_0^\infty \frac{e^{-\frac{\lambda^2 z}{a}}}{\lambda \sqrt{\lambda^2 + \frac{s}{a}}} J_1(\lambda a) d\lambda \]

\[ = \int_0^\infty e^{-\frac{(\lambda + x)^2}{2\alpha}} (\beta^2 + x^2)^{-\frac{1}{2}} J_1(\alpha x) dx \] \[ \therefore \frac{s}{\alpha} = \beta^2, z = \alpha, \lambda = x \text{and} d\lambda = dx \]

Referring to formula (f.7) (Appendix-3)
Therefore equation (3.20) is
\[
\phi(0, z, s) = \frac{Q_f a}{k_s} \int_\ell \left( \frac{1}{2} \sqrt{\frac{s}{\alpha}} \left( \sqrt{z^2 + a^2} - z \right) \right) K_1 \left( \frac{1}{2} \sqrt{\frac{s}{\alpha}} \left( \sqrt{z^2 + a^2} + z \right) \right) \left( \frac{1}{2} \right) \, dz
\]

Let
\[
q = \sqrt{\frac{s}{\alpha}}, x = \frac{\sqrt{z^2 + a^2} - z}{2}, x' = \frac{\sqrt{z^2 + a^2} + z}{2}
\]

\[
\therefore x' = x, x' + x = \sqrt{z^2 + a^2}, xx' = \frac{a^2}{4}
\]

Then equation (3.21) reduces to
\[
\phi(0, z, s) = \frac{aQ_f}{k} \int_\ell \left( \frac{1}{s} I_1(qx)K_1(qx') \right) \left( \frac{1}{2} \right) \, dx
\]

Taking inverse Laplace transformation of equation (3.22) with respect to \( t \), it becomes
\[
L^{-1}(\phi(0, z, s)) = T(0, z, t) = \frac{aQ_f}{k} L^{-1}\left( \frac{1}{s} I_1(qx)K_1(qx') \right) \left( \frac{1}{2} \right)
\]

Referring formulae (f.5) (Appendix-3)
\[
T(0, z, t) = \frac{aQ_f}{k} L^{-1}\left( \frac{1}{s} I_1(qx)K_1(qx') \right)
\]
\[
= \frac{aQ_f}{k} L^{-1}\left( \frac{1}{s} \sqrt{\frac{2q}{\pi}} \frac{\sinh(qx)}{qx} \sqrt{\frac{2q}{\pi}} \frac{\pi}{2q} e^{-q^2} \right)
\]
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\begin{equation}
\frac{aQ_f}{k} L^{\omega} \left( \frac{1}{s} \sqrt{\frac{2}{\pi}} \text{Sinh}(q_f) \sqrt{\frac{\pi}{2q_f}} e^{-\omega^2} \right)
= \frac{aQ_f}{k} L^{\omega} \left( \frac{1}{s} \sqrt{\frac{2}{\pi}} \text{Sinh}(q_f) e^{-\omega^2} \right)
= \frac{aQ_f}{k} \frac{1}{\sqrt{2\pi}} L^{\omega} \left( \frac{1}{s} \text{Sinh}(q_f) e^{-\omega^2} \right)
= \frac{aQ_f}{k} \frac{1}{\sqrt{2\pi}} L^{\omega} \left( \frac{1}{s} \left( \frac{e^{\omega^2} - e^{-\omega^2}}{2} \right) e^{-\omega^2} \right)
= \frac{aQ_f}{k} \frac{1}{\sqrt{2\pi}} L^{\omega} \left( \frac{1}{s} \left( e^{-\omega^2} - e^{-\omega^2} \right) \right)
= \frac{aQ_f}{k} \frac{1}{\sqrt{2\pi}} L^{\omega} \left( \frac{e^{-\omega^2} - e^{-\omega^2}}{2} \right)
\end{equation}

\begin{equation}
T(0,z,t) = \frac{Q_f}{k} \left( L^{\omega} \left( \frac{e^{-\omega^2}}{s} \right) - L^{\omega} \left( \frac{e^{-\omega^2}}{s} \right) \right) - \frac{a^2 e^{-\omega^2}}{2s^2} - \frac{a^2 e^{-\omega^2}}{2s^2} \left( \frac{e^{-\omega^2}}{2s^2} \right) \tag{3.24}
\end{equation}

According to formulae (f.6) (Appendix-3) equation (3.24) becomes

\begin{equation}
T(0,z,t) = \frac{Q_f}{k} \left( 2 \sqrt{\frac{\alpha t}{\pi}} e^{-\frac{z^2}{4\alpha t}} - (z)/\text{erfc} \left( -\frac{z}{2\sqrt{\alpha t}} \right) - 2 \sqrt{\frac{\alpha t}{\pi}} e^{-\frac{(z^2+a^2)}{4\alpha t}} + \sqrt{z^2+a^2} \text{erfc} \left( \frac{\sqrt{z^2+a^2}}{2\sqrt{\alpha t}} \right) \right)
\end{equation}

\begin{equation}
T(0,z,t) = \frac{Q_f}{k} \left( \sqrt{\frac{4\alpha t}{\pi}} e^{-\frac{z^2}{4\alpha t}} - e^{-\frac{(z^2+a^2)}{4\alpha t}} \right) + \sqrt{z^2+a^2} \text{erfc} \left( \frac{\sqrt{z^2+a^2}}{2\sqrt{\alpha t}} \right) - (z)/\text{erfc} \left( \frac{z}{2\sqrt{\alpha t}} \right) \tag{3.25}
\end{equation}

\begin{equation}
\therefore T(0,z,t) = T_{\text{max}}
\end{equation}

\begin{equation}
\therefore T_{\text{max}} = \frac{Q_f}{k} \left( \sqrt{\frac{4\alpha t}{\pi}} e^{-\frac{z^2}{4\alpha t}} - e^{-\frac{(z^2+a^2)}{4\alpha t}} \right) + \sqrt{z^2+a^2} \text{erfc} \left( \frac{\sqrt{z^2+a^2}}{2\sqrt{\alpha t}} \right) - (z)/\text{erfc} \left( \frac{z}{2\sqrt{\alpha t}} \right) \tag{3.25}
\end{equation}

From equation (3.16), i. e.

\begin{equation}
T(r,z,t)=T_{\text{max}} e^{-\frac{r^2}{4\alpha t}}
\end{equation}

\begin{equation}
T(r,z,t)=\frac{Q_f}{k} \left( \sqrt{\frac{4\alpha t}{\pi}} e^{-\frac{z^2}{4\alpha t}} - e^{-\frac{(z^2+a^2)}{4\alpha t}} \right) + \sqrt{z^2+a^2} \text{erfc} \left( \frac{\sqrt{z^2+a^2}}{2\sqrt{\alpha t}} \right) - (z)/\text{erfc} \left( \frac{z}{2\sqrt{\alpha t}} \right) e^{-\frac{r^2}{4\alpha t}} \tag{3.26}
\end{equation}
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\[ T(r, z, t) = \frac{Q_f}{k} \left[ \left( \frac{4\alpha t}{\pi} \right) \left( e^{-r^2} - e^{-z^2} \right) + 2\sqrt{\alpha t} \left\{ x'eIrfc(x') \right\} - 2\sqrt{\alpha t} \left\{ x' erfc(x) \right\} \right] e^{-\frac{r^2}{4\alpha t}} \]

where

\[ x = \frac{z}{2\sqrt{\alpha t}}, \quad y^2 = z^2 + a^2 \quad and \quad x' = \frac{y}{2\sqrt{\alpha t}} \]

\[ T(r, z, t) = \frac{2Q_f \sqrt{\alpha t}}{k} \left[ \left( \frac{1}{2} \right) \left( e^{-r^2} - e^{-z^2} \right) + \left\{ x'eIrfc(x') \right\} - \left\{ x' erfc(x) \right\} \right] e^{-\frac{r^2}{4\alpha t}} \]

\[ T(r, z, t) = \frac{2Q_f \sqrt{\alpha t}}{k} \left[ \left( \frac{1}{2} \right) \left( e^{-r^2} - e^{-z^2} \right) + \left\{ x'eIrfc(x') \right\} - \left\{ x' erfc(x) \right\} \right] e^{-\frac{r^2}{4\alpha t}} \]

Using (f8.3) (Appendix 3)

\[ T(r, z, t) = \frac{2Q_f \sqrt{\alpha t}}{k} \left[ \left( \frac{1}{2} \right) \left( e^{-r^2} - e^{-z^2} \right) + \left\{ x'eIrfc(x') \right\} - \left\{ x' erfc(x) \right\} \right] e^{-\frac{r^2}{4\alpha t}} \]

\[ T(r, z, t) = \frac{2Q_f \sqrt{\alpha t}}{k} \left[ \left( \frac{1}{2} \right) \left( e^{-r^2} - e^{-z^2} \right) + \left\{ x'eIrfc(x') \right\} - \left\{ x' erfc(x) \right\} \right] e^{-\frac{r^2}{4\alpha t}} \]

Using (f8.4) (Appendix 3)

Substituting values of \( x \) and \( x' \)

\[ T(r, z, t) = \frac{2Q_f \sqrt{\alpha t}}{k} \left[ \left( \frac{1}{2} \right) \left( e^{-r^2} - e^{-z^2} \right) + \left\{ x'eIrfc(x') \right\} - \left\{ x' erfc(x) \right\} \right] e^{-\frac{r^2}{4\alpha t}} \]

Equation (3.27) is the final equation to calculate the temperature at the desired location.

Equation (3.27) is the final equation to calculate the temperature at the desired location.
3.6 Determination case depth of hardening

Considering the heat flowing in only one direction i.e. in ‘z’, therefore the governing equation is given as

\[
\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial z^2}
\]

---- (3.28)

Using Similarity transformation, equation (3.27) can be written as

\[
\frac{\partial T}{\partial t} \approx \frac{\nabla T}{t}
\]

and

\[
\alpha \frac{\partial^2 T}{\partial z^2} \approx \alpha \frac{\partial}{\partial z} \left( \frac{\partial T}{\partial z} \right)
\]

\[
\alpha \frac{\partial^2 T}{\partial z^2} \approx \alpha \frac{1}{\delta} \frac{\nabla T}{\delta}
\]

\[
\alpha \frac{\partial^2 T}{\partial z^2} \approx \alpha \frac{\nabla T}{\delta^2}
\]

---- (3.30)

Using equations (3.29) and (3.30) equation (3.28) can be written as

\[
\frac{\nabla T}{t} \approx \alpha \frac{\nabla T}{\delta^2}
\]

\[
\delta = \sqrt{\alpha t}
\]

---- (3.31)

Equation (3.31) determines the approximate depth of hardened zone which depends on thermal diffusivity and interaction time.

The model thus developed is used for temperature estimation at various points under process. The experiments are required to validate the theoretical results of the model. The details about these experiments are the subject matter of next chapters to follow i.e. Chapter 4 and Chapter 5.