Introduction

0.1 Motivation and survey of literature

In 1963 [32], Levine introduced semi-open sets in topological spaces. Since then, many papers were devoted to many weak forms of open sets, namely pre-open sets [34], $\alpha$-open sets [38], $\beta$-open sets [1], feebly open sets [33] etc. These open sets can be defined using some combinations of interior operators and closure operators of a topology. Császár Á. observed the similarities of these generalized open sets and pointed out that these can be defined using more generalized class of functions.

A collection $\mu$ of subsets of a set $X$ is said to form a generalized topology on $X$ if $\emptyset \in \mu$ and arbitrary union of elements in $\mu$ is again in $\mu$ and the pair $(X, \mu)$ is called the generalized topological space [9].

Császár defined a map $\gamma : P(X) \to P(X)$, from the powerset of the under-
lying set $X$ into itself possessing the property of monotonicity, i.e., $A \subseteq B \Rightarrow \gamma(A) \subseteq \gamma(B)$ for every $A, B \in P(X)$. A subset $A \subseteq X$ is $\gamma$-open [8] if and only if $A \subseteq \gamma(A)$. Then if $\tau$ is a topology on $X$ and we denote the interior of $A \subseteq X$ with respect to $\tau$ by $iA$ and the closure of $A$ by $cA$, we obtain as important particular cases the collection of all open sets ($\gamma = i$), the collection of all semi-open sets ($\gamma = ci$), the collection of all pre-open sets ($\gamma = ic$), the collection of all $\beta$-open sets ($\gamma = cic$) and the collection of all $\alpha$-open sets ($\gamma = ici$) [9]. Thus these generalized forms of open sets can be generalized to $\gamma$-open sets and the collection of all $\gamma$-open sets in $X$ constitute a generalized topology in $X$ [8].

A generalized topology need not contain $X$ and need not be closed under finite intersection. Note that every topology is a generalized topology and a generalized topology need not be a topology. Hence we get a bigger arena to explore.

Many articles have been published in the topic related to the properties of generalized topologies such as compactness, countability, separation axioms, product, quotient etc. For more details, see [13, 35, 40, 43]. Discussion on generalized topology and preorders can be seen in [24, 42]. Generalized topological spaces has applications in evolutionary theory and combinatorial chemistry [41].

In this dissertation we consider the collection of all generalized topologies on a set $X$ denoted by $LGT(X)$. Comparison of different topologies on the same basic set has been an interesting problem ever. In 1963, Garett Birkhoff, in his paper, “On the combination of topologies”, compared different topologies by ordering the family of all topologies on a given set and considering the resulting
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This dissertation comprises of 5 chapters. The introductory chapter, Chapter 0 deals with the motivation and review of literature of the study of generalized topologies and in Chapter 1 preliminary definitions and results for the development of the thesis are given.

Basakaran, Murugalingam and Sivaraj [4] proved that the family of all generalized topologies on a nonempty set is a lattice, neither distributive nor complemented. They use the notation $\mathcal{G}(X)$ for the lattice of generalized topologies on a set $X$. They proved a characterization theorem for the existence of complement of a generalized topology on a set $X$. Also the direct sum of two generalized topologies is discussed in [4] and characterized the same. As an extension of this paper we discuss in Chapter 2 some properties of $LGT(X)$ and define simple expansion of a generalized topology [2]. Simple expansion of topologies has been studied previously by many mathematicians and the similar concept can be generalized to generalized topologies.

Let $X$ be a non empty set, $\mu \in LGT(X)$ and $A$ be a subset of $X$ which does not belong to $\mu$. Then the simple expansion of $\mu$ by $A$, denoted by $\mu(A)$, is
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defined as

$$\mu(A) = \mu \cup \{G \cup A : G \in \mu\}.$$  

We can prove easily that a simple expansion of $\mu$ forms a generalized topology. Also it is obvious from the definition that $\mu(A)$ is the smallest generalized topology containing $\mu$ and $A$.

We prove several equivalent conditions for a simple expansion of generalized topology by a subset $A$ of $X$ to be an upper neighbor of the generalized topology. Using these we compare $LT(X)$ and $LGT(X)$ and we answer the following problem: given a generalized topology on $X$, when does it possess a topological upper neighbor and vice versa. We provide examples for generalized topologies which do not possess upper neighbors. Given a generalized topological space $(X, \mu)$ with a property $P$, when will a simple expansion of $(X, \mu)$ possess the same property $P$, we discuss this in the same chapter. The main result we prove in Chapter 2 is the determination of automorphism group of $LGT(X)$.

Determination of automorphism group is interesting and important in the lattice of topologies, $LT(X)$. It is proved that if $X$ contains one or two elements or $X$ is infinite, the group of automorphisms of $LT(X)$ is isomorphic to the symmetric group on $X$. If $X$ is finite and contains more than two elements, the group of automorphisms of $LT(X)$ is isomorphic to the direct product of the symmetric group on $X$ with the two element group [17, 22]. This result is important because this has the following consequence. If $X$ is infinite, then the only lattice automorphisms of $LT(X)$ are elements of permutation group.
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$S(X)$, i.e., those which permute elements of $X$. Therefore if $P$ is any topological property then a topology possessing the property $P$ can be identified from the lattice structure of $LT(X)$.

Fuzzy set theory was introduced by Zadeh in 1965 [47]. According to him a fuzzy set is defined as a class of objects with a continuum of grades of membership. He assigns a grade of membership ranging from 0 to 1. Generalizing the lattice $[0, 1]$, Gougen [20] introduced the concept of $L$-fuzzy sets, where $L$ can be a semigroup, a poset, a lattice or a boolean ring. Using fuzzy sets, Chang [7] introduced a new branch of mathematics called fuzzy topology as a generalization of ordinary topology and Heba I. M. [23] introduced fuzzy generalized topology as a generalization of generalized topology.

The $L$-fuzzy generalized topological space is defined in the following way.

Let $X$ be a nonempty ordinary set, $L$ an $F$-lattice and $\mu \subseteq L^X$. Then $\mu$ is called an $L$-fuzzy generalized topology or fuzzy generalized topology on $X$, and $(L^X, \mu)$ is called an $L$-fuzzy generalized topological space or fuzzy generalized topological space, if $\mu$ satisfies the following conditions:

1. $\emptyset \in \mu$,

2. $\forall A \subseteq \mu, \bigvee_{A \in A} A \in \mu$.

For $L = \{0, \frac{1}{2}, 1\}$, Baby Chacko [6] determined the automorphism group of the lattice $LFT(X, L)$ of all $L$-fuzzy topological spaces on $X$. Madhavan Namboothiri [37] determined the group of automorphisms of the lattice $LFT(X, L)$.
in the cases when $L$ is a finite chain and when $L$ is the diamond-type lattice. We consider the same problem in $L$-fuzzy generalized topological space.

We determine the automorphism group of lattice of fuzzy generalized topologies, $LFGT(X, L)$ on a set $X$ and when $L$ is a finite chain and $L$ is the diamond-type lattice in Chapter 3.

Many investigations have been done in the study of topological property homogeneity in topological space. John Ginsburg in his paper [19] proved a simple representation theorem for finite topological spaces which are homogeneous. In Chapter 4 we discuss homogeneity in generalized topological spaces and in $L$-fuzzy generalized topological spaces. In the first section of this chapter, we characterize completely homogeneous generalized topological spaces. In the following sections we discuss homogeneous generalized topological spaces in a cyclic ordered set and completely homogeneous $L$-fuzzy generalized topological spaces. We try to find out new homogeneous generalized topological spaces by considering the join of homogeneous generalized topologies and discusses the properties.

We conclude the thesis with Chapter 5, some unsolved problems are discussed in this chapter and a bibliography is provided.