Chapter 4

Homogeneous generalized topological spaces

4.1 Introduction

Homogeneity in topological spaces is studied by many mathematicians. John Ginsburg in his paper [19] proved a simple representation theorem for finite topological spaces which are homogeneous. In the first section we characterize completely homogeneous generalized topological spaces. In the following sections we deal with homogeneous generalized topological spaces in a cyclic ordered set. We try to find out new homogeneous generalized topological spaces by considering the join of homogeneous generalized topologies and discuss the properties.

Let $X$ be a nonempty set and $\mu$ be a generalized topology on $X$. We denote the union of all open sets in $(X, \mu)$ by $M_\mu$. Let us recall the definition of
homogeneous generalized topological space.

**Definition 4.1.1.** [18] A generalized topological space \((X, \mu)\) is said to be homogeneous if for any two points \(x, y \in M_\mu\) there exists a \((\mu, \mu)\)-homeomorphism \(f : (X, \mu) \to (X, \mu)\) such that \(f(x) = y\) and \((X, \mu)\) is called completely homogeneous if every bijection on \(X\) is a homeomorphism on \((X, \mu)\).

### 4.2 Completely homogeneous generalized topological spaces

In this section we try to characterize completely homogeneous generalized topologies and here we prove results without loss of generality for completely homogeneous strong generalized topologies only. If \(\mu\) is a generalized topology on \(X\) which is not strong, then the results we prove here still hold if we replace \(X\) by \(M_\mu\).

We use some set theoretic results throughout this section. Consider a nonempty set \(X\) and \(A\) and \(B\) are subsets of \(X\). Then there exists a bijection on \(X\), which maps \(A\) onto \(B\) if and only if \(|A| = |B|\) and \(|X \setminus A| = |X \setminus B|\). If \(X\) is an infinite set, it is possible to choose subsets \(A\) and \(B\) of \(X\) such that \(A \cup B = X, A \cap B = \emptyset\), and \(|A| = |X| = |B|\) since \(\alpha + \alpha = \alpha\) for any infinite cardinal \(\alpha\) [27].

Throughout this chapter \(X\) will denote a nonempty ordinary set unless otherwise stated.

**Examples of completely homogeneous generalized topologies.**
4.2. Completely homogeneous generalized topological spaces

1. \(\{\emptyset\}, \{\emptyset, X\}\) and \(P(X)\) on any set \(X\).

2. \(\mu = \{\emptyset, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}\}\) on \(X = \{a, b, c\}\).

3. \(\tau = \{G \subseteq X : G \text{ is infinite}\} \cup \{\emptyset\}\) is a completely homogeneous generalized topology on an infinite set \(X\).

**Lemma 4.2.1.** Let \((X, \mu)\) be a completely homogeneous generalized topological space and \(C\) be a subset of \(X\) such that \(|C| < |X|\). If \(C\) is open in \((X, \mu)\), then every subset \(B\) of \(X\) such that \(|B| = |C|\) is also open in \((X, \mu)\).

**Proof.** Let \(B \subseteq X\) and \(|B| = |C|\). Since \(|C| < |X|\), we have \(|X \setminus C| = |X \setminus B|\). Then there exists a bijection \(f\) on \(X\), which map \(C\) onto \(B\), consequently \(f\) is an open map since every bijection is a homeomorphism in a completely homogeneous generalized topological space and hence \(f(C) = B\) is open in \((X, \mu)\). \(\square\)

**Lemma 4.2.2.** Let \((X, \mu)\) be a completely homogeneous generalized topological space and let \(C \subseteq X\), \(C \neq \emptyset\), is open in \((X, \mu)\). Then supersets of \(C\) are also open in \((X, \mu)\).

**Proof.** Let \(C \subsetneq D \subseteq X\), then there exists an element \(y \in D\) and \(y \notin C\). Let \(x \in C\). Consider the bijection \(f\) on \(X\) which map \(x\) onto \(y\) and \(y\) onto \(x\) and \(f\) is the identity map on all other elements. But every bijection is a homeomorphism on \(X\) and hence \(f\) is a homeomorphism on \(X\). Since \(f\) is an open map, \(f(C) = (C \setminus \{x\}) \cup \{y\}\) is open in \((X, \mu)\). Then \(C \cup \{y\}\) is open since it is the union of two open sets, \(C \cup \{y\} = C \cup (C \setminus \{x\} \cup \{y\})\). Thus \(D\) is open since \(D\) can be written as \(D = \bigcup_{y \in D} (C \cup \{y\})\). Hence the result. \(\square\)
4.2. Completely homogeneous generalized topological spaces

Clearly the converse of previous lemma is not true. For example consider the generalized topology $\mu = \{\emptyset, \{a, b\}, \{b, c\}, \{a, b, d\}, \{b, c, d\}, \{a, b, c\}, X\}$ on the set $X = \{a, b, c, d\}$. It can be easily verified that the supersets of nonempty open sets are again open in $(X, \mu)$, but is not completely homogeneous generalized topological space.

Larson determined the completely homogeneous topologies in his paper [30]. He proved the following theorem.

**Theorem 4.2.1.** [30] The only completely homogeneous topologies on a set $X$ are:

1. The indiscrete topology
2. The discrete topology
3. Topologies of the form $\{G \subseteq X : |X \setminus G| < m\} \cup \{\emptyset\}$, where $\aleph_0 \leq m \leq |X|$.

Next is a characterization theorem for completely homogeneous generalized topological spaces with a nonempty open subset of cardinality strictly less than that of $X$.

**Theorem 4.2.2.** Let $(X, \mu)$ be a generalized topological space and $C$ be a nonempty open subset of $X$ such that $|C| < |X|$. Then $\mu$ is completely homogeneous generalized topology if and only if $\mu = \{G \subseteq X : |G| \geq m\} \cup \{\emptyset\}$ where $m < |X|$.

**Proof.** Assume $(X, \mu)$ is completely homogeneous. If $(X, \mu)$ is a topological space, then we may use the preceding theorem by Larson. We observe that the only completely homogeneous topologies on a finite set are indiscrete and discrete topologies and if $X$ is infinite, then $\mu$ is either discrete or every nonempty
open set has cardinality the same as that of $X$. Therefore, since $\mu$ contain $C$ and by Theorem 4.2.1, $\mu$ is completely homogeneous if and only if $\mu$ is $P(X) = \{G \subseteq X : |G| \geq 1\} \cup \{\emptyset\}$, if $(X, \mu)$ is a topological space.

Let $(X, \mu)$ be a completely homogeneous generalized topological space and not a topological space. Now consider the set $S = \{|G| : \emptyset \neq G \in \mu \text{ and } |G| < |X|\}$. The set $S$ is nonempty since $|C| \in S$. Let $m$ be the smallest element in $S$. Then there exists a set $D \subset X$ such that $|D| = m < |X|$ and $D$ is open in $(X, \mu)$. By Lemma 4.2.1, if $B \subseteq X$ and $|B| = |D|$, then $B$ is also open in $(X, \mu)$. Also by Lemma 4.2.2, supersets of $B$ is also open for every $B \subseteq X$ such that $|B| = |D|$. On the other hand, nonempty subsets of cardinality less than $m$ are not open. Thus $\mu$ is of the form $\{G \subseteq X : |G| \geq m\} \cup \{\emptyset\}$, where $m < |X|$. Conversely, if $\mu = \{G \subseteq X : |G| \geq m\} \cup \{\emptyset\}$ for some $m < |X|$, then it can be easily verified that $\mu$ is a completely homogeneous generalized topology on $X$. \qed

Now consider the generalized topological space in which every non empty open set has cardinality same as that of whole set. Next we enquire when does this generalized topology completely homogeneous. First we prove some Lemmas.

**Lemma 4.2.3.** Let $\mu$ be a completely homogeneous generalized topology on an infinite set $X$. Let $G$ be an open subset of $X$ with $|G| = |X|$ and $|G^c| = |X|$. Then every $H \subseteq X$ such that $|H| = |G|$ is open in $(X, \mu)$.

*Proof.* Let $H \subseteq X$ and $|H| = |G|$. Since $H$ is an infinite set, there exist disjoint subsets $A, B \subseteq H$ such that $|A| = |B| = |H|$ and $A \cup B = H$. Then $B \subseteq A^c$ and $|H| = |B| \leq |A^c| \leq |X| = |H|$. Hence $|A^c| = |H|$. But $|H| = |G| = |X| = |G^c|$ getting $|A^c| = |G^c|$. Also $|A| = |H| = |G|$ getting $|A| = |G|$. Then there exists a bijection $f$ on $X$, which map $A$ onto $G$. Since $(X, \mu)$ is a completely homogeneous
4.2. Completely homogeneous generalized topological spaces

generalized topological space, \( f \) is a homeomorphism. Consequently \( A \) is an open set since \( A = f^{-1}(G) \) and \( G \) is open. But \( H \) is a superset of \( A \). Hence by Lemma 4.2.2, \( H \) is open in \((X, \mu)\).

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**Lemma 4.2.4.** Let \( \mu \) be a completely homogeneous generalized topology on an infinite set \( X \). Let \( G \) be a subset of \( X \) with \( |G| = |X| \) and \( |G^c| < |X| \). If \( G \) is open in \((X, \mu)\), then for every \( H \subseteq X \) such that \( |H| = |G| \) and \( |H^c| \leq |G^c| \) are open in \((X, \mu)\).

**Proof.** Let \( H \) be a subset of \( X \) such that \( |H| = |G| \) and \( |H^c| \leq |G^c| \). If \( |H^c| = |G^c| \), then there exists a bijection, say \( f \), on \( X \) mapping \( G \) onto \( H \). Since every bijection is a homeomorphism, \( f(G) = H \) is open in \((X, \mu)\).

Now assume \( |H^c| < |G^c| \). Consider a subset \( A \subseteq H \) such that \( |A \cup H^c| = |G^c| \). But \( A \cup H^c = (H \setminus A)^c \). Therefore \( |(H \setminus A)^c| = |G^c| \).

**Case 1:** \( G^c \) is a finite set.

Then \( H^c \) is finite and consequently \( A \) has to be finite and since \( |H \setminus A| + |A| = |H| \), we have \( |H \setminus A| = |H| = |G| \). Thus we obtain \( |H \setminus A| = |G| \) and \( |(H \setminus A)^c| = |G^c| \).

Then there exists a bijection on \( X \) mapping \( G \) onto \( H \setminus A \) and by proceeding as earlier we get \( H \setminus A \) is open in \((X, \mu)\). But \( H \setminus A \subseteq H \), therefore by Lemma 4.2.2, \( H \) is also open in \((X, \mu)\).

**Case 2:** \( G^c \) is an infinite set.

Note that \( |(H \setminus A)^c| = |G^c| \), i.e., \( |H^c \cup A| = |G^c| \) implying \( |H^c| + |A| = |G^c| \).

Since \( |G^c| \) is infinite and \( |H^c| < |G^c| \), we have \( |A| = |G^c| \). But \( |G^c| < |X| = |H| \), resulting \( |A| < |H| \). Consider \( |H \setminus A| + |A| = |H| \), consequently \( |H \setminus A| = |H| \) since \( |H| \) is infinite. Thus we have \( |H \setminus A| = |G| \) and \( |(H \setminus A)^c| = |G^c| \) and by similar arguments as in Case 1, we can prove that \( H \) is an open subset of \( X \). Hence the proof is complete. \( \square \)
The previous lemmas enable us to prove the following characterization theorem.

The following definition is adopted from [14].

**Definition 4.2.1.** The successor of a cardinal $m$ is the least cardinal greater than $m$. A cardinal is said to be a limit cardinal if it is not the successor of a cardinal.

**Theorem 4.2.3.** Let $\mu$ be a generalized topology on an infinite set $X$ and every $\emptyset \neq G \in \mu$ has cardinality as that of $X$. Then $\mu$ is a completely homogeneous generalized topology if and only if $\mu$ is of one of the following form.

1. $\{G \subseteq X : |G| = |X|\} \cup \{\emptyset\}$.
2. $\{G \subseteq X : |G^c| \leq m\} \cup \{\emptyset\}$, where $m < |X|$.
3. $\{G \subseteq X : |G^c| < m\} \cup \{\emptyset\}$, where $m \leq |X|$ and $m$ is a limit cardinal, $m \neq 0$.

**Proof.** Let $(X, \mu)$ be a completely homogeneous generalized topological space in which every $\emptyset \neq G \in \mu$ has $|G| = |X|$.

By Lemma 4.2.3, if for some $G \in \mu$ has $|G^c| = |X|$, then every $H \subseteq X$ such that $|H| = |G|$ is open in $(X, \mu)$. In other words, $\{H \subseteq X : |H| = |X|\} \subseteq \mu$. Moreover by the assumption every nonempty open set has cardinality the same as that of $X$. Therefore $\mu = \{G \subseteq X : |G| = |X|\} \cup \{\emptyset\}$.

Now suppose for every $\emptyset \neq G \in \mu$, $|G| = |X|$ and $|G^c| < |X|$. Consider the set $F = \{|G^c| : G \in \mu, G \neq \emptyset\}$. Since $F$ is bounded by $|X|$, supremum of $F$ exists and let $m = sup F$.

**Case 1:** There exists $\emptyset \neq K \in \mu$ such that $|K^c| = m$
Then for every $\emptyset \neq G \in \mu$, $|G^c| \leq m$, i.e., $\mu \subseteq \{G \subseteq X : |G^c| \leq m\} \cup \{\emptyset\}$. Now by Lemma 4.2.4, every $G \subseteq X$ such that $|G^c| \leq |K^c|$, is also open in $(X, \mu)$. Hence $\{G \subseteq X : |G^c| \leq m\} \subseteq \mu$. Also note that here $m \neq |X|$. Hence $\mu = \{G \subseteq X : |G^c| \leq m\} \cup \{\emptyset\}$, where $m < |X|$. 

**Case 2:** For every open set $\emptyset \neq G \in \mu$, $|G^c| \neq m$ and $m \neq 0$.

For every $\emptyset \neq G \in \mu$, $|G^c| < m$, i.e., $\mu \subseteq \{G \subseteq X : |G^c| < m\} \cup \{\emptyset\}$. Now since $m = \sup F$, given any $\alpha < m$, there exists $H \in \mu$ such that $|H^c| = \alpha$. Then every set $M \subseteq X$, with $|M^c| = \alpha$, is open in $(X, \mu)$. Moreover by Lemma 4.2.4, every set $U \subseteq X$ with $|U^c| < \alpha$ is also open in $(X, \mu)$. This is true for every cardinal number $\alpha < m$. Hence $\{G \subseteq X : |G^c| < m\} \cup \{\emptyset\} \subseteq \mu$ and thus we get $\mu = \{G \subseteq X : |G^c| < m\} \cup \{\emptyset\}$, where $m \leq |X|$. 

If $m$ is not a limit cardinal then there exists a cardinal $n$ such that $m$ is the successor of $n$. Therefore $\mu$ can be written as $\mu = \{G \subseteq X : |G^c| \leq n\} \cup \{\emptyset\}$. Hence if $m$ is a limit cardinal and $m \neq 0$, then $\mu$ takes the form $\{G \subseteq X : |G^c| < m\} \cup \{\emptyset\}$.

Now the converse part of the theorem, we can easily verify that the generalized topologies listed in the theorem are completely homogeneous. Hence the proof.

To conclude this section, the completely homogeneous strong generalized topologies on an arbitrary nonempty set $X$ are listed in the following theorem.

**Theorem 4.2.4.** The completely homogeneous strong generalized topologies on an arbitrary nonempty set $X$ are

1. $\{G \subseteq X : |G| \geq m\} \cup \{\emptyset\}$, where $m \leq |X|$.
4.3. On homogeneous generalized topological spaces

2. \( \{ G \subseteq X : |G^c| < m \} \cup \{ \emptyset \} \), where \( m \leq |X| \) and \( m \) is a limit cardinal, \( m \neq 0 \).

3. \( \{ G \subseteq X : |G^c| \leq m \} \cup \{ \emptyset \} \), where \( m < |X| \).

Note that \( P(X) \) and \( \{ \emptyset, X \} \) can be obtained from (1) and (3) of the above list for \( m = 1 \) and \( m = 0 \) respectively. Also we may obtain generalized topologies of the form (3) from (2) if \( m \) is a limit cardinal and \( m \) has a successor.

4.3 On homogeneous generalized topological spaces

Here we consider a large collection of homogeneous generalized topologies on cyclically ordered set and we study the properties of the same. First let us go through the following examples.

**Example 4.3.1.** Let \( X = \{ a, b, c, d \} \). Some homogeneous generalized topologies on \( X \) are,

1. \( \{ \emptyset \} \)
2. \( \{ \emptyset, \{ a, b \}, \{ b, c \}, \{ c, d \}, \{ a, d \}, \{ a, b, c \}, \{ b, c, d \}, \{ a, d, c \}, \{ a, b, d \}, X \} \)
3. \( \{ \emptyset, \{ a, b, c \}, \{ b, c, d \}, \{ c, d, a \}, \{ d, a, b \}, X \} \)
4. \( \{ \emptyset, \{ a, b \}, \{ c, d \}, \{ a, b, c \}, \{ b, c, d \}, \{ a, d, c \}, \{ a, b, d \}, X \} \)
4.3. On homogeneous generalized topological spaces

Note that in examples 2 and 3, there is a cyclicity in the sets and many homogeneous generalized topologies are obtained in this way. A cyclically ordered set is defined as follows.

**Definition 4.3.1.** [44] A set $X$ is said to be cyclically ordered if there is a ternary relation $[a < b < c]$ on $X$ which satisfies

1. For any distinct $a, b, c \in X$ we have either $[a < b < c]$ or $[b < a < c]$, but not both.
2. $[a < b < c]$, if and only if $[b < c < a]$, if and only if $[c < a < b]$, for any $a, b, c \in X$.
3. If $[a < b < c]$ and $[a < c < d]$, then $[a < b < d]$.

A cyclic interval of length $n$ is an $n$-tuple $(x_1, x_2, \ldots, x_n)$ in which every three tuple $(x_i, x_j, x_k)$ satisfies above three axioms, for every $i < j < k$ where $i, j, k \in \{1, 2, \ldots, n\}$ and also there exists no element $x \in X$ such that $(x_{i-1}, x, x_i)$, $i = 1, 2, \ldots, n$, are in cyclic order. Let $C$ denotes a cyclic interval of length $n$, $C = [x_1 < x_2 < \ldots < x_n]$, let us denote the set $\{x_1, x_2, \ldots, x_n\}$ also by $C$ if there is no confusion. A cyclic subinterval $C'$ is a subset of $C$ which itself is a cyclic interval.

Let $X$ be a cyclically ordered set. Then two intervals $C_1$ and $C_2$ of $X$ are said to be $k$ connected if $|C_1 \cap C_2| = k$ considering $C_1$ and $C_2$ as underlying sets and the intervals $C_1$ and $C_2$ are said to be disjoint if they are disjoint as subsets of $X$ or if they are 0 connected.
4.3. On homogeneous generalized topological spaces

Lemma 4.3.1. [18] Let \((X, \mu)\) be a generalized topological space and let \(M_\mu\) denotes the union of open sets in \((X, \mu)\). Then the following are equivalent.

1. \((X, \mu)\) is homogeneous.
2. \((M_\mu, \mu)\) is homogeneous.

Definition 4.3.2. Consider a finite cyclically ordered set \(X\) and \(A\) be a cyclic interval of \(X\). Let \(A = [x_1 < x_2 < \ldots < x_n]\). For an integer \(k\) such that \(1 \leq k \leq |A|\), we define cyclic subintervals \(C'_i\)'s of \(A\), where \(C_i = [x_i < x_{i\oplus 1} < \ldots < x_{i\oplus (k-1)}] \) for \(i = 1, 2, \ldots, n\) and \(\oplus\) denotes the addition modulo \(n\). Consider the generalized topology generated by the sets \(C_1, C_2, \ldots, C_n\). Note that \(C'_i\)'s are subintervals of length \(k\), with \(k \geq 1\) and \(C_i\) and \(C_{i\oplus 1}\) are \(k - 1\) connected for every \(i = 1, 2, \ldots, n\) and let us denote this generalized topology by \(\mu_k(A)\).

We discuss the properties of the generalized topology \((X, \mu_k(A))\) in this section. We use the cycles in group theory in the proofs of some of the theorems and these cycles are different from the cyclic interval we discussed in this section.

Theorem 4.3.1. Let \(X\) be a finite cyclically ordered set and \(A \subseteq X\) be a cyclic interval of \(X\). Then \(\mu_k(A)\), for \(1 \leq k \leq |A|\), is a homogeneous generalized topology on \(X\).

Proof. Let \(A = \{x_1, x_2, \ldots, x_n\}\) and by the definition of \(\mu_k(A)\), there exist intervals \(C_1, C_2, \ldots, C_n\), where each \(C_i\), for \(i = 1, 2, \ldots, n\), is of length \(k\) and each \(C_i\) and \(C_{i\oplus 1}\) are \(k - 1\) connected, such that the sets \(B = \{C_1, C_2, \ldots, C_n\}\) generate the generalized topology \(\mu_k(A)\). Now let \(x_i, x_j \in A\) and we need a homeomorphism \(h\) on \((A, \mu_k(A))\) which map \(x_i\) onto \(x_j\). Let \(S(X)\) denotes the group of all permutations on \(X\). Consider the subgroup \(G\) of \(S(X)\) generated by the cycle
4.3. On homogeneous generalized topological spaces

\( g = (x_1 \ x_2 \ldots \ x_n) \in S(X) \). Then it is easy to verify that \( g \) and all of its powers are homeomorphisms on \( A \). Define \( h \) as \( h = g^{j-i} \) if \( i < j \) and \( g^{n-(i-j)} \) if \( i > j \). Then \( h \in G \) and \( h \) is a homeomorphism on \( A \) which map \( x_i \) onto \( x_j \). Thus \( \mu_k(A) \) is a homogeneous generalized topology on \( X \).

Given a nonempty set \( X \), we can give several cyclic order for \( X \) to obtain homogeneous generalized topologies. Considering the collection of all generalized topologies on \( X \), we saw that it form a complete lattice. Thus we can talk about the join of two generalized topologies. See the following examples.

**Example 4.3.2.** Let \( X = \{1, 2, 3, 4, 5\} \) with cyclic order \([1 < 2 < 3]\) and \([4 < 5]\). Then \( A = \{1, 2, 3\} \subseteq X \) is a cyclic interval of \( X \). Let \( k = 2 \). Then \( \mu_2(A) = \{\emptyset, \{1, 2\}, \{2, 3\}, \{3, 1\}, \{1, 2, 3\}\} \) is a homogeneous generalized topology on \( A \), in fact it is completely homogeneous.

**Example 4.3.3.** Let \( X = \{1, 2, 3, 4, 5, 6, 7\} \) and \( A = [1 < 2 < 3] \) and \( B = [3 < 4 < 5] \) are cyclic intervals with respect to two different cyclic orders on \( X \). Then \( \mu_2(A) = \{\emptyset, \{1, 2\}, \{2, 3\}, \{3, 1\}, \{1, 2, 3\}\} \) and \( \mu_2(B) = \{\emptyset, \{3, 4\}, \{4, 5\}, \{5, 3\}, \{3, 4, 5\}\} \). Even though \( \mu_2(A) \) and \( \mu_2(B) \) are homogeneous it can be easily seen that \( \mu_2(A) \lor \mu_2(B) \) is not a homogenous generalized topology on \( X \). See Theorem 4.3.2.

**Example 4.3.4.** Let \( X = \{1, 2, 3, 4, 5\} \) and \( A = [1 < 2 < 3 < 4 < 5] \) and \( B = [3 < 2 < 4 < 1 < 5] \) are cyclic intervals with respect to two different cyclic orders on \( X \). Then \( \mu_k(A) \) and \( \mu_k(B) \) are homogeneous for any \( k \) such that \( 1 \leq k \leq 5 \). See Note 4.3.1.

**Remark 4.3.1.** Minimal open sets of \( \mu_k(A) \) are the cyclic intervals in the base, namely \( C_1, C_2, \ldots, C_n \). Also if \( k = |A| - 1 \). then \( \mu_k(A) \) is a completely homogeneous generalized topology on \( A \).
Note that given any finite set $X$, we obtain several homogeneous generalized topologies on $X$ by giving some cyclic order to elements of $X$.

**Remark 4.3.2.** Homeomorphism preserves cyclic order.

Let $X$ be a cyclically ordered set and $A = [x_1 < x_2 < \ldots < x_n]$ is a cyclic interval of $X$ with generalized topology $\mu_k(A)$. Let $h$ be a homeomorphism on $(A, \mu_k(A))$. Let $C_1, C_2, \ldots, C_n$ are cyclic intervals which generate $\mu_k(A)$. By Remark 4.3.1, $\{C_1, C_2, \ldots, C_n\}$ is a collection of minimal open sets [18]. Then $\{h(C_1), h(C_2), \ldots, h(C_n)\}$ is again a collection of minimal open sets. Also $C_i$ and $C_j$ are $k - 1$ connected implies $h(C_i)$ and $h(C_j)$ are $k - 1$ connected. Hence $[h(x_1) < h(x_2) < \ldots < h(x_n)]$ is a cyclic interval, in fact $h$ map a cyclic interval of length $n$ onto a cyclic interval of same length.

**Note 4.3.1.** Thus for each cyclic interval $A \subseteq X$ and each integer $k$ such that $1 \leq k \leq |A|$, $\mu_k(A)$ is a homogeneous generalized topology on $A$ or $X$. Now fix $k$ and change the cyclic order on $A$. Let $A$ and $B$ are cyclic intervals of $X$ such that $|A| = |B|$. Then it is easy to verify that $(X, \mu_k(A))$ is homeomorphic to $(X, \mu_k(B))$. Thus varying cyclic order on $A$ can no longer make non homeomorphic homogeneous generalized topologies. But varying $k$ in $\mu_k(A)$ gives non homeomorphic homogeneous generalized topologies on $A$ or $X$. Here we try to find out new homogeneous generalized topologies.

**Theorem 4.3.2.** Let $F$ and $G$ be two disjoint cyclic intervals of a finite cyclically ordered set $X$ and consider the generalized topologies $\mu_k(F)$ and $\mu_{k'}(G)$ on $X$ where $k$ and $k'$ are integers vary in the range $1 \leq k \leq |F|$ and $1 \leq k' \leq |G|$. Then the join of $\mu_k(F)$ and $\mu_{k'}(G)$ is a homogeneous generalized topology on $X$ if and only if

1. Cardinality of $F$ and $G$ are same and
4.3. On homogeneous generalized topological spaces

2. \( k = k' \).

Proof. Let \( F = \{ x_1, x_2, \ldots, x_m \} \) and \( G = \{ y_1, y_2, \ldots, y_n \} \). Let \( B = \{ C_1, C_2, \ldots, C_m \} \) and \( B' = \{ D_1, D_2, \ldots, D_n \} \) are collections of cyclic sub intervals of \( F \) and \( G \) respectively which satisfy properties in Definition 4.3.2, where \( C'_i \)s are of length \( k \) and \( D'_j \)s are of length \( k' \) and each \( C_i \) and \( C_{i+1} \) are \( k - 1 \) connected for \( i = 1, 2, \ldots, m \). Also each \( D_j \) and \( D_{j+1} \) are \( k' - 1 \) connected for \( j = 1, 2, \ldots, n \).

Assume \( \mu = \mu_k(F) \cup \mu_{k'}(G) \) is a homogeneous generalized topology on \( X \). Let \( x \in F \) and \( y \in G \). Then there exist a homeomorphism \( h \) on \( (X, \mu) \) such that \( h(x) = y \). Since \( x \in F \), \( x \in C_i \) for some \( i \in \{1,2,\ldots,m\} \). Note that here \( \min(X, \mu) = B \cup B' \). Then \( h(C_i) \) is a minimal open set containing \( y \), since homeomorphism maps minimal open sets onto minimal open sets. Thus \( h(C_i) \in B' \) implies \( h(C_i) = D_l \) for some \( l \in \{1,2,\ldots,n\} \). Since \( h \) is a bijection \( |C_i| = |h(C_i)| = |D_l| \). But \( |D_l| = |D_j| \) for every \( j \in \{1,2,\ldots,n\} \). Hence \( k = k' \) and also \( h(\bigcup_{i=1}^m C_i) = \bigcup_{j=1}^n D_j \). Also by Remark 4.3.2, \( h \) preserves cyclic order. Thus \( h(F) \) is a cyclic interval and \( h(F) = G \), since \( F \cap G = \emptyset \), we have \( |F| = |G| \).

Next assume the converse. Then \( n = m \) and \( k = k' \). Let \( a, b \in F \cup G \).

Case 1: \( a, b \in F \). Let \( a = x_i \) and \( b = x_j \) for some \( i, j \in \{1,2,\ldots,m\} \). Define \( h = g^{j-i} \) if \( i < j \) and \( h = g^{n-(i-j)} \) if \( i > j \) where \( g = (x_1 x_2 \ldots x_n) \in S(X) \) where \( S(X) \) is the group of all permutations on \( X \). Then \( h \) is a homeomorphism on \( F \cup G \) and hence \( h \) is a homeomorphism on \( X \).

Case 2: \( a, b \in G \). Similar to Case 1.

Case 3: \( a \in F \) and \( b \in G \). Let \( a = x_i \) and \( b = y_j \) for some \( i, j \in \{1,2,\ldots,m\} \). Define \( h \) by \( h(x_{i \oplus n}) = y_{j \oplus n} \) where \( n \) is a natural number. Then it is easy to check that \( h \) is a homeomorphism on \( X \). Hence the proof is complete.

\[ \square \]

Remark 4.3.3. Above theorem can be extended to a finite number of disjoint
subintervals of $X$.

**Lemma 4.3.2.** Let $(X, \mu)$ be a homogeneous generalized topological space. Then the number of minimal open sets containing $x$ is same for every $x \in X$.

*Proof.* Let $\{U_i\}_{i \in I}$, where $I$ is an indexing set, be the collection of all minimal open sets containing $x$. Let $y \in X$ be arbitrary and $y \neq x$. Then there exists a homeomorphism $h$ on $X$ such that $h(x) = y$. Then $h(U_i)$ is a minimal open set containing $y$ for every $i \in I$. Also these are the only minimal open set containing $y$. If not, suppose $G$ is a minimal open set containing $y$ such that $G$ is not of the form $h(U_i)$. Now consider $h^{-1}(G)$, this is a minimal open set containing $x$ so $h^{-1}(G) = U_i$ for some $i$ implies $G = h(U_i)$, a contradiction to our assumption. \qed

Thus if elements of $\min(X, \mu)$ has finite cardinality, say $m$, and the number of minimal open sets containing $x$ is $k$, then we obtain the following result.

**Proposition 4.3.1.** Let $\mu$ be a homogeneous generalized topology on a finite set $X$ and for each $U \in \min(X, \mu)$ has cardinality $m$. Let $k$ denote the number of minimal open sets containing $x$. Then $m.\mid \min(X, \mu) \mid = n.k$, where $n = \mid X \mid$.

**Remark 4.3.4.** Let $A$ be a cyclic interval of a finite set $X$ and consider the generalized topology $\mu_k(A)$ where $1 \leq k \leq \mid A \mid$. Then the number of minimal open set containing $x \in A$ is $k$.

**Theorem 4.3.3.** Let $F$ and $G$ be cyclic subintervals of a finite cyclically ordered set $X$ such that $F \cap G$ is nonempty. Consider the generalized topologies $\mu_k(F)$ and $\mu_{k'}(G)$ on $X$ where $1 \leq k \leq \mid F \mid$ and $1 \leq k' \leq \mid G \mid$. If the join of $\mu_k(F)$ and $\mu_{k'}(G)$ is a homogeneous generalized topology on $X$ then $F = G$. 

93
4.3. On homogeneous generalized topological spaces

Proof. Let \( B = \{C_1, C_2, \ldots, C_m\} \) and \( B' = \{D_1, D_2, \ldots, D_n\} \) are collections of cyclic intervals of \( F \) and \( G \) respectively which satisfy properties in 4.3.2, where \( C_i \)'s are of length \( k \) and \( D_j \)'s are of length \( k' \) and each \( C_i \) and \( C_{i+1} \) are \( k - 1 \) connected for \( i = 1, 2, \ldots, m \). And each \( D_j \) and \( D_{j+1} \) are \( k' - 1 \) connected for \( j = 1, 2, \ldots, n \).

Assume that the join of \( \mu_k(F) \) and \( \mu_{k'}(G) \), say \( \mu \), is a homogeneous generalized topology on \( X \). Then \( B \cup B' \cup \{\emptyset\} \) form a base for a homogeneous generalized topology \( \mu \) on \( X \). Suppose \( F \neq G \). Let \( a \in F \). Without loss of generality let us assume that there exist an element \( b \in G \) such that \( b \notin F \). i.e., \( G \not\subset F \).

Case 1: \( k \neq k' \)
Let \( h \) be a homeomorphism on \((X, \mu)\) mapping \( a \) onto \( b \). If \( C_i \) is a minimal open set containing \( a \) for some \( i \in \{1, 2, \ldots, m\} \), then \( h(C_i) = D_j \) for some \( j \in \{1, 2, \ldots, n\} \), is a minimal open set containing \( b \). Since \( h \) is a bijection \( |C_i| = |D_j| \) and hence \( k = k' \), which is a contradiction. Similar is the case if we assume \( F \not\subset G \). Thus \( F = G \).

Case 2: \( k = k' \)
Given \( F \cap G \neq \emptyset \), choose \( c \in F \cap G \). Then By Lemma 4.3.2, number of minimal open set containing every \( x \in F \cup G \) is constant. Then the minimal open set containing \( c \) in \( B \) and \( B' \) are same, otherwise the number of minimal open set containing \( c \) is strictly greater than the number of minimal open set containing \( b \in G \), since \( b \notin F \). Thus for each element \( x \in F \cap G \), minimal open set containing \( x \) in the collection \( B \) is same as that in \( B' \). Since \( c \in F \) there exists a \( p \in \{1, 2, \ldots, m\} \) such that \( c \in C_p \in B \Rightarrow C_p \in B' \). But by Remark 4.3.4, there are exactly \( k \) minimal open sets containing \( c \) in the collection \( B \), without loss of generality let \( C_1, C_2, \ldots, C_k \) are the minimal open sets containing \( c \) which implies \( C_1, C_2, \ldots, C_k \in B' \) consequently, \( C_i \subseteq F \cap G \) for every \( i \in \{1, 2, \ldots, k\} \).
But $C_k$ and $C_{k+1}$ are $k-1$ connected implies $C_{k+1}$ is a minimal open set for all elements in $C_k \cap C_{k+1} \subseteq C_k \subseteq F \cap G$, consequently, $C_{k+1} \in B'$. Proceeding like this we get $C_i \in B'$ for every $i \in \{1, 2, \ldots, m\}$. That is $B \subseteq B'$ and hence $F \subseteq G$. Similarly we can prove that $B' \subseteq B$ implying $G \subseteq F$. Hence $F = G$.

\[ \square \]

**Remark 4.3.5.** Converse of above theorem is not true. For example, let $X = \{a,b,c,d,e,f\}$, $F = \{a,b,c,d,e\}$ with order $[a < b < c < d < e]$ and $k = 2$ and $G = \{a,b,c,d,e\}$ with order $[a < c < b < d < e]$ and $k' = 2$. Consider $\mu_2(F)$ and $\mu_2(G)$ with base $B = \{\{a,b\}, \{b,c\}, \{c,d\}, \{d,e\}, \{e,a\}, \emptyset\}$ and $B' = \{\emptyset, \{a,c\}, \{c,b\}, \{b,d\}, \{d,e\}, \{e,a\}\}$ respectively. Consider the generalized topology $\mu_2(F) \lor \mu_2(G)$, then we can not find a homeomorphism mapping $a$ onto $b$. Therefore $\mu_2(F) \lor \mu_2(G)$ is not homogeneous.

**Remark 4.3.6.** In Theorem 4.3.3, $F = G$ does not imply that cyclic orders on $F$ and $G$ are same. That is there are generalized topologies $\mu_k(F)$ and $\mu_{k'}(G)$, with $F = G$, cyclic orders on $F$ and $G$ are different and generalized topology $\mu_k(F) \lor \mu_{k'}(G)$ is homogeneous. Let $X = \{a,b,c,d,e\}$ and $F = G = \{a,b,c,d\}$. $[a < b < c < d]$ is the cyclic order in $F$ and $[a < c < b < d]$ is the cyclic order in $G$. Let $k = 2$ and $k' = 3$. Then $\mu_k(F) = \{\emptyset, \{a,b\}, \{b,c\}, \{c,d\}, \{d,a\}, \{a,b,c\}, \{b,c,d\}, \{c,d,a\}, \{a,d,b\}\}$, $F$ and $\mu_{k'}(G) = \{\emptyset, \{a,c\}, \{c,b\}, \{b,d\}, \{d,a\}, \{a,d,c\}\}$, $G$. Then $\mu_k(F) \lor \mu_{k'}(G)$ is homogeneous.
4.4 Completely homogeneous fuzzy generalized topologies

In this section we introduce the concept of homogeneous spaces and completely homogeneous spaces in fuzzy generalized topologies and discuss few properties of level generalized topologies.

Throughout this section $L$ will denote an $F$-lattice.

**Definition 4.4.1.** [23] Let $\mu_1, \mu_2$ be $L$-fuzzy generalized topologies on $X$ and $Y$ respectively. Let $f : X \rightarrow Y$. Then the function $f$ is called continuous if for every $A \in \mu_2$, $f^{-1}(A) \in \mu_1$, where $f^{-1}$ is the $L$-fuzzy reverse mapping from $L^Y$ to $L^X$ induced from $f : X \rightarrow Y$. Also $f$ is called homeomorphism if it is bijective and the induced $L$-fuzzy map, $f$ and $L$-fuzzy reverse map, $f^{-1}$ are continuous.

We introduce the concept of homogeneity in $L$-fuzzy generalized topological spaces.

**Definition 4.4.2.** Let $X$ be a nonempty set and $L$ be an $F$-lattice. Then the $L$-fuzzy generalized topological space $(L^X, \mu)$ is called homogeneous if for every pair $x, y \in X$, there exists a bijection on $X$ mapping $x$ onto $y$, which induces a homeomorphism on $(L^X, \mu)$ and $(L^X, \mu)$ is called completely homogeneous if every bijection on $X$ induces a homeomorphism on $(L^X, \mu)$.

Note that a necessary and sufficient condition for a permutation $h$ of a set $X$ to be an $L$-fuzzy homeomorphism of $(L^X, \mu)$ onto itself is that $f \in \mu$ if and only if $f \circ h \in \mu$. 

96
4.4. Completely homogeneous fuzzy generalized topologies

**Definition 4.4.3.** [47] Let $X$ be a nonempty set and $L$ be a complete lattice. Consider the $L$-fuzzy space $L^X$, for $A \in L^X$ and $a \in L$, we define $a$-level (or $a$-stratification) of $A$ as the ordinary set $\{ x \in X : A(x) \geq a \}$ denoted by $A[a]$.

**Proposition 4.4.1.** Let $L^X$ be an $L$-fuzzy space and $\mu$ be an $L$-fuzzy generalized topology on $X$. Then the set $G[a](\mu) = \{ f[a] : f \in \mu \}$, where $a \in L$ and $a \neq 0$, is a generalized topology on $X$.

*Proof.* The level set corresponds to $\emptyset \in \mu$ is $\emptyset$, therefore $\emptyset \in G[a](\mu)$. Let $\{ f_i[a] \}_{i \in I}$ be an arbitrary collection of elements in $G[a](\mu)$. Then, $\bigcup_{i \in I} f_i[a] = \bigcup_{i \in I} \{ x \in X : f_i(x) \geq a \} = \{ x \in X : \bigvee_{i \in I} f_i(x) \geq a \}$. Since $\bigvee_{i \in I} f_i \in \mu$, we have $\bigcup_{i \in I} f_i[a] \in G[a](\mu)$. Thus $G[a](\mu)$ is a generalized topology on $X$.

Let $\mu$ be an $L$-fuzzy generalized topology on $X$. Then the collection $G[a](\mu) = \{ f[a] : f \in \mu \}$ for $a \in L$ and $a \neq 0$, is called level generalized topology with respect to $a$.

**Theorem 4.4.1.** Let $L^X$ be an $L$-fuzzy space and $\mu$ be a completely homogeneous $L$-fuzzy generalized topology on $X$. Then all the level generalized topologies are also completely homogeneous.

*Proof.* Let $h$ be a bijection on $X$, since $(L^X, \mu)$ is completely homogeneous, $h$ will induce a homeomorphism on $(L^X, \mu)$. Note that $h$ is a homeomorphism of $(L^X, \mu)$, for $f \in L^X$, $h(f)$ and $h^{-1}(f)$ are in $\mu$, where $h(f)(y) = \bigvee \{ f(x) : x \in X, h(x) = y \}$ for all $y \in X$ and $h^{-1}(f)(x) = f(h(x))$. Let $a \in L$ and $G[a](\mu)$ be a level generalized topology on $X$ and let $U \in G[a](\mu)$. Then $U = f[a]$ for some $f \in \mu$. It is enough to show that $h(U) \in G[a](\mu)$ and $h^{-1}(U) \in G[a](\mu)$. Consider $h(U) = h(f[a]) = \{ h(x) : f(x) \geq a \} = \{ x \in X : f(h^{-1}(x)) \geq a \} = \{ x \in X :
4.4. Completely homogeneous fuzzy generalized topologies

\[ f \circ h^{-1}(x) \geq a \] = \{ x \in X : h(f)(x) \geq a \} = h(f)_a. \text{ But } h(f) \in \mu \text{ and thus } h(U) \in G_a(\mu).

Similarly \( h^{-1}(U) = h^{-1}(f_a) = \{ h^{-1}(x) : f(x) \geq a \} = \{ x \in X : f(h(x)) \geq a \} = (h^{-1}(f))_a(\mu). \) Since \( h^{-1}(f) \in \mu, \) we have \( h^{-1}(U) \in G_a(\mu). \) Thus \( h \) is a homeomorphism on \( (X, G_a(\mu)). \) Since \( h \) and \( G_a \) are arbitrary, all level generalized topologies are completely homogeneous. \( \square \)

Remark 4.4.1. Converse of Theorem 4.4.1 is not true. For example, consider the set \( X = \{a, b, c\} \) and \( L = \{0, \frac{1}{2}, 1\} \) with usual order and \( 0' = 1, 1' = 0 \) and \( (\frac{1}{2})' = \frac{1}{2}. \) Then \( L^X \) is an \( L \)-fuzzy space and consider the \( L \)-fuzzy generalized topology \( \mu \) having base \( B = \{0, a_1, b_1, c_1, f\} \) where \( f(a) = \frac{1}{2}, f(b) = \frac{1}{2} \) and \( f(c) = 1. \) Then \( G_{\frac{1}{2}}(\mu) \) and \( G_{\frac{1}{2}}(\mu) \) are \( P(X) \) which is obviously completely homogeneous. But as you see here \( \mu \) is not completely homogeneous.