2 Nonlinear generalization of
damped harmonic oscillator:
nonstandard conserved
Hamiltonian structures

2.1 Introduction

In the introductory chapter we discussed some of the important methods available in the literature to investigate/study the integrability of ordinary differential equations (ODEs). In particular, we discussed in some detail the recently developed modified Prelle-Singer (PS) procedure for the second order ODEs. The application of this modified PS procedure has lead to the discovery of many interesting results in the field of nonlinear ODEs. One such interesting result is the identification of conservative Hamiltonian structure for a typical dissipative system, namely the damped harmonic oscillator. The Hamiltonian obtained through this procedure is found to be not of the standard Hamiltonian form [87, 90, 93], which is written as the sum of the kinetic and potential energies. The problem of quantizing dissipative systems such as the damped harmonic oscillator is a long standing one [94] and the identification of conservative Hamiltonian structure for the damped harmonic oscillator can lead to possible quantization [85]. In a parallel study the modified PS proce-
2.2 Hamiltonian structure of DHO and MEE

In this section we briefly recall the Hamiltonian dynamics associated with the two dissipative systems, namely the damped harmonic oscillator and the modified Emden type equation, and then show how they are interrelated. Note that the former one is a linear system and latter one is a nonlinear system. One may essentially consider the transformation to be introduced as a linearizing transformation of the latter.

2.2.1 Damped harmonic oscillator

To start with let us consider the damped harmonic oscillator (using a different notation for convenience of comparison)

\[ y'' + \alpha y' + \lambda y = 0, \quad \left( \dot{r} = \frac{d}{d\tau} \right) \]  

(2.1)
where $\alpha$ and $\lambda$ are arbitrary parameters. Recently, the following time independent integral of motion for the system (2.1) has been identified [85] using the modified PS procedure,

$$I = \begin{cases} 
\frac{(r-1)}{r-2} \left( y' + \frac{\alpha y}{r} \right) \left( y + \frac{(r-1)}{r} \alpha y \right)^{(1-r)}, & \alpha^2 > 4\lambda \\
\frac{y'}{(y'+\frac{1}{2}\alpha y)} - \log[y'+\frac{1}{2}\alpha y], & \alpha^2 = 4\lambda \\
\frac{1}{2} \log[y^2 + \alpha yy' + \lambda y^2] + \frac{\alpha}{2\omega} \tan^{-1} \left[ \frac{\alpha y' + 2\lambda y}{2\omega} \right], & \alpha^2 < 4\lambda \\
y' + \alpha y, & \lambda = 0,
\end{cases} \tag{2.2}$$

where $r = \frac{\alpha}{2\lambda}(\alpha \pm \sqrt{\alpha^2 - 4\lambda})$ and $\omega = \frac{1}{2}\sqrt{4\lambda - \alpha^2}$, for the overdamped, critically damped and the underdamped oscillations, respectively. Using the time independent integral of motion and the relation $L = p\dot{x} - H$, the following time independent Lagrangian and Hamiltonian are obtained [85] as

$$L = \begin{cases} 
\left( y' + \frac{(r-1)}{r} \alpha y \right)^{(2-r)}, & \alpha^2 > 4\lambda \\
\log \left[ y' + \frac{1}{2} \alpha y \right], & \alpha^2 = 4\lambda \\
\frac{1}{2\omega} \left( \alpha \tan^{-1} \left[ \frac{\alpha y' + 2\lambda y}{2\omega y'} \right] - \frac{2y'}{y} \tan^{-1} \left[ \frac{\alpha y + 2y'}{2\omega y} \right] \right) + \frac{1}{2} \log \left[ y^2 + \alpha yy' + \lambda y^2 \right], & \alpha^2 < 4\lambda \\
y' \log(y') - y' - \alpha y, & \lambda = 0 \tag{2.3}
\end{cases}$$
and

\[ H = \begin{cases} 
\frac{(r - 1)}{(r - 2)} \left( \frac{r - 2}{p} \right)^{r-2} - \frac{(r - 1)}{r} \alpha py, & p > 0, \alpha^2 > 4\lambda \\
\log(p) - \frac{1}{2} \alpha py, & p > 0, \alpha^2 = 4\lambda \\
\frac{1}{2} \log|y^2 \sec^2(\omega py)| - \frac{\alpha}{2} py, & \alpha^2 < 4\lambda \\
e^p + \alpha y, & \lambda = 0
\end{cases} \] (2.4)

where the canonically conjugate momentum \( p \) is defined by

\[ p = \begin{cases} 
\left( y' + \frac{r - 1}{r} \alpha y \right)^{(1-r)}, & \alpha^2 \geq 4\lambda \\
\frac{1}{\omega y} \tan^{-1} \left[ \frac{2y' + \alpha y}{2\omega y} \right], & \alpha^2 < 4\lambda \\
p = y', & \lambda = 0
\end{cases} \] (2.5)

From the Hamiltonian (2.4)-(2.5) one can straightforwardly write down the canonical equations of motion for all the parametric choices. Integrating the resultant system of first order equations we obtain the known exact solutions straightforwardly. Note that no multivaluedness arises in the system due to the constraints on the momentum.

2.2.1.A General solution and Dynamics

Case 1

Let us consider the Hamiltonian of the simple case of purely damped system
Figure 2.1: Phase plane portrait of the purely damped system $y'' + \alpha y = 0$.

$(\lambda = 0)$. The canonical equations read

$$y' = e^p, \quad p' = -\alpha. \quad (2.6)$$

Rewriting the above equations as a second order equation in $y$ we get back the original equation $y'' + \alpha y = 0$. We note here that the volume preservation condition, the divergence of the flow function vanishing,

$$\Lambda = \frac{\partial f_1}{\partial y} + \frac{\partial f_2}{\partial p} = 0, \quad (2.7)$$

is indeed satisfied. We also note that the planar system (2.6) does not admit any critical point. One can trivially integrate the canonical equations (2.6) to get the known general solution

$$y(t) = C_0 - \frac{1}{\alpha} e^{-\alpha(t-t_0)}. \quad (2.8)$$

The phase plane portrait of the purely damped system (2.6) is shown in Fig. 2.1.

**Case 2.3** $\alpha^2 \geq 4\lambda$
Figure 2.2: Phase plane portrait of the over damped (a) and the critically damped (b) harmonic oscillator.

Substituting the corresponding Hamiltonians from (2.4) into the canonical equations $y' = \frac{\partial H}{\partial p}$, $p' = -\frac{\partial H}{\partial y}$, one gets the following equivalent system of first order ordinary differential equations for the damped harmonic oscillator in the overdamped and critically damped ($\alpha^2 > 4\lambda$ and $\alpha^2 = 4\lambda$) parametric regimes,

$$y' = (p)^{\frac{1}{\alpha - 1}} - \frac{(r - 1)}{r} \alpha y, \quad p' = \frac{(r - 1)}{r} \alpha p. \quad (2.9)$$

One can easily check that the second order equivalence of this equation coincides exactly with (2.1) and that the standard form of the solutions for the overdamped and the critically damped cases follow naturally by integrating the above system of first order ODEs. The phase plane portrait of the overdamped and the critically damped cases are shown in Fig. 2.2.

We mention here that the divergence of the flow function of the underlying canonical equations of motion is zero, $\Lambda = \frac{\partial f_1}{\partial y} + \frac{\partial f_2}{\partial p} = 0$, which in turn confirms that the damped harmonic oscillator has a non-standard conservative Hamiltonian description.

**Case 4 $\alpha^2 < 4\lambda$**

Finally, for the underdamped case, one can write down the Hamilton equations
Figure 2.3: Phase plane portrait of the underdamped harmonic oscillator.

of motion in the form

\[ y' = \frac{1}{2} y(2\omega \tan(\omega y) - \alpha), \quad p' = \frac{1}{2} p(\alpha - 2\omega \tan(\omega y)) - \frac{1}{y}. \quad (2.10) \]

Again rewriting this equation as a single second order equation in the variable \( y \) one ends up in the form (2.1). The divergence of the flow function in this case is also zero. Integrating the canonical equations (2.10) we arrive at the known general solution. The phase plane portrait of the underdamped harmonic oscillator is shown in Fig. 2.3. The existence of time independent Hamiltonians (2.4) in a non-standard form and the basic definition of the divergence of flow function lead us to the conclusion that the damped harmonic oscillator has a non-standard conservative Hamiltonian description as well, in spite of its known dissipative nature.

### 2.2.2 Modified Emden type equation

In a parallel investigation it was found that the MEE,

\[ \ddot{x} + \alpha x \dot{x} + \beta x^3 = 0, \quad (2.11) \]
admits a time independent conservative Hamiltonian description \[86\]. The MEE, which is also known as the Painlevé - Ince equation, is an extensively studied and physically significant equation in the contemporary nonlinear dynamics literature \[48, 95–99\]. On the other hand the MEE with linear external forcing,

\[ \ddot{x} + \alpha \dot{x} + \beta x^3 + \gamma x = 0, \]  

(2.12)

possesses certain unusual nonlinear dynamical properties for the parametric choice \( \beta = \frac{a^2}{b} \). For this parametric choice, equation (2.12) is found to admit the following general solution

\[ x(t) = \frac{A \sin(\omega t + \delta)}{1 - \left( \frac{k}{\omega} \right) A \cos(\omega t + \delta)}, \quad 0 \leq A < \frac{3\omega}{k}, \quad \omega = \sqrt{\gamma}. \]  

(2.13)

This system is known to admit nonisolated periodic orbits of conservative Hamiltonian type \[93\], which is shown in Fig. 2.4. These periodic orbits exhibit the unexpected property that the frequency of oscillations is completely independent of amplitude and continues to remain as that of the linear harmonic oscillator \[93\]. Such systems are also known as isochronous systems.

The more general system (2.12) with \( \alpha, \beta \) and \( \gamma \) being arbitrary parameters also admits the following time independent integrals of motion \[100\],

\[
I = \left\{ \begin{array}{ll}
\frac{r-1}{(r-2)} \left( \frac{\alpha x^2}{2r} + \frac{r \gamma}{\alpha (r-1)} \right) \left( \frac{\alpha (r-1)}{2r} x^2 + \frac{r \gamma}{\alpha} \right)^{(1-r)}, & \alpha^2 > 8\beta \\
\frac{\dot{x}}{(\dot{x} + \frac{1}{4} \alpha (x^2 + \frac{4\gamma}{\alpha^2})^{-1}} - \log[\dot{x} + \frac{1}{4} \alpha (x^2 + \frac{4\gamma}{\alpha^2})], & \alpha^2 = 8\beta \\
\frac{1}{2} \log[\dot{x}^2 + \frac{\alpha}{2} \dot{x} (x^2 + \frac{\gamma}{\beta}) + \frac{\beta}{2} (x^2 + \frac{\gamma}{\beta})^2] + \frac{\alpha}{2\omega} \tan^{-1} \left[ \frac{\alpha \dot{x} + 2\beta (x^2 + \frac{\gamma}{\beta})}{2\omega \dot{x}} \right], & \alpha^2 < 8\beta,
\end{array} \right.
\]

(2.14)
Figure 2.4: Solution and phase space plots of equation (2.12) for the case $\beta = \frac{\alpha^2}{\beta}$, $\gamma < 0$: (a) periodic oscillations and (b) phase space portrait

where $r = \frac{\alpha}{4\beta}(\alpha \pm \sqrt{\alpha^2 - 8\beta})$ and $\omega = \frac{1}{2}\sqrt{8\beta - \alpha^2}$. From the time independent integrals we have identified the following Lagrangian,

$$L = \begin{cases} 
\left(\frac{\dot{x}}{2r}(\alpha x^2 + \frac{\gamma}{\beta})\right)^{(2-r)}, & \alpha^2 > 8\beta \\
\log\left(\frac{\dot{x} + \frac{\alpha}{4}(x^2 + \frac{\beta}{\gamma})}{\sqrt{2}}\right), & \alpha^2 = 8\beta \\
\frac{1}{\omega} \left(\frac{1}{\omega x^2} \left(\frac{4\ddot{x} + \alpha x^2}{x}\right) - \alpha \tan^{-1}\left(\frac{\alpha \dot{x} + 2\beta x^2}{\omega x}\right)\right) - \frac{1}{2} \log(2x^2 + \alpha x^4), & \alpha^2 < 8\beta
\end{cases} (2.15)$$

and Hamiltonian for the system (2.12),
\[ H = \begin{cases} \frac{(r - 1)}{(r - 2)} \frac{(r - 2)}{(r - 1)} (p) \frac{(r - 2)}{(r - 1)} - \frac{(r - 1)}{2r} \alpha p x^2 - \frac{pr \gamma}{\alpha}, & \alpha^2 > 8\beta \\ \log(p) - \frac{1}{2} \alpha p \left( \frac{x^2}{2} + \frac{4\gamma}{\alpha^2} \right), & \alpha^2 = 8\beta \\ \frac{1}{2} \log \left( \frac{x^2 + \frac{\gamma}{\beta}}{2} \right) \sec^2 \left( \frac{\omega p}{2} \left( x^2 + \frac{\gamma}{\beta} \right) \right) - \frac{\alpha}{4} p \left( x^2 + \frac{\gamma}{\beta} \right), & \alpha^2 < 8\beta \end{cases} \] (2.16)

where the canonically conjugate momentum \( p \) is defined by

\[ p = \begin{cases} \left( \dot{x} + \frac{(r - 1)}{2r} \alpha x^2 + \frac{r \gamma}{\alpha} \right)^{(1-r)}, & \alpha^2 \geq 8\beta \\ \frac{2}{\omega \left( x^2 + \frac{\gamma}{\beta} \right)} \tan^{-1} \left[ \frac{4\dot{x} + \alpha \left( x^2 + \frac{\gamma}{\beta} \right)}{2\omega \left( x^2 + \frac{\gamma}{\beta} \right)} \right], & \alpha^2 < 8\beta \end{cases} \] (2.17)

The Hamilton equations of motion follow straightforwardly from (2.16) as

**Cases 1 & 2**

\[ \dot{x} = p^{1-r} - \frac{\alpha (r - 1) x^2}{2r} - \frac{r \gamma}{\alpha}, \quad \dot{p} = \frac{\alpha (r - 1) px}{r} \].

**Case 3**

\[ \dot{x} = \frac{1}{4\beta} \left( \beta x^2 + \gamma \right) \left( 2\omega \tan \left[ \frac{1}{2} p \omega \left( x^2 + \frac{\gamma}{\beta} \right) \right] - \alpha \right), \quad \dot{p} = \frac{x (p \alpha (\beta x^2 + \gamma) - 4\beta - 2 p \omega (\beta x^2 + \gamma) \tan \left[ \frac{1}{2} p \omega (x^2 + \frac{\gamma}{\beta}) \right])}{2(\beta x^2 + \gamma)}. \] (2.19)

Note that in all the above three cases, the divergence of the flow function \( \Lambda \) is zero, corresponding to a conservative Hamiltonian description. However, unlike the damped harmonic oscillator one cannot integrate the canonical equations (2.18) and (2.19) and obtain the solutions straightforwardly. To overcome this difficulty one should introduce suitable canonical transformations and change the Hamilton equations (2.18) and (2.19) into simpler forms so that they can be integrated.
2.2.3 Transformation connecting DHO and MEE

By comparing the structure of the integral of motion, Hamiltonian function and canonical equations of the damped harmonic oscillator with that of the equation (2.12), namely equation (2.2) with (2.14), (2.4)-(2.5) with (2.16)-(2.17), one can identify that the two systems are transformed into each other through the nonlocal transformation

\[ y = \frac{x^2}{2} + \frac{\gamma}{\lambda}, \quad d\tau = x dt, \]  

(2.20)

with the identification \( \lambda = 2\beta \). One can also check directly that the above transformation also maps the damped harmonic oscillator equation (2.1) onto the MEE with forcing, equation (2.12), and vice versa. Consequently one can treat the transformation as a linearizing transformation (albeit nonlocal) of the nonlinear equation (2.12).

Note that the transformation (2.20) is not the only possible linearizing transformation at least for specific parametric choices. We mention here that one can also transform the nonlinear system (2.12) with \( \beta = \frac{\alpha^2}{\gamma} \) into a linear harmonic oscillator equation \( (\ddot{U} + \gamma U = 0) \) by introducing another nonlocal transformation of the form [88, 90]

\[ U = xe^{\int_0^\tau \frac{\alpha}{\gamma} d\tau}, \quad \tau = t. \]  

(2.21)

The nonlocal transformation (2.21) is different from (2.20) in the following respect. In (2.20) the nonlocality is introduced in the independent variable whereas in (2.21) the nonlocality is introduced in the dependent variable. We may also add that the nonlocal transformation (2.20) has some similarity with the well known Kustaanheimo-Stiefel transformation used in atomic physics [101]. Even though both the nonlocal transformations (2.20) and (2.21) map the nonlinear equation into a linear one and vice versa, the nonlocal transfor-
mation of the type (2.20) is much useful in identifying the Hamiltonian structures associated with the nonlinear system whereas the nonlocal transformation of the type (2.21) is more useful in constructing general solution for the transformed nonlinear system. For more details about the nonlocal transformation of the type (2.21) we refer to Chapter 4 and Chapter 5. In the following we confine our attention to the nonlocal transformation of the form (2.20) and its generalization.

2.3 A general class of nonlinear damped oscillator: Hamiltonian description

We find that the transformation (2.20) is a specific case of a rather general nonlocal transformation of the form

\[ y = \int f(x)dx, \quad d\tau = \frac{f(x)}{g(x)}dt. \tag{2.22} \]

For example, restricting \( f(x) = x \) and \( g(x) = 1 \) in (2.22) one gets exactly (2.20).

The nonlocal transformation (2.22) modifies the damped harmonic oscillator equation (2.1) to the general class of nonlinear oscillators of the form,

\[ \ddot{x} + \frac{g'(x)}{g(x)} \dot{x}^2 + \alpha \frac{f(x)}{g(x)} \dot{x} + \lambda \frac{f(x)}{g(x)^2} \int f(x)dx = 0, \tag{2.23} \]

where \( f(x), g(x) \) are arbitrary function of \( x \).

Applying the above nonlocal transformation (2.22) to the damped harmonic oscillator equation (2.1) we obtain the following time-independent integral of motion for the nonlinear system (2.23), that is.

Case 1. \( \alpha^2 > 4\lambda \)

\[ I = \frac{(r-1)}{(r-2)} \left( g(x)\dot{x} + \frac{\alpha}{r} \int f(x)dx \right) \left( g(x)\dot{x} + \frac{(r-1)}{r} \alpha \int f(x)dx \right)^{(1-r)} \tag{2.24} \]
2.3 A general class of nonlinear damped oscillator: Hamiltonian description

Case 2. $\alpha^2 = 4\lambda$

\[
I = \frac{g(x)\dot{x}}{g(x)\dot{x} + \frac{1}{2} \alpha \int f(x)dx} - \log[g(x)\dot{x} + \frac{1}{2} \alpha \int f(x)dx] \tag{2.25}
\]

Case 3. $\alpha^2 < 4\lambda$

\[
I = \frac{1}{2} \log[g(x)^2\dot{x}^2 + \alpha g(x)\dot{x} \int f(x)dx + \lambda \int f(x)dx^2] \\
+ \frac{\alpha}{2\omega} \tan^{-1} \left[ \frac{\alpha g(x)\dot{x} + 2\lambda \int f(x)dx}{2\omega g(x)\dot{x}} \right], \tag{2.26}
\]

where $\omega = \frac{1}{2} \sqrt{4\lambda - \alpha^2}$ and $r = \frac{\alpha}{2\omega}(\alpha \pm \sqrt{\alpha^2 - 4\lambda})$.

Now applying the nonlocal transformation (2.22) to the above Lagrangian (2.3), one can readily obtain the Lagrangian associated with the general nonlinear damped oscillator equation (2.23) as

\[
L = \begin{cases} 
\left( g(x)\dot{x} + \frac{(r-1)}{r} \alpha \int f(x)dx \right)^{2-r}, & \alpha^2 > 4\lambda \\
\log \left[ g(x)\dot{x} + \frac{1}{2} \alpha \int f(x)dx \right], & \alpha^2 = 4\lambda \\
\frac{\alpha}{2\omega} \tan^{-1} \left[ \frac{\alpha g(x)\dot{x} + 2\lambda \int f(x)dx}{2\omega g(x)\dot{x}} \right] \\
- \frac{g(x)\dot{x}}{\omega \int f(x)dx} \tan^{-1} \left[ \frac{\alpha \int f(x)dx + 2g(x)\dot{x}}{2\omega \int f(x)dx} \right] \\
\frac{1}{2} \log \left[ g(x)^2\dot{x}^2 + \alpha g(x)\dot{x} \int f(x)dx + \lambda \left( \int f(x)dx \right)^2 \right], & \alpha^2 < 4\lambda.
\end{cases} \tag{2.27}
\]

It may be noted that all the above Lagrangians are of non-standard type, that is of the forms which cannot be written in the standard form as ‘kinetic energy’ minus ‘potential energy’. In particular, one can readily check that the non-standard Lagrangian

\[
L = \frac{1}{\dot{x} + \frac{2}{3} \int b(x)dx}, \tag{2.28}
\]
deduced by Musielak [87] for the nonlinear ODE

\[ \ddot{x} + b(x)\dot{x} + \frac{2}{9}b(x) \int b(x)dx = 0, \] (2.29)

follows from the above general form (2.27) by choosing \( f(x) = b(x), \ g(x) = 1, \ \alpha = 1 \) and \( \lambda = \frac{2}{9} \) in equation (2.23). Note that the above equation (2.29) itself includes the MEE discussed in Refs. [90, 93] as a special case with \( b(x) = kx \).

The Hamiltonian for the equation (2.23) can also be constructed by simply substituting the transformation (2.22) into (2.4) so that the latter reads

\[ H = \begin{cases} \frac{(r-1)}{(r-2)} \left( \frac{p}{g(x)} \right)^{r-2} - \frac{\alpha(r-1)}{r} \frac{p}{g(x)} \int f(x)dx, & \alpha^2 > 4\lambda \\ \frac{\alpha}{2g(x)} p \int f(x)dx + \log \left[ \frac{g(x)}{p} \right], & \alpha^2 = 4\lambda \\ \frac{1}{2} \log \left[ \left( \int f(x)dx \right)^2 \sec^2 \left( \frac{\omega p}{g(x)} \int f(x)dx \right) \right] - \frac{\alpha p}{2g(x)} \int f(x)dx, & \alpha^2 < 4\lambda, \end{cases} \] (2.30)

where \( p \) is the canonically conjugate momentum defined by

\[ p = \begin{cases} \frac{g(x)}{\left[ g(x)\dot{x} + \frac{(r-1)}{r} \alpha \int f(x)dx \right]^{r-1}}, & \alpha^2 \geq 4\lambda \\ \frac{g(x)}{\omega \int f(x)dx} \tan^{-1} \left[ \frac{2g(x)\dot{x} + \alpha \int f(x)dx}{2\omega \int f(x)dx} \right], & \alpha^2 < 4\lambda. \end{cases} \] (2.31)

One can easily check that the canonical equations of motion take the form

\[ \begin{align*}
\alpha^2 \geq 4\lambda : \quad & \ddot{x} = \frac{1}{g(x)} \left[ \left( \frac{p}{g(x)} \right)^{\frac{1}{r-1}} - \frac{(r-1)}{r} \alpha \int f(x)dx \right], \\
\dot{p} = & \frac{(r-1)}{r} \alpha p f(x) + \frac{p g'(x)}{g(x)^2} \left[ \left( \frac{p}{g(x)} \right)^{\frac{1}{r-1}} - \frac{\alpha(r-1)}{r} \int f(x)dx \right]. \end{align*} \] (2.32)
\( \alpha^2 < 4\lambda \quad : \quad \dot{x} = \frac{\int f(x)dx}{2g(x)} \left[ 2\omega \tan \left( \frac{p\omega}{g(x)} \int f(x)dx \right) - \alpha \right], \)

\[ \dot{p} = \frac{1}{2g(x)^2 \int f(x)dx} \left[ f(x)g(x) \left( 2g(x) - \left( \alpha - 2\omega \tan \left( \frac{p\omega}{g(x)} \int f(x)dx \right) \right) \right] \]

\[ \times p \int f(x)dx \right) + p \left( \int f(x)dx \right)^2 \left( \alpha - 2\omega \tan \left( \frac{p\omega}{g(x)} \int f(x)dx \right) \right) g'(x). \quad (2.33) \]

One can straightforwardly check that the second order equivalence of equations (2.32) and (2.33) coincides exactly with (2.23) in the appropriate parametric regimes. Further, we find that the flow function for the canonical equations of motion is zero in all the three parametric regions, equations (2.32)-(2.33), which in turn confirms that (2.23) or (2.32)-(2.33) has a Hamiltonian description. Further, the integrability of (2.23) is automatically ensured by the existence of the time independent Hamiltonian (2.30) in the Liouville sense.

We also note here that equation (2.23) can be transformed into an Abel equation of the second kind [102].

\[ \omega w' + \frac{g'(x)}{g(x)} w^2 + \alpha \frac{f(x)}{g(x)} w + \lambda \frac{f(x)}{g(x)^2} \int f(x)dx = 0, \quad (2.34) \]

through the transformation \( w(x) = \dot{x} \). For the choice \( g(x) = \text{constant} \), one can deduce the time independent first integrals of (2.34) using the transformation [103].

\[ w = \frac{\lambda \xi}{\alpha \xi} \int f(x)dx. \quad (2.35) \]

Using the above transformation, equation (2.34) reduces to

\[ \xi' = \frac{f(x)}{\int f(x)dx} \left( \frac{\alpha^2}{\lambda \xi^2} + \frac{\alpha^2}{\lambda \xi} + 1 \right) \xi. \quad (2.36) \]

Integrating the above relation we get

\[ I \equiv \int \left( \frac{\alpha^2}{\lambda \xi^2} + \frac{\alpha^2}{\lambda \xi} + 1 \right) \xi = \log(\int f(x)dx). \quad (2.37) \]
Using the relation $\xi = \frac{1}{a^2} \int f(x)dx$, one can rewrite the above integral of motion in terms of $x$ and $\dot{x}$ and in the case where $\int f(x)dx$ is an invertible function of $x$, then the solution will lead to quadratures in terms of certain complicated integrals which however cannot be evaluated in general [104].

Therefore one should adopt a different procedure to construct the general solution of (2.23). The effective way to proceed further is to transform the Hamiltonian (2.30) to a simpler form through suitable canonical transformations as demonstrated for the case of the MEE in [86]. However, to adopt this procedure one should specify the explicit forms of $f(x)$ and $g(x)$. We demonstrate the procedure with a specific example in the following section.

### 2.4 Generalized MEE: Hamiltonian structure and general solution

To illustrate the ideas given in section 3, we focus our attention on the case $f(x) = x^q$, $g(x) = 1$ so that equation (2.23) now becomes

$$\ddot{x} + \alpha x^q \dot{x} + \beta x^{2q+1} = 0, \quad (2.38)$$

where $\beta = \frac{\lambda}{(q+1)}$. The reason for choosing this form of $f(x)$ for illustration is that it provides a natural generalization of the damped harmonic oscillator and the modified Emden type equation.

Substituting $f(x) = x^q$ in equations (2.24) - (2.26) we obtain the following time independent integral of motion for the equation (2.38), that is

**Case 1.** $\alpha^2 > 4\beta(q + 1)$

$$I = \frac{(r - 1)}{(r - 2)} \left(\dot{x} + \frac{\alpha}{r} x^{q+1}\right) \left(\dot{x} + \frac{(r - 1)}{r} \frac{\alpha}{r} x^{q+1}\right)^{(1-r)} \quad (2.39)$$
2.4 Generalized MEE: Hamiltonian structure and general solution

Case 2. $\alpha^2 = 4\beta (q + 1)$

$$I = \frac{\dot{x}}{\dot{x} + \frac{\alpha}{2} x^{q+1}} - \log \left[ \frac{\dot{x}}{\dot{x} + \frac{\alpha}{2} x^{q+1}} \right]$$  \hspace{1cm} (2.40)

Case 3. $\alpha^2 < 4\beta (q + 1)$

$$I = \frac{1}{2} \log \left[ \dot{x}^2 + \frac{\alpha}{2} \dot{x} x^{q+1} + \frac{\beta}{(q + 1)} x^{2(q+1)} \right] + \frac{\alpha}{2\omega} \tan^{-1} \left[ \frac{\alpha \dot{x} + 2\beta x^{q+1}}{2\omega \dot{x}} \right].$$  \hspace{1cm} (2.41)

where $r = \frac{\alpha}{2\beta (q+1)}(\alpha \pm \sqrt{\alpha^2 - 4\beta (q+1)})$, $\omega = \frac{1}{2} \sqrt{4\beta (q + 1) - \alpha^2}$ and $\hat{\alpha} = \frac{\alpha}{q+1}$.

The Hamiltonian for the equation (2.38) can be deduced from equation (2.30) which turns out to be

$$H = \begin{cases} 
\frac{(r-1)}{r-2} p^{(r-2)} - \frac{r-1}{r} \alpha p x^{q+1}, & \alpha^2 > 4\beta (q + 1) \\
\log(p) - \frac{\hat{\alpha}}{2} p x^{q+1}, & \alpha^2 = 4\beta (q + 1) \\
\frac{\hat{\alpha}}{2} p x^{q+1} - \frac{1}{2} \log \left[ x^{2(q+1)} \sec^2 \frac{\omega}{(q + 1)} x^{q+1} p \right], & \alpha^2 < 4\beta (q + 1),
\end{cases} \hspace{1cm} (2.42)$$

where the canonically conjugate momentum is defined by

$$p = \begin{cases} 
\frac{1}{(r-1)} \left( \dot{x} + \frac{(r-1)}{r} \hat{\alpha} x^{q+1} \right)^{(1-r)}, & \alpha^2 > 4\beta (q + 1) \\
\frac{(q+1)}{\omega x^{q+1}} \tan^{-1} \left[ \frac{\alpha x^{q+1} + 2(q + 1) \dot{x}}{2\omega x^{q+1}} \right]. & \alpha^2 < 4\beta (q + 1)
\end{cases} \hspace{1cm} (2.43)$$

Here we note that the integrals (2.39)-(2.41) can also be derived systematically through various methods available in the recent literature. To name a few, we cite Prelle-Singer procedure [74], method of reducing Liénard equation to the Abel equation form [103], factorization method [105, 106] and so
on. However, all these methods provide only time independent integrals but neither the integrals nor the Hamilton canonical equations can be integrated straightforwardly.

### 2.4.1 Method of obtaining general solution

In this section we transform the Hamiltonian (2.42) into a simpler form, by introducing a suitable canonical transformation and then obtain the solutions.

**Case 1: \( \alpha^2 > 4\beta(q + 1) \)**

The Hamiltonian for this parametric regime is given by

\[
H = \frac{(r - 1)}{(r - 2)} p^{\frac{r-2}{r-1}} - \frac{(r - 1)}{r} \hat{a} p x^{q+1}, \quad r = \frac{\alpha^2 \pm \alpha \sqrt{\alpha^2 - 4\beta(q + 1)}}{2\beta(q + 1)} \tag{2.42}
\]

By introducing a canonical transformation

\[
x = q \left( \frac{U}{P} \right)^{\frac{1}{q}}, \quad p = \frac{1}{2} \left( U^{q-1} P^{q+1} \right)^{\frac{1}{q}} \tag{2.44}
\]

the Hamiltonian (2.42) can be transformed into the form

\[
H = \sigma_1 P^{n_1} U^{m_1} + \eta_1 U^2 = E, \tag{2.45}
\]

where we have defined \( \sigma_1 = \frac{(r-1)}{(r-2)} \left( \frac{1}{2} \right)^{\frac{r-2}{r-1}}, \quad n_1 = \frac{(q+1)(r-2)}{q(r-1)}, \quad m_1 = \frac{(q-1)(r-2)}{q(r-1)} \) and \( \eta_1 = \frac{(1-r)}{2r} \hat{a} q^{q+1} \).

The canonical equations of motion for the transformed Hamiltonian (2.45) now reads

\[
\dot{U} = \sigma_1 n_1 U^{n_1-1}, \tag{2.46a}
\]

\[
\dot{P} = -\left( \sigma_1 m_1 P^{m_1} U^{m_1-1} + 2\eta_1 U \right). \tag{2.46b}
\]

One may observe that for the choice \( n_1 = 1 \) equation (2.46) becomes uncoupled
and one can integrate the resultant equations straightforwardly. In the following, first we consider the general case and then discuss the particular case \( n_1 = 1 \) which corresponds to the parametric choice \( \beta = \frac{\alpha^2}{(q+2)^2} \).

Substituting (2.46a) into (2.45) and rewriting the latter, we get

\[
E = \frac{\sigma_1 U^{n_1}}{(\sigma_1 n_1)^{n_1^{-1}}} \left[ U^{\frac{n_1}{n_1 - 1}} U^{\frac{n_1 - 1}{n_1 - 1}} \right] + \eta_1 U^2. \tag{2.47}
\]

Rewriting equation (2.47) we obtain

\[
\dot{U} = \left[ \frac{1}{\dot{\sigma}_1} \left( E - \eta_1 U^2 \right) \right]^{\frac{n_1 - 1}{n_1}} U^{\frac{m_1}{n_1}}, \quad n_1 \neq 1 \tag{2.48}
\]

where \( \dot{\sigma}_1 = \frac{\sigma_1}{(\sigma_1 n_1)^{n_1^{-1}}} \). Integrating the above equation one obtains,

\[
t - t_0 = \int \frac{\dot{\sigma}_1 U^m dU}{(E - \eta_1 U^2)^n}, \tag{2.49}
\]

where \( m = \frac{-m_1}{n_1} \) and \( n = \frac{n_1 - 1}{n_1} \). The above integral can be split into two cases depending upon the values of \( m \) and \( n \), that is

\[
t - t_0 = \begin{cases} 
\frac{U^{m-1}}{\eta_1 (2n - m - 1)(E - \eta_1 U^2)^{n-1}} \\
+ \frac{(m - 1)E}{\eta_1 (2n - m - 1)} \int \frac{U^{m-2}}{(E - \eta_1 U^2)^n} dU & m, n > 0 \\
\frac{1}{E(1 - m)U^{m-1}(E - \eta_1 U^2)^{n-1}} \\
+ \frac{\eta_1 (m + 2n - 3)}{E(1 - m)} \int \frac{1}{U^{m-2}(E - \eta_1 U^2)^n} dU & m < 0.
\end{cases} \tag{2.50}
\]

The integrals on the right hand sides of (2.50) can be integrated again and again until all the integrals are exhausted, thereby giving the general solution in an implicit form.

Now we analyze the case \( n_1 = 1 \) in (2.46). For this choice, equation (2.46a)
can be integrated to yield $U$ in the form

$$U = [(1 - m_1)(\sigma_1 t + c_{1})]^{\frac{1}{1 - m_1}},$$  \hspace{1cm} (2.51)

where $c_{1}$ is an integration constant. From equation (2.45) one can express $P$ in terms of $U$ and $E$ and substituting the latter into the first expression in (2.44) one gets

$$x = q \left( \frac{\sigma_1 U^{m_1+1}}{E - \eta_1 U^2} \right)^{\frac{1}{q}},$$  \hspace{1cm} (2.52)

where $\sigma_1 = \frac{q+1}{q} \left( \frac{1}{2} \right)^{\frac{q+1}{q}}$, $\eta_1 = -\alpha \frac{q^{q+1}}{2(q+2)}$ and $m_1 = \frac{q-1}{q+1}$. Substituting (2.51) into (2.52) and simplifying the resultant expression we arrive at

$$x(t) = \left( \frac{2^{q-1} q^{q-1} (q+1)^2 (q+2) (\sigma_1 t + c_1)^q}{2E(q+2) + \alpha 2^{q+1} q^{q+1} (\sigma_1 t + c_1)^{q+1}} \right)^{\frac{1}{q}}.$$  \hspace{1cm} (2.53)

with $E$ and $c_1$ are two arbitrary constants. We mention here that the linearizable case $\alpha^2 = 9\beta$ belongs to this case and the solution exactly agrees with the known one in the literature [84, 107, 108].

**Case 2 : $\alpha^2 = 4\beta(q + 1)$**

Using the same canonical transformation, equation (2.44), one can transform the Hamiltonian, $H = \log(p) - \frac{\alpha}{2} p^2 q^1 + 1$, into the form

$$H = \frac{(q-1)}{q} \log(U) + \frac{q+1}{q} \log(P) + \eta_2 U^2 = E$$  \hspace{1cm} (2.54)
where \( \eta_2 = -\frac{a}{4} q^{(q+1)} \). The corresponding canonical equations become

\[
\dot{U} = \frac{(q+1)}{qP}, \\
\dot{P} = \left( \frac{1-q}{q} \right) \frac{1}{U} + 2\eta_2 U.
\] (2.55a)

From (2.55a) one can express \( P \) in terms of \( \dot{U} \) and substituting this into (2.54) one can bring the latter to the form

\[
\dot{E} = \frac{(q-1)}{q} \log(U) - \frac{(q+1)}{q} \log(\dot{U}) + \eta_2 U^2
\] (2.56)

which in turn gives us

\[
\dot{U} = \exp \left[ \frac{\eta_2 U^2}{(q+1)} \right] U^{\frac{q-1}{q+1}} E_1,
\] (2.57)

where \( E_1 = \exp \left[ \frac{-q\dot{E}}{(q+1)} \right] \). Integrating the above equation, one obtains the solution in the form

\[
\begin{align*}
  t - t_0 &= \frac{1}{E_1} \int \frac{dU}{\exp \left[ \frac{\eta_2 U^2}{(q+1)} \right] U^{\frac{q-1}{q+1}}} \\
  &= -\frac{1}{2E_1} U^{\frac{2}{q+1}} \left( \frac{q+1}{\eta_2 U^2} \right)^{\frac{1}{q+1}} \Gamma \left[ \frac{1}{q+1}, \frac{\eta_2 U^2}{q+1} \right],
\end{align*}
\] (2.58)

where \( \Gamma \) is the gamma function [109]. For the choice \( q = 1 \) the integral (2.58) can be evaluated in terms of error function [86, 109].

**Case 3 :** \( \alpha^2 < 4\beta(q+1) \)

Now we focus our attention on the underdamped case. Using the canonical transformation (2.44), we rewrite the underlying Hamiltonian in the form

\[
H = \frac{\dot{\mathcal{a}}}{4} q^{(q+1)} U^2 - \log \left[ \left( \frac{U}{P} \right)^{\frac{q+1}{q}} \right] \sec \left[ \frac{\omega}{2(q+1)} U^2 \right].
\] (2.59)
The associated canonical equations read

\[
\dot{U} = \frac{q + 1}{qP}, \quad (2.60a)
\]
\[
\dot{P} = \frac{2q\omega U^2 \tan\left[\frac{\omega U^2}{2(q+1)}\right] - ((q + 1)q^{q+2}U^2 \alpha - 2(q + 1)^2)}{2q(q + 1)U}. \quad (2.60b)
\]

From (2.60a) one can write \( P = \frac{q + 1}{qU} \) and substituting this in the Hamiltonian (2.59) and simplifying we get

\[
\tilde{H} = \log[U \tilde{U}^{\frac{q+1}{4}} \sec\left[\frac{\omega}{2(q + 1)} U^2\right]] + \eta_2 U^2 = E. \quad (2.61)
\]

Rewriting equation (2.61) we have

\[
\tilde{U} = \frac{1}{U} \exp\left[\frac{q}{q+1}(E - \eta_2 U^2)\right]. \quad (2.62)
\]

Integrating equation (2.62), we get

\[
t - t_0 = \int \frac{U \sec\left[\frac{\omega}{2(q + 1)} U^2\right]^{\frac{3}{q+1}}}{\exp\left[\frac{q}{q+1}(E - \eta_2 U^2)\right]} dU
= \frac{q}{q+1} \frac{q(\omega - 2iq_1)}{2(q + 1)\omega}, \frac{(3q + 2)\omega - 2iqq_1}{2(q + 1)\omega}, -\exp\left[\frac{i\omega U^2}{(q + 1)}\right]
\times \frac{(q + 1)^2}{q(2(q + 1)\eta_2 + i\omega)} \left[1 + \exp\left[\frac{i\omega q^2}{(q + 1)}\right]\right]^{\frac{q}{q+1}}(\cosh[q_2] - \sinh[q_2]), \quad (2.63)
\]

where \( q_1 = \eta_2(q + 1) \), \( q_2 = \frac{q(E - \eta_2 U^2)}{q+1} \) and \( F \) is the hypergeometric function [109].
2.5 Damped Mathews-Lakshmanan oscillator

2.5.1 Mathews-Lakshmanan oscillator

Mathews and Lakshmanan studied the following nonlinear oscillator [1, 110]

$$\ddot{x} - \frac{\lambda_1 x}{1 + \lambda_1 x^2} \dot{x}^2 + \frac{\lambda x}{1 + \lambda_1 x^2} = 0,$$

(2.64)

and found the system to possess simple trigonometric solution,

$$x(t) = A \sin(\Omega t + \theta), \quad \Omega^2 = \frac{\lambda}{1 + \lambda_1 A^2}. \quad (2.65)$$

Note here that the system (2.64) admits oscillatory solution and the frequency of oscillations depends on the amplitude unlike the equation (2.12) (with $\beta = \frac{\alpha^2}{9}$). The system (2.64) is also found to admit the time independent Hamiltonian

$$H = p^2 (1 + \lambda_1 x^2) + \frac{\lambda x^2}{1 + \lambda_1 x^2}. \quad (2.66)$$

We note here that Eq. (2.64) and its Hamiltonian (2.66) are related to the harmonic oscillator

$$U'' + \lambda U = 0, \quad \left( t = \frac{d}{d\tau} \right) \quad (2.67)$$

and its corresponding Hamiltonian through the transformation

$$U = \frac{x}{\sqrt{1 + \lambda_1 x^2}}, \quad d\tau = \frac{dt}{(1 + \lambda_1 x^2)}. \quad (2.68)$$

One can arrive at the general solution (2.65) of Eq. (2.64) by integrating the above transformation as shown below.

Substituting $U = C \sin(\sqrt{\lambda} \tau + \delta)$ and rewriting the first relation of (2.68), we
get

\[ x = \frac{C \sin(\sqrt{\lambda} \tau + \delta)}{1 - C^2 \lambda_1 \sin^2(\sqrt{\lambda} \tau + \delta)}. \quad (2.69) \]

Substituting the above expression of \( x \) in the second relation of (2.68), we get

\[ dt = \frac{d\tau}{1 - C^2 \lambda_1 \sin^2(\sqrt{\lambda} \tau + \delta)}. \quad (2.70) \]

Integrating Eq. (2.70) and rewriting we get

\[ \tau = \frac{1}{\sqrt{\lambda}} \left( \tan^{-1} \left( \frac{\tanh((B + t)\sqrt{\lambda(C^2\lambda_1 - 1)})}{(C^2\lambda_1 - 1)} \right) \right). \quad (2.71) \]

Substituting the above expression in (2.69) and simplifying with suitable redefinition of the arbitrary constants \( B \) and \( C \), we get

\[ x = A \sin(\Omega t + \theta), \quad \Omega^2 = \frac{\lambda}{1 + \lambda_1 A^2}. \quad (2.72) \]

We consider a more generalized form of the Mathew-Lakshmanan oscillator by including damping and obtain its Hamiltonian structure and its general solution in the next subsection.

### 2.5.2 Damped Mathews-Lakshmanan oscillator: Hamiltonian structure

Let us consider the following damped version of the Mathews-Lakshmanan oscillator equation [110],

\[ \ddot{x} = -\frac{\lambda_1 x}{1 + \lambda_1 x^2} \dot{x}^2 + \frac{\alpha}{1 + \lambda_1 x^2} \dot{x} + \frac{\lambda x}{1 + \lambda_1 x^2} = 0, \quad (2.73) \]
where we have included a nonlinear damping term to the Mathews-Lakshmanan oscillator. Comparing the above equation with (2.23), we find

\[ g(x) = \frac{1}{\sqrt{1 + \lambda_1 x^2}}, \quad f(x) = \frac{1}{(1 + \lambda_1 x^2)^{3/2}}. \] (2.74)

The integral of motion is then

\[ I = \begin{cases} 
\frac{(r - 1)}{(r - 2)} \frac{(r \dot{x} + \alpha x)}{r \sqrt{1 + \lambda_1 x^2}} \left( \frac{r \dot{x} + \alpha x (r - 1)}{r \sqrt{1 + \lambda_1 x^2}} \right)^{1-r}, & \alpha^2 > 4\lambda \\
\frac{2 \dot{x}}{2 \dot{x} - x} - \log \left[ \frac{2 \dot{x} + \alpha x}{2 \sqrt{1 + \lambda_1 x^2}} \right], & \alpha^2 = 4\lambda \\
\frac{\alpha}{2 \omega} \tan^{-1} \left[ \frac{\alpha x + 2 \lambda x}{2 \omega x} \right] + \frac{1}{2} \log \left[ \frac{\dot{x}^2 + \alpha \dot{x} + \lambda x^2}{(1 + \lambda_1 x^2)} \right], & \alpha^2 < 4\lambda.
\end{cases} \] (2.75)

Substituting the above forms of \( g(x) \) and \( f(x) \) in the expression for the Hamiltonian (2.30), we get

\[ H = \begin{cases} 
\frac{(r - 1)}{(r - 2)} \left( p \sqrt{1 + \lambda_1 x^2} \right)^{\frac{r-2}{2}} - \frac{\alpha (r - 1)}{r} px, & \alpha^2 > 4\lambda \\
\frac{\alpha}{2} px + \log \left[ p \sqrt{1 + \lambda_1 x^2} \right], & \alpha^2 = 4\lambda \\
\frac{1}{2} \log \left[ \frac{x^2}{(1 + \lambda_1 x^2) \sec^2 \left[ \omega px \right]} \right] - \frac{\alpha p}{2} x, & \alpha^2 < 4\lambda,
\end{cases} \] (2.76)

where the canonically conjugate momentum

\[ p = \begin{cases} 
\frac{1}{\sqrt{1 + \lambda_1 x^2}} \left( \frac{r \dot{x} + \alpha x (r - 1)}{r \sqrt{1 + \lambda_1 x^2}} \right)^{1-r}, & \alpha^2 \geq 4\lambda \\
\frac{1}{\omega x} \tan^{-1} \left[ \frac{\alpha + 2 \dot{x}}{2 \omega x} \right], & \alpha^2 < 4\lambda.
\] (2.77)
Note here that the Hamiltonian (2.76) of the Eq. (2.73) is of nonstandard type unlike the standard Hamiltonian of the undamped Mathews-Lakshmanan oscillator (2.64).

### 2.5.3 General solution

**Case 1:**

The general solution of Eq. (2.73) is found by transforming the above Hamiltonian using suitable canonical transformation. The Hamiltonian of the overdamped case can be reduced to the following simple form using the canonical transformation $U = \frac{1}{\lambda_1} \sinh^{-1}(\sqrt{\lambda_1}x)$ and $P = p\sqrt{1 + \lambda_1 x^2}$

$$H = \frac{(r-1)}{(r-2)} P^{(r-2)/(r-1)} - \frac{\alpha(r-1)}{r\sqrt{\lambda_1}} P \tanh(\sqrt{\lambda_1}U) \equiv E. \quad (2.78)$$

The Hamilton equations associated with the above Hamiltonian read as

$$\dot{U} = P^{\frac{1}{r-2}} - \frac{(r-1)}{r} \frac{\alpha}{\sqrt{\lambda_1}} \tanh(\sqrt{\lambda_1}U), \quad \dot{P} = \frac{P(r-1)}{r} \alpha \text{sech}^2(\sqrt{\lambda_1}U). \quad (2.79)$$

From the canonical equations we find $U = \frac{1}{\sqrt{\lambda_1}} \text{sech}^{-1} \left( \frac{r\dot{P}}{P(r-1)\alpha} \right)^{\frac{1}{2}}$ and substituting this in (2.78) and rewriting we get,

$$\dot{P} = \frac{1}{r} \sqrt{\alpha^2 P^2(r-1)^2 - r^2 \lambda_1 \left[ E - \frac{(r-1)}{(r-2)} P^{(r-2)/(r-1)} \right]^2}. \quad (2.80)$$

Integrating the above equation we get

$$t - t_0 = \int \frac{rdP}{\sqrt{\alpha^2 P^2(r-1)^2 - r^2 \lambda_1 \left[ E - \frac{(r-1)}{(r-2)} P^{(r-2)/(r-1)} \right]^2}}, \quad (2.81)$$

where $t_0$ and $E$ are arbitrary constants.

**Case 2:**

In order to find the general solution of the critically damped case we trans-
form the corresponding Hamiltonian using the canonical transformation \( P = \log x, U = px \) to the form

\[
H = \frac{\alpha}{2} U + \log(U \sqrt{e^{-2P} + \lambda_1}). \tag{2.82}
\]

The Hamilton equations read

\[
\dot{U} = \frac{-1}{1 + \lambda_1 e^{2P}}, \quad \dot{P} = -\left( \frac{1}{U} + \frac{\alpha}{2} \right). \tag{2.83}
\]

From the above equations we find

\[
P = \frac{1}{2} \log \left( \frac{-1}{\lambda_1} \left( \frac{1}{U} + 1 \right) \right), \tag{2.84}
\]

and substituting this in the transformed Hamiltonian (2.82) we get

\[
H = \frac{\alpha}{2} U + \log(u \sqrt{\frac{\lambda_1}{1 + U}}) \equiv E. \tag{2.85}
\]

Rewriting the above equation we get,

\[
\dot{U} = \lambda_1 U^2 e^{\alpha U - 2E} - 1. \tag{2.86}
\]

Separating the variables and integrating we get

\[
t - t_0 = \int \frac{dU}{\lambda_1 U^2 e^{\alpha U - 2E} - 1}, \tag{2.87}
\]

where \( t_0 \) and \( E \) are arbitrary constants.

**Case 3:**

Using the canonical transformation \( P = \frac{1}{2} \log(x^2), U = px \), one can rewrite the Hamiltonian for the underdamped case \((\alpha^2 < 4\lambda)\) as

\[
H = \frac{1}{2} \left( \log[\sec^2(\omega U)] - \log[e^{-2P} + \lambda_1] - \alpha U \right) \equiv E. \tag{2.88}
\]
Following the same procedure illustrated in the previous example, one can arrive at the following equation

\[ \dot{U} = 1 - \cos^2(\omega U) \exp[2E + \alpha U]. \quad (2.89) \]

Obviously this can be rewritten as the quadrature

\[ t - t_0 = \int \frac{dU}{1 - \cos^2(\omega U) \exp[2E + \alpha U]}. \quad (2.90) \]

where \( t_0 \) and \( E \) are arbitrary constants. One finds that the above integration can be easily performed for the choice \( \alpha = 0 \) and one obtains the known solution (2.65). For the choice \( \alpha \neq 0 \) the general solution is given in terms of the quadrature.

### 2.6 Conclusion

In this chapter, we have investigated a class of nonlinear dissipative systems which admits non-standard time independent Hamiltonian description through a novel nonlocal transformation. The procedure is simple and straightforward. By introducing a nonlocal transformation in the 'source equation', namely the DHO equation, we are able to generate a class of 'target equations', namely the nonlinear generalizations of DHO. We have used the same nonlocal transformation to deduce the time independent Hamiltonian for the nonlinear equation. The nonlocal transformation introduced here is different from the one adopted in Ref. [88] and has certain salient features in identifying the time independent Hamiltonian structure. To illustrate the procedure two specific systems is considered, namely the generalized MEE and the damped Mathews-Lakshmanan oscillator equation and their general solutions are obtained by integrating their corresponding canonical equations of motion after introducing a suitable transformation. The associated Lagrangian description for both the systems has also been briefly discussed.