Chapter IV

BOUNDARY LAYER SLIP FLOW AND HEAT TRANSFER OVER A FLAT PLATE

a. Magnetic Effects in an electrically conducting fluid
b. When the Medium is Porous

This part of the Chapter has been published in Chinese Physics Letters 28 (2011) 024701.

This part of the Chapter has been published in Journal of Petroleum Science and Engineering 78 (2011) 304-309.
a. Magnetic Effects in an electrically conducting fluid

4.1a Introduction

Boundary layer flows play a central role in many aspects of fluid mechanics because the whole dynamics is initiated from the boundary surface. From both theoretical and experimental standpoints, forced convection over a flat plate has been widely studied over the past few decades. Early investigations mainly engaged in finding the similarity characteristics within the boundary layer framework. The development of velocity boundary layer structure on a flat plate was first investigated by Blasius (1908) and the heat transfer for this problem was solved by Pohlhausen (1921a). Howarth (1938) numerically studied the various aspects of the Blasius flow problem. The existence of a solution for flow past a flat plate was established by Abu-Sitta (1994). Wang (2004) reported an approximate solution of the classical Blasius equation using Adomian decomposition method. A numerical investigation of the classical Blasius flat plate problem was presented by Cortell (2005). Recently, Batallar (2008) and Cortell (2008) extended the Blasius and Sakiadis problems, respectively by studying the effects of thermal radiation on the boundary layer flow and heat transfer.

The interaction between electrically conducting fluid and a magnetic field has been studied intensively by many researchers due to its important applications in different electrical appliances. Hydro-magnetic behaviour of boundary layer along a fixed or moving plate in presence of transverse magnetic field is a basic and important problem in this area. Due to the effects of magnetic field on the boundary layer, the study of MHD flow and heat transfer is always interesting. In recent years, a significant number of investigations have been carried out to understand the effects of magnetic field on flow and heat transfer of electrically conducting fluids (viz. liquid metals, water mixed with a little acid, and many others) past a plate. It is well known that, by the change in the velocity of the fluid or liquid metal, the magnetic field influences the heat transfer and, thus, the temperature distribution. Moreover, the MHD flow and heat transfer for a viscous fluid over a plate has enormous applications.
in many engineering problems such as petroleum industries, plasma studies, geothermal energy extractions, the boundary layer control in the field of aerodynamics and many others. Specially to control the behaviour of the boundary layer several artificial methods have been developed and out of that, the application of MHD principle is an important method for affecting the flow field in the desired direction by altering the structure of the boundary layer. The effect of a magnetic field on free convection heat transfer on isothermal vertical plate was discussed by Sparrow and Cess (1961). Gupta (1963) studied laminar free convection flow of an electrically conducting fluid past a vertical plate with uniform surface heat flux and variable wall temperature in presence of a magnetic field. Riley (1964) investigated the flow of an electrically conducting fluid on a vertical plate in presence of strong magnetic field applied normal to the flow. Watanabe and Pop (1995) explained the hall effects on MHD boundary layer flow over a continuous moving flat plate. Damseh et al. (2006) obtained the similarity solution for forced convection flow with magnetic field and thermal radiation.

The study of hydrodynamic flow and heat transfer over a porous flat plate becomes much more interesting due to its ever increasing applications on boundary layer flow control using suction or blowing through porous boundary wall. In fact, suction tends to stabilize the boundary layer flow. On the contrary, the wall shear stress and hence the friction drag due to fluid viscosity is reduced by blowing [Ishak et al. (2007)].

The no-slip boundary condition is known as the central tenets of the Navier-Stokes theory. But there are situations wherein such condition is not appropriate. The fluids exhibiting boundary slip find applications in technology such as in the polishing of artificial heart valves and internal cavities. Recently, micro-scale fluid dynamics in the Micro-Electro-Mechanical Systems (MEMS) received much attention in research. Due to the micro-scale dimensions, the fluid flow behaviour belongs to the slip flow regime and greatly differs from traditional flow.

In all the aforesaid investigations, the no-slip condition at the boundary wall had assumed. The non-adherence of the fluid to a solid boundary, also known as velocity slip, is a phenomenon that has been observed under certain circumstances. The assumption of no-slip condition does no longer valid and should be replaced by a
partial slip boundary condition relating to the shear rate at the boundary. The partial slip condition had been used in studies of fluid flow past permeable wall by Beavers and Joseph (1967). The use of slip condition is also very important for the flow in microdevices [Gad-el-Hak (1999)]. In literature, there is a scarcity of the study of the slip flow over a flat plate. Martin and Boyd (2006) considered the momentum and heat transfer in a laminar boundary layer flow over a flat plate with slip boundary condition. Cao and Baker (2009) studied the mixed convective flow and heat transfer from a vertical plate taking velocity slip and temperature jump boundary conditions and gave local non-similar solutions to the boundary-layer equations. Recently, Aziz (2010) studied the boundary layer slip flow over a flat plate with constant heat flux condition at the surface and in this work also the local similarity was appeared in the slip boundary condition. The effects of slip boundary condition on flow of Newtonian fluid due to a stretching sheet were explained by Andersson (2002) and Wang (2002). Recently, Pal and Talukdar (2010) presented an analytical solution of unsteady magnetohydrodynamic convective heat and mass transfer past a vertical permeable plate with thermal radiation and chemical reaction in presence of slip at the boundary.

Motivated by the above studies, in the 1st part of the chapter we investigate the slip effect on MHD boundary layer flow over a flat plate. Thermal slip is also considered which gives interesting features regarding such flow. The no-slip condition is replaced by Navier's slip condition where the amount of relative slip is proportional to the local shear stress. The slip model of Andersson (2002) is taken here in some modified form. A self-similar set of equations are obtained. No local similarity is appeared at the boundary conditions. The equations with the boundary conditions are then solved numerically using well known shooting method. Computed numerical results are plotted and the characteristics of flow and heat transfer are analyzed thoroughly.

4.2a Mathematical formulation of the problem

We consider the steady two-dimensional laminar flow of an electrically conducting viscous incompressible fluid and heat transfer over a flat plate in presence of transverse magnetic field. The outline of the physical problem is given in Fig4.1a.
Using boundary layer and MHD approximations, the equations for MHD flow and temperature are written in usual notation as:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4.1a}
\]

\[
\frac{u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} + \frac{\sigma B_x^2}{\rho} (U_\infty - u) \tag{4.2a}
\]

and

\[
\frac{u}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\kappa}{\rho C_p} \frac{\partial^2 T}{\partial y^2}, \tag{4.3a}
\]

where \(u\) and \(v\) are the velocity components in \(x\)- and \(y\)-directions respectively, \(\nu(=\mu/\rho)\) is the kinematic fluid viscosity, \(\rho\) is the fluid density, \(\mu\) is the coefficient of fluid viscosity, \(\sigma\) is the constant electrical conductivity of the fluid, \(U_\infty\) is the free stream velocity, \(T\) is the temperature, \(\kappa\) is the thermal conductivity of the fluid and \(C_p\) is the specific heat. \(B_x(x)\) is the magnetic field in the \(y\)-direction and is given by \(B_x(x) = B_0(x)^{1/2}\). We have considered very simplified flow where the magnetic Reynolds number \(R_{M}\) is very small. In this case, the induced field can be neglected. Basically, we want to solve the problem in an inductionless situation.

The appropriate boundary conditions with partial slip for the velocity and the temperature are given by

\[
u = L_1(\partial u/\partial y), \quad u = 0 \text{ at } y = 0; \quad u \to U_\infty \text{ as } y \to \infty \tag{4.4a}
\]

and

\[
T = T_\infty + D_1(\partial T/\partial y) \text{ at } y = 0; \quad T \to T_\infty \text{ as } y \to \infty. \tag{4.5a}
\]

Here \(L_1 = L_0(Re_x)^{1/2}\) is the velocity slip factor and \(D_1 = D_0(Re_x)^{1/2}\) is the thermal slip factor with \(L_0\) and \(D_0\) being initial values of velocity and thermal slip factors having same dimension of length and \(Re_x\) being the local Reynolds number and \(Re_x = U_\infty \nu / \mu\), \(T_\infty\) is the temperature of the plate and \(T_\infty\) is the free stream temperature, both assumed to be constants.

It is to be worth mentioned that Maxwell (1879) used the slip boundary condition where the fluid velocity at the solid surface is assumed to be proportional to the sheer rate at the surface and the constant of proportionality has dimension of length. Our model for slip agrees with this. In addition, constant of proportionality, in our case, varies with local Reynolds number. This proposed slip model is physically realistic.

We now introduce the stream function \(\psi(x,y)\) as
\[ u = \frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \psi}{\partial x}. \]  

(4.6a)

Now for relations in (4.6a), the continuity equation (4.1a) is satisfied automatically. Using (4.6a), the momentum equation (4.2a) and the temperature equation (4.3a) take the following forms:

\[ \frac{\partial \psi}{\partial y} \frac{\partial^3 \psi}{\partial x \partial y^2} - \frac{\partial \psi}{\partial x} \frac{\partial^3 \psi}{\partial y^2} = \nu \frac{\partial^3 \psi}{\partial y^3} + \frac{\sigma B^2}{\rho} \left( \frac{\partial U_\infty}{\partial y} \right) \]

and

\[ \frac{\partial \psi}{\partial x} \frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial y} \frac{\partial T}{\partial y} = \frac{\kappa}{\rho c_p} \frac{\partial^3 T}{\partial y^3}. \]

(4.7a)

(4.8a)

The boundary conditions of (4.4a) for the velocity components reduce to

\[ \frac{\partial \psi}{\partial y} = 0 \text{ at } y = 0; \quad \frac{\partial \psi}{\partial y} \to U_\infty \text{ as } y \to \infty. \]

(4.9a)

Next, we introduce the dimensionless variables for \( \psi \) and \( T \) as given below:

\[ \psi = \sqrt{U_\infty} \nu \chi f(\eta) \text{ and } T = \frac{T_\infty + (T_0 - T_\infty) \theta(\eta)}{\rho c_p}, \quad \eta = \frac{\nu}{\sqrt{Re}}. \]

(4.10a)

where the similarity variable \( \eta \) is defined as \( \eta = \frac{\nu}{\sqrt{Re}} \).

In view of relations in (4.10a) we finally obtain following self-similar equations as

\[ f'' + \frac{1}{2} f' M(1 - f') = 0 \]

and

\[ \theta' + \frac{1}{2} Pr f \theta' = 0, \]

(4.11a)

(4.12a)

where \( M = \frac{\sigma B^2}{\rho U_\infty} \) is the magnetic parameter and \( Pr = \frac{\mu c_p}{\kappa} \) is the Prandtl number.

The boundary conditions (4.9a) and (4.5a) reduce to the following forms:

\[ f(\eta) = 0, f'(\eta) = \delta'(\eta) \text{ at } \eta = 0; \quad f'(\eta) \to 1 \text{ as } \eta \to \infty \]

and

\[ \theta(\eta) = 1 + \beta \theta(\eta) \text{ at } \eta = 0; \quad \theta(\eta) \to 0 \text{ as } \eta \to \infty, \]

(4.13a)

(4.14a)

where \( \delta = \frac{L_0 U_\infty}{\nu} \) is the velocity slip parameter and \( \beta = D_0 U_\infty / \nu \) is the thermal slip parameter.

### 4.3a Numerical method for solution

The nonlinear coupled differential equations (4.11a) and (4.12a) along with the boundary conditions (4.13a) and (4.14a) form a two point boundary value problem (BVP) and are solved using shooting method, by converting it into an initial value
problem (IVP). In this method we have to choose a suitable finite value of \( \eta \to \infty \), say \( \eta_c \). We set following first-order systems

\[
f' = p, \quad p' = q, \quad q' = -\frac{1}{2} f q - M(1 - p) \tag{4.15a}
\]

and

\[
\theta' = z, \quad z' = -\frac{1}{2} Pr f z \tag{4.16a}
\]

with the boundary conditions

\[
f(0) = 0, \quad p(0) = \delta q(0), \quad \theta(0) = 1 + \beta z(0). \tag{4.17a}
\]

To solve (4.15a) and (4.16a) with (4.17a) as an IVP we must need values for \( q(0) \) i.e. \( f(0) \) and \( z(0) \) i.e. \( \theta(0) \) but no such values are given. The initial guess values for \( f(0) \) and \( \theta(0) \) are chosen and applying fourth order Runge-Kutta method, an approximate solutions are obtained. We compare the calculated values of \( f(\eta) \) and \( \theta(\eta) \) at \( \eta_c = 20 \) with the given boundary conditions \( f(\eta_c) = 1 \) and \( \theta(\eta_c) = 0 \) and adjust values of \( f(0) \) and \( \theta(0) \) using Secant method to give better approximation for the solution. The step-size is taken as \( \Delta \eta = 0.01 \). The process is repeated until we get the results correct up to the desired accuracy of \( 10^{-6} \) level.

### 4.4a Results and discussion

The numerical computations is performed for several values of dimensionless parameters involved in the equations viz. the magnetic parameter \( (M) \), the velocity slip parameter \( (\delta) \), thermal slip parameter \( (\beta) \) and the Prandtl number \( (Pr) \). To illustrate the computed results, some figures are plotted and physical explanations are given.

At first, for the verification of the accuracy of the applied numerical method, we compare our results corresponding to the velocity and shear stress profiles for \( M = 0 \) and \( \delta = 0 \) (i.e. in the absence of the magnetic field and slip at the boundary) with the available published results of Howarth (1938) in Fig4.2a and are found in excellent agreement.

Now, we present the influence of the magnetic parameter \( M \) on the velocity and temperature profiles in the presence of slip and in the absence of slip at the boundary. Fig4.3a shows the variation in velocity field for several values of \( M \) while the temperature distribution is demonstrated in Fig4.4a. For both slip and no-slip
cases, the velocity $f'(\eta)$ along the plate increases and consequently the thickness of the boundary layer decreases. Thus for both cases the magnetic force enhances the fluid motion in the boundary layer because the last term of the momentum equation $(U_0-u)$ remains positive in the boundary layer region. Here the Lorentz force associated with the magnetic field makes the boundary layer thinner. From Fig4.4a, it is noticed that the temperature $\Theta(\eta)$ at a point decreases with $M$ for slip as well as no-slip conditions. Increase in the magnetic interaction parameter $M$ causes to decrease the thermal boundary layer thickness.

Next, we shall pay our attention to notice how the velocity slip parameter affects the velocity, the shear stress, the temperature and the temperature gradient profiles. The velocity $f'(\eta)$ and shear stress $f'(\eta)$ profiles for various values of the velocity slip parameter $\delta$ are depicted in Fig4.5a and Fig4.6a respectively. With increasing values of $\delta$, the fluid velocity increases monotonically. Due to the slip condition at the plate, the velocity of fluid adjacent to the plate has some positive value and accordingly the thickness of boundary layer decreases. But, the shear stress profile decreases with $\delta$. As the slip parameter increases in magnitude, permitting the more fluid to slip past the plate, the flow gets accelerated for distances closer to the plate however, for distances far away from the plate, the opposite behaviour is true. Opposite is the case for shear stress which is obvious. Fig4.7a and Fig4.8a exhibit the temperature $\Theta(\eta)$ and temperature gradient $\partial\Theta(\eta)$ profiles respectively for different values of $\delta$. From the figures, it is observed that the temperature decreases with the increase in velocity slip parameter $\delta$. The enhanced velocity due to slip near the plate is the cause of increasing heat transfer. An interesting behaviour for the temperature gradient profiles is perceived. The magnitude of temperature gradient increases with $\delta$ upto $\eta=2.705$ and after that point it decreases.

Fig4.9a demonstrates the effect of Prandtl number on temperature distribution. The temperature (at a fixed $\eta$) as well as the thermal boundary layer thickness rapidly decrease with increasing values of $Pr$ under both slip and no-slip conditions. An increase in Prandtl number means an increase of fluid viscosity which causes a decrease in the flow velocity and the temperature decreases. This is consistent with the fact that the thermal boundary layer thickness decreases with increasing Prandtl
number. For large value of \( Pr = 2 \) the temperature profile shows the negative nature. This is due to substantial increase in the heat transfer from the plate.

In Fig4.10a and Fig4.11a, the effects of thermal slip parameter \( \theta(\eta) \) on temperature and temperature gradient \( \theta'(\eta) \) are displayed respectively. As the thermal slip increases, less heat is transferred from the plate to the fluid and hence the temperature and the magnitude of temperature gradient decrease. In all temperature gradient profiles a common character is observed. The magnitude of the gradient profiles are initially increasing and for large \( \eta \), the curves decreases and ultimately goes to zero. As the momentum equation is independent of \( \theta \), no effect of thermal slip \( \beta \) on the velocity profiles is noticed.

Skin-friction coefficient is one of the interesting physical quantities in evaluating the viscous stress acting on the surface of the plate. Skin-friction coefficient \( f'(0) \) is plotted against the slip parameter \( \delta \) for several values of the magnetic parameter \( M \) in Fig4.12a. It is apparent that the skin-friction coefficient decreases rapidly and approaches zero as the slip starts to increase and the magnetic field affects in opposite way i.e. with increasing \( M \) it increases. The skin-friction coefficient is the maximum at the no-slip condition which is similar to the observations of Cao and Baker (2009). In Fig4.13a and Fig4.14a, the temperature gradient at the plate \( -\theta'(0) \) which is proportional to the rate of heat transfer from the plate is plotted against \( \delta \) and \( \beta \) respectively for the same values of \( M \). The rate of heat transfer increases with the increase of velocity slip as well as magnetic field but most importantly it decreases with \( \beta \). The negative value of \( \theta'(0) \) physically explains that there is heat flow from the plate.
Fig 4.1a Physical description of the MHD slip flow.

Fig 4.2a Velocity $f'(\eta)$ and shear stress $f''(\eta)$ profiles for $M=0$ and $\delta=0$. 

Present study ———
Howarth (1938) • • •
Boundary Layer Slip Flow and Heat Transfer over a Flat Plate

Fig 4.3a Velocity profiles $f(\eta)$ for various values of $M$ with slip and no-slip boundary conditions.

Fig 4.4a Temperature profiles $\theta(\eta)$ for various values of $M$ with slip and no-slip boundary conditions.
Boundary Layer Slip Flow and Heat Transfer over a Flat Plate

Fig 4.5a Velocity profiles $f(\eta)$ for various values of $\delta$.

Fig 4.6a Shear stress profiles $f'(\eta)$ for various values of $\delta$. 

$M = 0.5$, $Pr = 0.5$, $\beta = 0.1$

$\delta = 0, 0.2, 0.5, 1, 2, 4$

$\delta = 0, 0.2, 0.5, 1, 2, 4$
Fig 4.7a Temperature profiles $\theta(\eta)$ for various values of $\delta$.

Fig 4.8a Temperature gradient profiles $\theta'(\eta)$ for various values of $\delta$. 
Boundary Layer Slip Flow and Heat Transfer over a Flat Plate

Fig4.9a Temperature profiles $\theta(\eta)$ for various values of $Pr$ with slip and no-slip boundary conditions.

Fig4.10a Temperature profiles $\theta(\eta)$ for various values of $\beta$. 
Fig 4.11a Temperature gradient profiles $\theta(\eta)$ for various values of $\beta$.

Fig 4.12a Skin-friction coefficient $f'(0)$ against $\delta$ for various values of $M$. 

Equations and graphs showing the effects of various parameters on temperature gradients and skin-friction coefficients.
Fig 4.13a Temperature gradient at the plate $-\theta(0)$ against $\delta$ for various values of $M$.

Fig 4.14a Temperature gradient at the plate $-\theta(0)$ against $\beta$ for various values of $M$. 
b. When the Medium is Porous

4.1b Introduction

In the recent few years, considerable attention has been devoted to the study of boundary layer flow behavior and heat transfer characteristics of a Newtonian fluid past a plate embedded in a fluid saturated porous medium because of its extensive applications in engineering processes, especially in the enhanced recovery of petroleum resources and packed bed reactors [Pal and Shivakumara (2006) & Nield and Bejan (2006)]. A better understanding of convection through porous medium can benefit several areas like insulation design, underground nuclear waste storage sites, grain storage, heat exchangers, filtering devices, metal processing, catalytic reactors etc. Among the possible physical cases, in one case the temperature difference between the plate and the ambient fluid may be appreciably large. This physical concept has a number of geothermal and engineering applications. For example, the residual warm water discharged from a geothermal power plant is usually disposed off from wells through subsurface reinjection. This can be idealized as plane sources in porous medium [Prasad et al. (2011)]. Cheng and Minkowycz (1977) and Cheng (1977) studied the free convective flow in a saturated porous medium. Lai and Kulacki (1991) discussed the coupled heat and mass transfer by mixed convection from a vertical plate in a saturated porous medium. Many problems of Darcian and non-Darcian mixed convection about a vertical plate had been reported, as in Hsu and Cheng (1985), Pop and Takhar (1983), Vafai and Tien (1981). Hony et al. (1987) had studied analytically the non-Darcian effects on a vertical plate natural convection in porous media. Kaviany (1987) used the Darcy-Brinkman model to study the effects of boundary and inertia forces on forced convection over a fixed impermeable heated plate embedded in a porous medium. Chen and Ho (1988) studied the effects of flow inertia on vertical, natural convection in saturated porous media. Kumari et al. (1990) had investigated the non-Darcian effects on forced convection heat transfer over a flat plate in a highly porous medium. Rabadi and Hamdan (2000) presented the free convection flow from permeable inclined surface embedded in saturated porous media with variable permeability and thermal conductivity. Rashad (2008) discussed the
magneto-hydrodynamics and thermal radiation effects on heat and mass transfer in steady boundary layer flow over a vertical flat plate embedded in a fluid saturated porous media in presence of the thermophoresis particle deposition effect. Mukhopadhyay and Layek (2009) demonstrated the radiation effects on forced convective flow and heat transfer over a porous plate in porous medium. It is well known that Darcy’s law is an empirical formula relating the pressure gradient, the bulk viscous fluid resistance and the gravitational force for a forced convective flow in a porous medium. Deviations from Darcy’s law occur when the Reynolds number based on the pore diameter is within the range of 1 to 10 [Ishak et al. (2006b)]. The aim of this paper is to study the steady forced convection flow and heat transfer past a porous plate placed in a fluid saturated porous medium using the Darcy model.

The 2\textsuperscript{nd} part of the chapter deals with fluid flow and heat transfer over a flat porous plate embedded in a porous medium with partial slip condition at the boundary. Here also, the thermal slip is considered. Same slip models for velocity and thermal slip are considered. The self-similar equations along with appropriate the boundary conditions are then solved numerically using shooting method. Analyses of flow and heat transfer characteristics are rendered through physical and graphical point of views.

4.2b Mathematical formulation of the problem

Let us consider the steady two-dimensional laminar flow of a viscous incompressible fluid and heat transfer over a porous flat plate embedded in a porous medium. A physical sketch is given in Fig4.1b. Using boundary layer approximations, the equations for such type of flow and the energy equation may be written in usual notation as:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (4.1b)
\]

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} - \frac{k}{\kappa}(u - U_{\infty}) \quad (4.2b)
\]

and \[
u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{k}{\rho c_p} \frac{\partial^2 T}{\partial y^2}, \quad (4.3b)
\]
where \( u \) and \( v \) are the velocity components in \( x \) - and \( y \) -directions respectively, \( \nu \) (=\( \mu /\rho \)) is the kinematic fluid viscosity, \( \rho \) is the fluid density, \( \mu \) is the coefficient of fluid viscosity, \( k \) is the permeability of the porous medium, \( U_\infty \) is the free stream velocity, \( T \) is the temperature, \( \kappa \) is the thermal conductivity of the fluid and \( c_p \) is the specific heat.

The appropriate boundary conditions with partial slip for the velocity and the temperature are given by

\[
\begin{align*}
    u &= L_1(\partial u / \partial y), \quad v = v_w \text{ at } y = 0; \quad u \to U_\infty \text{ as } y \to \infty \quad (4.4b) \\
    T &= T_w + D_1(\partial T / \partial y) \text{ at } y = 0; \quad T \to T_\infty \text{ as } y \to \infty. \quad (4.5b)
\end{align*}
\]

Here \( L_1 = L_0(Re_x)^{1/2} \) is the velocity slip factor and \( D_1 = D_0(Re_x)^{1/2} \) is the thermal slip factor with \( L_0 \) and \( D_0 \) being initial values of velocity and thermal slip factors having same dimension of length and \( Re_x \) being the local Reynolds number and \( Re_x = U_\infty x / \nu \), \( T_w \) is the temperature of the plate and \( T_\infty \) is the free stream temperature, both assumed to be constants. Here \( v_w \) is prescribed suction or blowing through the porous plate and is given by \( v_w = v_0/(x)^{1/2} \), \( v_0 \) being constant with \( v_0 < 0 \) for suction and \( v_0 > 0 \) for blowing.

Now, we introduce the same similarity transformations (see 4.10a):

\[
\begin{align*}
    \psi &= \sqrt{U_\infty \nu x} f(\eta), \quad T = T_w + (T_\infty - T_w) \theta(\eta) \quad \text{and} \quad \eta = \frac{x}{\sqrt{Re_x}}, \quad (4.6b)
\end{align*}
\]

where \( \psi \) is the stream function with \( u = \frac{\partial \psi}{\partial \eta} \) \& \( v = -\frac{\partial \psi}{\partial \xi} \) and \( \eta \) is the similarity variable.

In view of relations in (4.6b) we finally obtain following self-similar equations as:

\[
\begin{align*}
    f'' + \frac{1}{2} f f' - \frac{1}{Da_x Re_x} (f' - 1) &= 0, \quad (4.7b) \\
    \text{and} \quad \theta'' + \frac{1}{2} Pr f \theta' &= 0, \quad (4.8b)
\end{align*}
\]

where \( Da_x = k / \xi^2 = k_0 / \xi \) is the local Darcy number, \( k = k_0 x \), \( k_0 \) is a constant and \( Pr = \mu c_p / \kappa \) the Prandtl number.

Equation (4.7b) can be written as:

\[
    f'' + \frac{1}{2} f f' - k_1 (f' - 1) = 0, \quad (4.9b)
\]

where \( k_1 = 1/(Da_x Re_x) \) is the permeability parameter of the porous medium [Mukhopadhyay and Layek (2009)].

The boundary conditions (4.4b) and (4.5b) reduce to the following forms:
Boundary Layer Slip Flow and Heat Transfer over a Flat Plate

\[ f(\eta) = S, f'(\eta) = -\eta \] at \( \eta = 0 \); \( f(\eta) \to 1 \) as \( \eta \to \infty \) \hfill (4.10b)

and \( \vartheta(\eta) = 1 + \beta \vartheta(\eta) \) at \( \eta = 0 \); \( \vartheta(\eta) \to 0 \) as \( \eta \to \infty \), \hfill (4.11b)

where \( S = (-2\nu_0/U_\infty)(Re_x)^{1/2} = -2\nu_0/(U_\infty \nu)^{1/2} \), \( S > 0 \) (i.e. \( \nu_0 < 0 \)) corresponds to suction and \( S < 0 \) (i.e. \( \nu_0 > 0 \)) corresponds to blowing, \( \delta = L_0 U_\infty / \nu \) is the velocity slip parameter and \( \beta = D_0 U_\infty / \nu \) is the thermal slip parameter.

4.3b Numerical method for solution

The nonlinear coupled differential equations (4.9b) and (4.8b) along with the boundary conditions (4.10b) and (4.11b) form a two point boundary value problem (BVP) and are solved using shooting method, by converting it into an initial value problem (IVP) as:

\[ f' = p, \quad p' = q, \quad q' = -\frac{1}{2} f q + k^*(p - 1) \] \hfill (4.12b)

and \( \vartheta' = z, \quad z' = -\frac{1}{2} Pr f z \) \hfill (4.13b)

with

\[ f(0) = 0, \quad p(0) = \vartheta(0), \quad \vartheta(0) = 1 + \beta z(0). \] \hfill (4.14b)

4.4b Results and discussion

The numerical computations are performed for several values of dimensionless parameters viz., the permeability parameter \( k^* \), the velocity slip parameter \( \delta \), thermal slip parameter \( \beta \) and the Prandtl number \( Pr \). For illustrating the obtained data, some figures are plotted and physical reasons are explained.

Now, we present the influence of the permeability parameter \( k^* \) on the velocity and temperature profiles in presence of slip and also in the absence of slip at the boundary. Fig4.2b shows the variation in velocity field for several values of \( k^* \) while the temperature distribution is demonstrated in Fig4.3b. For both slip and no-slip cases, the velocity \( f(\eta) \) along the plate increases and consequently the thickness of the momentum boundary layer decreases. With a rise in permeability of the medium, the regime becomes more porous. As a consequence, the Darcian body force
Boundary Layer Slip Flow and Heat Transfer over a Flat Plate

decreases in magnitude (as it is inversely proportional to the permeability). The Darcian resistance acts to decelerate the fluid particles in continua. This resistance diminishes as permeability of the medium increases. So progressively less drag is experienced by the flow and flow retardation is thereby decreased. Thus for both cases the permeability parameter enhances the fluid motion in the boundary layer. From Fig4.3b, it is noticed that the temperature $\theta(\eta)$ at a point decreases with $k^*$ for slip as well as no-slip conditions. Increase in the permeability parameter $k^*$ causes to decrease the thermal boundary layer thickness. The rate of heat transfer is enhanced with increase of $k^*$, since the momentum boundary layer thickness decreases.

We now focus our attention on the behaviour of velocity and temperature distribution for the variation of suction ($S>0$)/blowing ($S<0$) parameter in presence of slip and no-slip conditions at the boundary for flat plate embedded in porous medium. The velocity and temperature distributions for various values of $S$ are shown in Fig4.4b and Fig4.5b. With the increasing $S$ ($S>0$), fluid velocity is found to increase [Fig4.4] for both cases i.e. suction causes to increase the velocity of the fluid. By sucking fluid particles through porous wall the growth of fluid boundary layer is reduced. Since the effect of suction is to suck the fluid near the wall, the momentum boundary layer is reduced due to suction ($S>0$). Consequently the velocity increases. Hence the velocity gradient and so the skin friction increases with increasing $S$ ($S>0$). Opposite behaviour is noted for blowing ($S<0$). Fig4.6b demonstrates that the temperature $\theta(\eta)$ decreases with the increasing suction parameter $S$. The thermal boundary layer thickness decreases with the suction parameter $S$ ($S>0$) which causes an increase in the rate of heat transfer. The explanation for such behavior is that the fluid is brought closer to the wall and so the thermal boundary layer thickness reduces [Fig4.5b]. But the temperature increases with increasing blowing parameter ($S<0$).

Next, we shall pay our attention to notice how the velocity slip and thermal slip parameters affects the velocity, stream function and the temperature profiles when the fluid flows in porous medium. The velocity $f(\eta)$ and stream function $\psi(\eta)$ for various values of the velocity slip parameter $\delta$ are depicted in Fig4.6b and Fig4.7b respectively. With the increasing values of $\delta$, the fluid velocity increases monotonically. Similar to non-porous medium, also in porous medium, due to the slip
the velocity of fluid adjacent to the plate has some positive value and accordingly the thickness of momentum boundary layer decreases. The dimensionless stream function also increases with $\delta$ [Fig 4.7b]. Fig 4.8b exhibits the dimensionless temperature profiles $\theta(\eta)$ for different values of $\delta$. From this figure it is observed that the temperature decreases with the increase in velocity slip parameter $\delta$. Also, near the plate, the enhanced velocity due to slip causes the increase in heat transfer. In Fig 4.9b, the effects of thermal slip parameter on temperature distribution is displayed. As the thermal slip increases, less heat is transferred from the plate to the fluid and hence the temperature decreases. As the momentum equation is independent of $\theta$, no effect of thermal slip parameter $\beta$ on the velocity profiles is noticed.

Skin-friction coefficient is one of the interesting physical quantities in evaluating the viscous stress acting on the surface of the plate. Skin-friction coefficient $f'(0)$ is plotted against the slip parameter $\delta$ for several values of the permeability parameter $k^*$ in Fig 4.10b. It is apparent that the skin-friction coefficient decreases rapidly and approaches zero as the slip starts to increase and the permeability parameter affects conversely i.e. with increasing $k^*$ it increases. In Fig 4.11b and Fig 4.12b, the negative value of temperature gradient at the plate $-\theta'(0)$ which is proportional to the rate of heat transfer from the plate is plotted against $\delta$ and $\beta$ respectively for the same values of $k^*$. The rate of heat transfer increases with the increase of velocity slip and also with the increasing permeability parameter but it decreases with $\beta$. The negative value of $\theta'(0)$ physically explains that there is heat flow from the plate to the ambient fluid.
Fig 4.1b Physical Model of the slip flow through porous medium.

Fig 4.2b Velocity profiles $f(\eta)$ for various values of $k^*$ with slip and no-slip boundary conditions.
Fig 4.3b Temperature profiles $\theta(\eta)$ for various values of $k^*$ with slip and no-slip boundary conditions.

Fig 4.4b Velocity profiles $f(\eta)$ for various values of $S$ with slip and no-slip boundary conditions.
Fig 4.5b Temperature profiles $\theta(\eta)$ for various values of $S$ with slip and no-slip boundary conditions.

Fig 4.6b Velocity profiles $f(\eta)$ for various values of $\delta$. 
Fig4.7b Shear stress profiles $f'(\eta)$ for various values of $\delta$.

Fig4.8b Temperature profiles $\theta(\eta)$ for various values of $\delta$. 

$k^* = 0.2, S = 0.2, Pr = 0.3, \beta = 0.1$
Fig 4.9b Temperature profiles $\theta(\eta)$ for various values of $\beta$.

Fig 4.10b Skin-friction coefficient $f''(0)$ against $\delta$ for various values of $k^*$. 

$k^* = 0.2, S = 0.2, Pr = 0.3, \delta = 0.1$

$\beta = 0, 0.1, 0.3, 0.6, 1$

$k^* = 0, 0.1, 0.2, 0.4$

$S = 0.2, Pr = 0.3, \beta = 0.1$
Fig 4.11b Temperature gradient at the plate $-\theta'(0)$ against $\delta$ for various values of $k^*$. 

Fig 4.12b Temperature gradient at the plate $-\theta'(0)$ against $\beta$ for various values of $k^*$. 

$S = 0.2, Pr = 0.3, \beta = 0.1$
4.1c Introduction

The thermal buoyancy generated due to heating/cooling of a vertical plate has a large impact on flow and heat transfer characteristics. The importance of mixed convective phenomenon is increasing day by day due to the enhanced concern in science and technology about buoyancy induced motions in the atmosphere, bodies in water, quasi-solid bodies such as earth etc. combined forced and natural convection over a flat plate has been widely studied from both theoretical and experimental standpoint over the past few decades. Early investigations mainly involved in finding the similarity characteristics within the boundary layer framework. The effect of a magnetic field on free convection heat transfer on isothermal vertical plate was discussed by Sparrow and Cess (1961) & Gupta (1963) studied laminar free convection flow of an electrically conducting fluid past a vertical plate with uniform surface heat flux and variable wall temperature in presence of a magnetic field. Afzal and Hussain (1984) discussed the mixed convection over a horizontal plate. Yao (1987) investigated the two-dimensional mixed convection along a flat plate. Hossain and Takhar (1996) determined the effect of radiation on mixed convection along a vertical plate with uniform free stream velocity and surface temperature. Aydin and Kaya (2007) analyzed the mixed convection flow of a viscous dissipating fluid about a vertical flat plate. Cao and Baker (2009) presented local non-similar solutions to the boundary layer equations for mixed convection over a vertical isothermal plate.

In best of our knowledge, no previous study has been undertaken to examine the simultaneous effects of velocity slip and thermal slip upon mixed convection in case of boundary layer flow developed over a flat plate. In the last part, we investigate the slip effect on boundary layer mixed convection flow from a vertical plate. A self-similar set of equations are obtained and then solved numerically using shooting method. Computed numerical results for various values of physical parameters are plotted and analyzed.
4.2c Mathematical formulation of the problem

Let us consider the steady two-dimensional laminar mixed convective slip flow of a viscous incompressible fluid over a vertical plate. The physical sketch of the flow dynamics is given in Fig4.1c. Using boundary layer approximations, the equations for the flow and the temperature are written in usual notation as

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (4.1c)
\]

\[
\frac{u}{\partial x} + \frac{v}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} + g \beta' (T - T_\infty) \quad (4.2c)
\]

and

\[
\frac{\partial T}{\partial x} + \frac{\partial T}{\partial y} = \frac{\kappa}{\rho c_p} \frac{\partial^2 T}{\partial y^2}, \quad (4.3c)
\]

where \( u \) and \( v \) are velocity components in \( x \)- and \( y \)-directions respectively, \( \rho \) is the fluid density, \( \mu \) the coefficient of fluid viscosity, \( \nu (= \mu / \rho) \) is the kinematic fluid viscosity, \( \beta' \) is the volumetric coefficient of thermal expansion, \( g \) is the acceleration due to gravity, \( T \) is the temperature, \( T_\infty \) is the free stream temperature, \( \kappa \) is the thermal conductivity of the fluid, \( c_p \) is the specific heat.

The appropriate boundary conditions for the velocity components with partial slip and the temperature are given by

\[
u = L_1 (\partial u / \partial y), \quad v = 0 \text{ at } y = 0; \quad u \to U_\infty \text{ as } y \to \infty, \quad (4.4c)
\]

and

\[
T = T_\infty + D_1 (\partial T / \partial y) \text{ at } y = 0; \quad T \to T_\infty \text{ as } y \to \infty. \quad (4.5c)
\]

Here \( L_1 = L_0 (Re_{\infty})^{1/2} \) is the velocity slip factor and \( D_1 = D_0 (Re_{\infty})^{1/2} \) is the thermal slip factor with \( L_0 \) and \( D_0 \) being the initial values of velocity and thermal slip factors having same dimension of length and \( Re_{\infty} \) being the local Reynolds number and \( Re_{\infty} = U_\infty / \nu, \quad U_\infty \) the free stream velocity, \( T_{\infty} = T_{\infty} + T_0 / x \) is the variable temperature of the plate and \( T_0 \) is a constant that measures the rate of temperature increase along the plate.

Now, we introduce the same similarity transformations (see 4.10a and 4.6b):

\[
\psi = \sqrt{U_\infty \nu x} f(\eta), \quad T = T_\infty + (T_{\infty} - T_\infty) \vartheta(\eta) \text{ and } \eta = \frac{x}{\sqrt{Re_{\infty}}}, \quad (4.6c)
\]

where \( \psi \) is the stream function with \( u = \frac{\partial \psi}{\partial \eta} \) & \( v = -\frac{\partial \psi}{\partial x} \) and \( \eta \) is the similarity variable.

In view of relations in (4.6c) we finally obtain the following self-similar equations as
Boundary Layer Slip Flow and Heat Transfer over a Flat Plate

\[ f'' + \frac{1}{2} f' + \lambda \theta = 0 \quad (4.7c) \]

and \( \theta'' + Pr(\frac{1}{2} f' + f'\theta) = 0 \), \( (4.8c) \)

where \( \lambda = g^\beta T_0/U_\infty^2 \) is the mixed convection parameter and \( Pr = \mu c_p \kappa \) is the Prandtl number.

The boundary conditions (4.4c) and (4.5c) reduce to the following forms:

\[ f(\eta) = 0, f'(\eta) = \delta f'\eta(\eta) \text{ at } \eta = 0; \quad f'(\eta) \to 1 \text{ as } \eta \to \infty \quad (4.9c) \]

and \( \theta(\eta) = 1 + \beta \theta'(\eta) \text{ at } \eta = 0; \quad \theta'(\eta) \to 0 \text{ as } \eta \to \infty, \quad (4.10c) \)

where \( \delta = L_v U_\infty/\nu \) is the velocity slip parameter and \( \beta = D_0 U_\infty/\nu \) is the thermal slip parameter.

### 4.3c Numerical method for solution

The nonlinear coupled differential equations (4.7c) and (4.8c) along with the boundary conditions (4.9c) and (4.10c) form a two point boundary value problem (BVP) and are solved using shooting method, by converting into an initial value problem (IVP) as:

\[ f' = p, \quad p' = q, \quad q' = -\frac{1}{2} f q - \lambda \theta \quad (4.11c) \]

and \( \theta' = z, \quad z' = -Pr(\frac{1}{2} f z + p \theta) \quad (4.12c) \)

with

\[ f(0) = 0, \quad p(0) = \delta q(0), \quad \theta(0) = 1 + \beta z(0). \quad (4.13c) \]

### 4.4c Results and discussion

The numerical computations are executed for several values of dimensionless parameters involved in the equations viz. the mixed convection parameter (\( \lambda \)), the velocity slip parameter (\( \delta \)), the thermal slip parameter (\( \beta \)) and the Prandtl number (\( Pr \)). To illustrate the computed results, some figures are plotted and physical explanations are given.

The variations of the dimensionless velocity, shear stress and temperature profiles for different values of the mixed convection parameter \( \lambda \) in presence of slip and in the absence of slip at the boundary are represented in Fig4.2c, Fig4.3c and
Fig4.4c respectively. For the buoyancy aiding flow ($\lambda>0$), increase in mixed convection parameter will increase the velocity inside the boundary layer due to favourable buoyancy effects in both slip and no-slip cases (Fig4.2c) and consequently heat transfer rate from the plate will increase and the shear stress profile $f'(\eta)$ though initially increases but it decreases for large $\eta$. From Fig4.4c, it is found that for the increase of $\lambda$ the temperature distribution is suppressed in case of slip as well as no-slip condition and consequently the thermal boundary layer thickness becomes thinner. Velocity overshoot and temperature overshoot are noted near the plate.

Next, we discuss the characteristics of the velocity, shear stress and the temperature profiles for mixed convection when slip occurs at the boundary. For various values of slip parameter $\delta$, the velocity, shear stress and temperature profiles are depicted in Fig4.5c, Fig4.6c and Fig4.7c respectively. Both, the velocity $f(\eta)$ and shear stress $f'(\eta)$ profiles exhibit opposite character before and after some points. With increasing values of $\delta$, the velocity increases upto $\eta=2.72$ and then decreases. Also, the dimensionless shear stress decreases upto $\eta=3.52$ and after that it increases. As the slip parameter increases in magnitude, permitting more fluid to slip past the plate, the flow gets accelerated near the plate, for distances away from the plate the flow gets decelerated. Opposite is the case for shear stress which is quite natural. Such type of behaviour is due to the combined effects of mixed convection and velocity slip parameters. From Fig4.7c, it is observed that the temperature decreases significantly with the increase in slip parameter $\delta$ and also the thickness of the thermal boundary layer reduces.

Due to mixed convection flow, the Prandtl number $Pr$ also affects the velocity and shear stress profiles in addition to the temperature distribution. Fig4.8c, Fig4.9c and Fig4.10c demonstrate the effects of the $Pr$ to the velocity, shear stress and temperature distributions. The velocity $f(\eta)$ along the plate decreases with increase in $Pr$ for both slip and no-slip cases and the profile $f'(\eta)$ decreases upto a point, then increases. In both cases, as Prandtl number increases, the temperature at every location in the thermal boundary layer, decreases. The thickness of the boundary layer decreases as Prandtl number increases as in the classical case of an isothermal flat
plate with no-slip. An increase in Prandtl number means an increase of fluid viscosity which causes a decrease in the flow velocity and the temperature decreases.

Variations of velocity, shear stress and temperature due to thermal slip parameter are presented in Fig4.11c, Fig4.12c and Fig4.13c. As the thermal slip parameter increases, the velocity increases and shear stress at first increases and then after a point ($\eta = 1.26$) it decreases. This is due to the combined effects of mixed convection and velocity slip. From Fig4.13c we observed that with the increasing thermal slip, the temperature rises above the plate temperature $T_w$ (i.e. temperature overshoot is noted) before it decays to the ambient temperature $T_o$. In these cases the plate temperature becomes lower as the distance $x$ from the origin increases and the heat transfer is therefore directed from the fluid to the plate, rather than in the common case, from the plate to the fluid i.e. $\theta(0) > 0$ in these cases.

Fig4.14c and Fig4.15c demonstrate the effects of velocity slip and thermal slip on skin friction coefficient and temperature gradient at the plate. Skin friction coefficient and plate temperature gradient are found to increase with the increasing thermal slip parameter as well as with the increasing velocity slip parameter. As the thermal slip (parameter) increases an amount of heat is transferred from the fluid to the plate and the temperature gradient at the plate increases in this case. The heat transfer is augmented greater than the no-slip value as a result of the velocity slip.
Fig 4.1c Physical Model of the mixed convective slip flow.

Fig 4.2c Velocity profiles $f(\eta)$ for several values of $\lambda$ with slip and without slip.
**Fig 4.3c** Shear stress profiles $f''(\eta)$ for several values of $\lambda$ with slip and without slip.

**Fig 4.4c** Temperature profiles $\theta(\eta)$ for several values of $\lambda$ with slip and without slip.
Boundary Layer Slip Flow and Heat Transfer over a Flat Plate

Fig 4.5c Velocity profiles $f(\eta)$ for several values of $\delta$.

Fig 4.6c Shear stress profiles $f''(\eta)$ for several values of $\delta$. 
Fig 4.7c Temperature profiles $\theta(\eta)$ for several values of $\delta$.

Fig 4.8c Velocity profiles $f(\eta)$ for several values of $Pr$ with slip and without slip.
**Boundary Layer Slip Flow and Heat Transfer over a Flat Plate**

**Fig 4.9c** Shear stress profiles $f''(\eta)$ for several values of $Pr$ with slip and without slip.

**Fig 4.10c** Temperature profiles $\theta(\eta)$ for several values of $Pr$ with slip and without slip.
Fig4.11c Velocity profiles $f'\left(\eta\right)$ for several values of $\beta$.

Fig4.12c Shear stress profiles $f''\left(\eta\right)$ for several values of $\beta$. 

$\beta = 0, 0.3, 0.7, 1$

$\lambda = 0.2, \delta = 0.2, Pr = 0.5$
Fig4.13c Temperature profiles $\theta(\eta)$ for several values of $\beta$.

Fig4.14c Skin-friction coefficient $f''(0)$ against $\beta$ for various values of $\delta$. 
Fig4.15c Temperature gradient at the plate $\theta(0)$ against $\beta$ for various values of $\delta$. 

Legend: 
- $\delta = 0, 0.1, 0.2$ 
- $\lambda = 0.2$, $Pr = 0.5$