4.1. Introduction

Let F_q be the finite field with q elements, n be a positive integer with $\gcd(n,q) = 1$. A cyclic code $\mathcal{C}$ of length n over F_q is a linear subspace of F_q^n with the property that if $(a_0, a_1, \dots, a_{n-1}) \in \mathcal{C}$, then every cyclic shift $(a_{n-1}, a_0, \dots, a_{n-2})$ is in $\mathcal{C}$. Let $R_n = \frac{F_q[x]}{<x^n-1>}$. Then $F_q^n \cong R_n$ under the isomorphism $(a_0, a_1, \dots, a_{n-1}) \to a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$. Therefore, we can regard a cyclic code $\mathcal{C}$ as an ideal in R_n.

A minimal ideal in R_n is called an minimal cyclic code of length n over F_q. If $\mathcal{C}$ is a minimal cyclic code of length n over F_q and $v \in \mathcal{C}$, then the weight of v is defined to be the number of non-zero coordinates in v. We denote it by $wt(v)$. If A_w^n denotes the number of codewords of weight w in $\mathcal{C}$, then $A_0^{(n)}, A_1^{(n)}, \dots, A_n^{(n)}$ is called weight distribution of $\mathcal{C}$.

Ding [41] determined the weight distribution of $q-$ary minimal cyclic codes of length n provided $2 \le \frac{q^t-1}{n} \le 4$, $t = O_n(q) =$ multiplicative order of q modulo n. Sharma *et al*. [99, 100] computed the weight distribution of all minimal cyclic codes of length 2^m and p^m over F_q. Kumar *et al.* [67] computed the weight distribution of some minimal cyclic codes of length p^m and $2p^m$ by using different technique.

In this chapter, we determine the weight distribution of all minimal cyclic codes of length $2p^m$ over F_q, where $\gcd(2p,q) = 1$ and $m \ge 1$ is an integer. In Section 4.3 (Theorem 4.3.4-4.3.5) the weight distribution of $\mathbb{M}_{p^{m-j}}^{(2p^m)}$, for $1 \le j \le m$ are investigated. In Section 4.4 (Theorem 4.4.12) the weight distribution of $\mathbb{M}_1^{(2p^r)} (1 \le r \le m)$ is discussed. In Section 4.5, the weight distribution of minimal cyclic code of length 50 is obtained.

4.2. Cyclotomic Cosets Modulo $2p^m$

Let $S = \{0, 1, 2, \dots, 2p^m - 1\}$. For $a, b \in S$, say that $a \sim b$ if $a \cong bq^i (\text{mod } 2p^m)$ for some integer $i \ge 0$. This defines an equivalence relation on the set S. The equivalence classes due to this relation are called $q-$cyclotomic cosets modulo $2p^m$. The q – cyclotomic coset containing $s \in S$ is denoted by

$C_s = \{s, sq, sq^2, \dots, sq^{t_s-1}\}$, where t_s is the least positive integer such that $sq^{t_s} \equiv s \pmod{2p^m}$ and $|C_s|$ denotes the cardinality of C_s.

In this section, we describe the $q-$cyclotomic cosets modulo $2p^m$, where p and q are distinct odd primes and $o(q)_{2p^m} = \frac{\varphi(2p^m)}{d}$, d is a positive integer and φ is Euler's phi-function.

4.2.1. Theorem If p and q are odd primes such that $o(q)_{2p^m} = \varphi(2p^m)/d$, d is a positive integer, then $2(md+1)$ q- cyclotomic cosets (mod $2p^m$) are given by

(i) $C_0 = \{0\}$,

(ii) $C_{p^m} = \{p^m\}$.

For $0 \leq j \leq m-1, 0 \leq k \leq d-1$,

(iii) $C_{g^k p^j} = \{g^k p^j,\ g^k p^j q,\ g^k p^j q^2, \dots, g^k p^j q^{\frac{\varphi(2p^{m-j})}{d}-1}\}$,

(iv) $C_{2g^k p^j} = \{2g^k p^j,\ 2g^k p^j q,\ 2g^k p^j q^2, \dots, 2g^k p^j q^{\frac{\varphi(2p^{m-j})}{d}-1}\}$,

where g is primitive root modulo $2p^m$.

Proof. Trivial.

4.3. Weight Distribution of Minimal Cyclic Codes of Length $2p^m$

Definition 4.3.1. Let α be the primitive $2p^m$th root of unity in some extension of F_q. Then corresponding to the $q-$ cyclotomic coset C_s,

$$M_s^{(n)}(x) = \prod_{j \in C_s} (x - \alpha^j),$$

is called **minimal polynomial** of α^s over F_q.

Definition 4.3.2. Let $\mathbb{M}_s^{(2p^m)}$ be the minimal cyclic code of length $2p^m$ over F_q. It is well known that $\mathbb{M}_s^{(2p^m)}$ is the ideal in R_{2p^m} generated by $g(x) = \frac{x^{2p^m}-1}{M_s^{2p^m}(x)}$. Then $g(x)$ is called the **generating polynomial** of $\mathbb{M}_s^{(2p^m)}$.

Remark 4.3.3. If $C_{s_1}, C_{s_2}, \dots, C_{s_k}$ are all the distinct $q-$cyclotomic cosets modulo $2p^m$, then $\mathbb{M}_{s_1}^{(2p^m)}, \mathbb{M}_{s_2}^{(2p^m)}, \dots, \mathbb{M}_{s_k}^{(2p^m)}$ are precisely all the distinct minimal cyclic codes of length $2p^m$ over F_q.

Theorem 4.3.4. Let F_q be the finite field with q elements, p, q be two odd primes with gcd(p, q) = 1 and $m \geq 1$ be an integer. Let the multiplicative order of q modulo $2p^m$ is $\varphi(2p^m)$. Then

(i) The codes $\mathbb{M}_0^{(2p^m)}, \mathbb{M}_{p^m}^{(2p^m)}, \mathbb{M}_{g^k p^j}^{(2p^m)}$ and $\mathbb{M}_{2g^k p^j}^{(2p^m)}$, $0 \leq j \leq m-1, 0 \leq k \leq d-1$, are precisely all the distinct minimal cyclic codes of length $2p^m$ over F_q, where φ denote the Euler's Phi function.

(ii) All the nonzero codewords in $\mathbb{M}_0^{(2p^m)}$ and $\mathbb{M}_{p^m}^{(2p^m)}$ have weight $2p^m$.

(iii) The codes $\mathbb{M}_{g^k p^j}^{(2p^m)}$ and $\mathbb{M}_{2g^k p^j}^{(2p^m)}$ are equivalent to $\mathbb{M}_{p^j}^{(2p^m)}$ and $\mathbb{M}_{2p^j}^{(2p^m)}$ respectively, therefore they have same weight distribution.

Proof. (i) By Theorem 4.2.1, C_0, C_{p^m}, $C_{g^k p^j}$ and $C_{2g^k p^j}$ are all distinct $q-$cyclotomic cyclotomic cosets modulo $2p^m$. Therefore, by Remark 4.3.3 $\mathbb{M}_0^{(2p^m)}, \mathbb{M}_{p^m}^{(2p^m)}, \mathbb{M}_{g^k p^j}^{(2p^m)}$ and $\mathbb{M}_{2g^k p^j}^{(2p^m)}$, $0 \leq j \leq m-1, 0 \leq k \leq d-1$, are all the distinct minimal cyclic codes of length $2p^m$ over F_q.

(ii) By Definition 4.3.1, $x-1$ is minimal polynomial of $\mathbb{M}_0^{(2p^m)}$, therefore by Definition 4.3.2 ,

$$\frac{x^{2p^m}-1}{x-1} = 1 + x + x^2 + \cdots + x^{2p^m-1}.$$

is the generating polynomial of $\mathbb{M}_0^{(2p^m)}$. Thus every non$-$ zero codeword in $\mathbb{M}_0^{(2p^m)}$ has weight $2p^m$.

Now, $\mathbb{M}_{p^m}^{(2p^m)}$ is the minimal cyclic code corresponding to $q-$ cyclotomic coset C_{p^m}. Then by Definition 4.3.1, the minimal polynomial of α^{p^m} is $x-\alpha^{p^m}$, where α is primitive $2p^m$th root of unity.

Then, $\alpha^{p^m} = -1$.

By Definition 4.3.2, the generating polynomial of $\mathbb{M}_{p^m}^{(2p^m)}$ is

$$\frac{x^{2p^m}-1}{x+1} = -1 + x - x^2 + \cdots + x^{2p^m-1}.$$

Thus every non−zero codeword in $\mathbb{M}_{p^m}^{(2p^m)}$ has weight $2p^m$.

(iii) Trivial.

Theorem 4.3.5. (i) Let $1 \le j \le m$. The minimal cyclic code $\mathbb{M}_{p^{m-j}}^{(2p^m)}$ is the repetition code of the minimal cyclic code $\mathbb{M}_1^{(2p^j)}$ of length $2p^j$ corresponding to the $q-$ cyclotomic coset containing 1, repeated p^{m-j} times.

(ii) Let $w \ge 0$, then

$$A_w^{(2p^m)} = \begin{cases} 0, & \text{if } p^j \text{ does not divide } w; \\ A_{w'}^{2p^{m-j}}, & \text{if } w = 2p^j w', 0 \le w' \le 2p^{(m-j)}, \end{cases}$$

where $A_w^{2p^m}$ and $A_{w'}^{2p^{m-j}}$ denote the weight distribution of $\mathbb{M}_{p^j}^{(2p^m)}$ and $\mathbb{M}_1^{(2p^{m-j})}$ respectively.

Proof. Let α be the fixed $2p^m$th root of unity. By definition 4.3.2, the generating polynomial

polynomial of $\mathbb{M}_{p^{m-j}}^{(2p^m)}$ is $\frac{x^{2p^m}-1}{M_{p^{m-j}}^{2p^m}(x)}$,

where $M_{p^{m-j}}^{2p^m}(x) = \prod_{s\in C_{p^{m-j}}}(x-\alpha^s)$ and $C_{p^{m-j}}$ is cyclotomic coset modulo $2p^m$.

Now, $\frac{x^{2p^m}-1}{M_{p^{m-j}}^{2p^m}(x)} = \frac{x^{2p^m}-1}{\prod_{s\in C_{p^{m-j}}}(x-\alpha^s)}$

$$= \frac{\left(x^{(2p^j)^{p^{m-j}}}-1\right)}{\prod_{s\in C_{p^{m-j}}}(x-\alpha^s)}$$

$$= \frac{(x^{2p^j}-1)}{\prod_{s\in C_{p^{m-j}}}(x-\alpha^s)}\left(1 + x^{2p^j} + x^{4p^j} + \cdots + x^{(p^{m-j}-1)2p^j}\right).$$

For any $s \in C_{p^{m-j}}$, α^s are roots of $x^{2p^j} - 1$.

Consequently, $\prod_{s\in C_{p^{m-j}}}(x-\alpha^s)$ is an irreducible factor of $x^{2p^j} - 1$.

It is clear that, $\prod_{s\in C_{p^{m-j}}}(x-\alpha^s) = \prod_{s=0}^{\varphi(p^j)-1}\left(x-\alpha^{p^{m-j}l^s}\right)$. Let $\alpha^{p^{m-j}} = \beta$, then $\prod_{s=0}^{\varphi(p^j)-1}\left(x-\alpha^{p^{m-j}l^s}\right) = \prod_{s=0}^{\varphi(p^j)-1}\left(x-\beta^{l^s}\right)$, where β is the $2p^j$th root of unity

Similarly, the generating polynomial of $\mathbb{M}_1^{(2p^j)}$ is $\frac{x^{2p^j}-1}{M_1^{2p^j}(x)}$, where $M_1^{2p^j}(x) = \prod_{s\in C_1}(x-\beta^s)$, where β is the $2p^j$th root of unity

Also, $\prod_{s\in C_1}(x-\beta^s) = \prod_{s=0}^{\varphi(2p^j)-1}\left(x-\beta^{l^s}\right)$, where C_1 is cyclotomic coset modulo $2p^j$.

Consequently, $\prod_{s\in C_{2p^{m-j}}}(x-\alpha^s) = \prod_{s\in C_1}(x-\beta^s)$.

By the above discussion and Lemma 2.4.1, $\mathbb{M}_{2p^{m-j}}^{(2p^m)}$ is the repetition code of the minimal cyclic code $\mathbb{M}_1^{(2p^j)}$ of length $2p^j$ corresponding to the $q-$ cyclotomic coset containing 1, repeated p^{m-j} times.

(ii) Trivial.

Remark 4.3.6.

By Theorems 4.3.4 and 4.3.5, to discuss the weight distribution of all the minimal cyclic codes of length $2p^m$, it is sufficient to obtain the weight distribution of minimal cyclic code $\mathbb{M}_1^{(2p^j)}$ of length $2p^j$ corresponding to the $q-$ cyclotomic coset containing 1.

4.4. Weight Distribution of $\mathbb{M}_1^{(2p^r)} (1 \le r \le m)$

Case (i) The multiplicative order of q modulo $2p^m$ is $\varphi(2p^m)$.

Lemma 4.4.1. If the multiplicative order of q modulo $2p^m$ is $\varphi(2p^m)$, then the generating polynomial of $\mathbb{M}_1^{(2p^r)}$ is $x^{p^{r-1}(p+1)} + x^{p^r} - x^{p^{r-1}} - 1$ and the vectors

$e_{i+p^{r-1}(p+1)} + e_{i+p^r} - e_{i+p^{r-1}} - e_i, 1 \leq i \leq p^{r-1}(p-1)$ or $1 \leq i \leq \varphi(2p^r)$, constitute a basis of $\mathbb{M}_1^{(2p^r)}$ over F_q.

Proof. As multiplicative order of q modulo $2p^m$ is $\varphi(2p^m)$, therefore multiplicative order of q modulo $2p^r$ is $\varphi(2p^r)$, for $1 \leq r \leq m$.

Hence the $q-$ cyclotomic coset modulo $2p^r$ containing 1 is

$C_1 = \{1,\ q,\ q^2, \dots, q^{\varphi(2p^r)-1}\}$.

This is a reduced residue system modulo $2p^r$. Let α be a primitive $2p^r$th root of unity.

By Definition 4.3.2, the generating polynomial $g(x)$ of $\mathbb{M}_1^{(2p^r)}$ is $\frac{x^{2p^r}-1}{M_1^{2p^r}(x)}$, where $M_1^{2p^r}(x) = \prod_{\alpha \in C_1}(x - \alpha^j)$.

Now, we assert that $M_1^{2p^r}(x) = \frac{x^{2p^r}-1}{(x^{p^{r-1}}+1)(x^{p^r}-1)}$.

If α is primitive $2p^r$th root of unity, then α^j is again primitive $2p^r$th root of unity for each $j \in C_1$.

Note that $x^{2p^r} - 1 = (x^{p^r} - 1)(x^{p^r} + 1)$.

Since α is $2p^r$th root of unity, therefore $\alpha^{p^r} \neq 1$. So, α is a root of $(x^{p^r} + 1)$.

Note that, $(x^{p^r} + 1) = (x^{(p^{r-1})p} + 1)$

$$= (x^{p^{r-1}} + 1)(1 - x^{p^{r-1}} + x^{2p^{r-1}} - \dots + x^{(p-1)p^{r-1}})$$

Thus,

$$x^{2p^r} - 1 = (x^{p^r} - 1)(x^{p^{r-1}} + 1)(1 - x^{p^{r-1}} + x^{2p^{r-1}} - \dots + x^{(p-1)p^{r-1}}).$$

Consequently,

$$M_1^{2p^r}(x) = (1 - x^{p^{r-1}} + x^{2p^{r-1}} - \dots + x^{(p-1)p^{r-1}}).$$

Hence, $g(x) = (x^{p^r} - 1)(x^{p^{r-1}} + 1) = x^{p^{r-1}(p+1)} + x^{p^r} - x^{p^{r-1}} - 1.$

Now, by MacWilliams and Sloane [79] $\mathbb{M}_1^{(2p^r)}$ is the subspace of R_{2p^r} spanned by $g(x),\ xg(x), \dots, x^{(p-1)p^{r-1}-1}g(x)$.

But under the standard isomorphism $x^{i-1} \to e_i$ from R_{2p^r} to $F_q^{2p^r}$, $x^{i-1}g(x)$ corresponding to $e_{i+p^{r-1}(p+1)} + e_{i+p^r} - e_{i+p^{r-1}} - e_i$ for each i.

Remark 4.4.2.

Let V_i be the vector subspaces of $F_q^{2p^r}$ spanned by

$$e_{i+p^r+jp^{r-1}} + e_{i+p^r+(j-1)p^{r-1}} - e_{i+jp^{r-1}} - e_{i+(j-1)p^{r-1}}$$

for $1 \le i \le p^{r-1}$ and $1 \le j \le p-1$. Then by the above lemma,

$$\mathbb{M}_{\mathbf{1}}^{(\mathbf{2p^r})} \cong \boldsymbol{V_1} \oplus \boldsymbol{V_2} \oplus \dots \oplus \boldsymbol{V_{p^{r-1}}}.$$

Definition 4.4.3. A vector $v \in V_i$ is called nice vector if

$$v = \sum_{j=k}^{k+l} \alpha_j \left(e_{i+p^r+jp^{r-1}} + e_{i+p^r+(j-1)p^{r-1}} - e_{i+jp^{r-1}} - e_{i+(j-1)p^{r-1}}\right),$$

where $0 \ne \alpha_j \in F_q, k \ge 1, l \ge 0, k+l \le p-1$. The integer l is called the length of v denoted by $l(v)$, k is called initial point of v, denoted by $I(v)$ and $k+l$ is called the end point of v denoted by $E(v)$.

Definition 4.4.4. Let $v_1, v_2, \dots, v_t \in V_i$. We say that $v_1, v_2, \dots, v_t$ is a chain in V_i if each v_j, $1 \le j \le t$, is a nice vector and $I(v_j) \ge E(v_{j-1}) + 2$ for $2 \le j \le t$. Note that each vector $v \in V_i$ can be written as the sum of $v_1, v_2, \dots, v_t$ and $wt\left(\sum_{j=1}^t v_j\right) = \sum_{j=1}^t wt(v_j)$.

Remark 4.4.5. Any $v \in V_i$ can be written as $v = \sum_{j=1}^t v_j$, where $v_1, v_2, \dots, v_t$ is a chain in V_i and $wt\left(\sum_{j=1}^t v_j\right) = \sum_{j=1}^t wt(v_j)$.

Notations 4.4.6. Let Z denote the set of integers. For any $t, \lambda \in Z, t \ge 1$ and $\lambda \ge 2$, let

$B_t(\lambda) = \{(\lambda_1,\ \lambda_2, \dots, \lambda_t) \in Z^t : 2 \le \lambda_j \le p$ for all $j, \sum_{j=1}^t \lambda_j = \lambda\}$ and for any $(\lambda_1, \lambda_2, \dots, \lambda_t) \in B_t(\lambda)$, define $C_t(\lambda_1, \lambda_2, \dots, \lambda_t)$

$$= \left\{ (l_1, l_2, \dots, l_t) \in Z^t : l_j \geq \lambda_j - 2 \text{ for all } j, \sum_{j=1}^{t} l_j \leq p - 2t \right\}.$$

Given any $(l_1, l_2, \dots, l_t) \in C_t(\lambda_1, \lambda_2, \dots, \lambda_t)$, let

$A(\lambda_1, \lambda_2, \dots, \lambda_t; l_1, l_2, \dots, l_t) =$

$$a_{(l_1, l_2, \dots, l_t)} \binom{l_1}{\lambda_1 - 2} \binom{l_2}{\lambda_2 - 2} \cdots \binom{l_t}{\lambda_t - 2} (q-1)^t (q-2)^{\lambda - 2t} = \eta(\text{say}),$$

where

$$a_{(l_1, l_2, \dots, l_t)}$$

$$= \sum_{k_1=1}^{p - \sum_{i=1}^{t} l_i - 2t + 1} \; \sum_{k_2 = k_1 + l_1 + 2}^{p - \sum_{i=2}^{t} l_i - 2(t-1) + 1} \cdots \sum_{k_{t-1} = k_{t-2} + l_{t-2} + 2}^{p - \sum_{i=t-1}^{t} l_i - 3} \; \sum_{k_t = k_{t-1} + l_{t-1} + 2}^{p - l_t - 1} 1 .$$

Lemma 4.4.7.

(i) If $0 \neq v \in V_i$, then $4 \leq wt(v) \leq 2p$.

(ii) If $v \in V_i$ is nice vector of length l, then $4 \leq wt(v) \leq 2l + 4$.

Proof. (i) Let $v \in V_i$. Then

$$v = \sum_{j=1}^{p-1} \alpha_j \left(e_{i + p^r + j p^{r-1}} + e_{i + p^r + (j-1) p^{r-1}} - e_{i + j p^{r-1}} - e_{i + (j-1) p^{r-1}} \right)$$

$$= \alpha_1 \left(e_{i + p^r + p^{r-1}} + e_{i + p^r} - e_{i + p^{r-1}} - e_i \right)$$

$$+ \alpha_2 \left(e_{i + p^r + 2p^{r-1}} + e_{i + p^r + p^{r-1}} - e_{i + 2p^{r-1}} - e_{i + p^{r-1}} \right) + \cdots$$

$$+ \alpha_{p-2} \left(e_{i + p^r + (p-2) p^{r-1}} + e_{i + p^r + (p-3) p^{r-1}} - e_{i + (p-2) p^{r-1}} - e_{i + (p-3) p^{r-1}} \right)$$

$$+ \alpha_{p-1} \left(e_{i + p^r + (p-1) p^{r-1}} + e_{i + p^r + (p-2) p^{r-1}} - e_{i + (p-1) p^{r-1}} - e_{i + (p-2) p^{r-1}} \right)$$

$$= \alpha_1 \left(e_{i + p^r} - e_i \right) + \left\{ \alpha_1 \left(e_{i + p^r + p^{r-1}} - e_{i + p^{r-1}} \right) + \alpha_2 \left(e_{i + p^r + p^{r-1}} - e_{i + p^{r-1}} \right) \right\}$$

$$+ \cdots + \left\{ \alpha_{p-2} \left(e_{i + p^r + (p-2) p^{r-1}} - e_{i + (p-2) p^{r-1}} \right) + \alpha_{p-1} \left(e_{i + p^r + (p-2) p^{r-1}} - e_{i + (p-2) p^{r-1}} \right) \right\}$$

$$+\alpha_{p-1}\left(e_{i+p^r+(p-1)p^{r-1}} - e_{i+(p-1)p^{r-1}}\right)$$

$$= \alpha_1\left(e_{i+p^r} - e_i\right) + \alpha_{p-1}\left(e_{i+p^r+(p-1)p^{r-1}} - e_{i+(p-1)p^{r-1}}\right)$$

$$+(\alpha_1 + \alpha_2)\left(e_{i+p^r+(p-2)p^{r-1}} - e_{i+(p-2)p^{r-1}}\right) + \cdots +$$

$$(\alpha_{p-1} + \alpha_{p-2})\left(e_{i+p^r+(p-2)p^{r-1}} - e_{i+(p-2)p^{r-1}}\right)$$

$$= \alpha_1\left(e_{i+p^r} - e_i\right) + \alpha_{p-1}\left(e_{i+p^r+(p-1)p^{r-1}} - e_{i+(p-1)p^{r-1}}\right)$$

$$+\sum_{j=1}^{p-2}(\alpha_j + \alpha_{j+1})\left(e_{i+p^r+jp^{r-1}} - e_{i+jp^{r-1}}\right), \quad (1)$$

$\alpha_j \in F_q$. If $v \neq 0$, then at least one $\alpha_j \neq 0$.

Thus from (1), we have $wt(v) \geq 4$.

For maximum weight we assume $\alpha_j \neq 0$, for $j = 1, 2, \ldots, p-2$.

Thus from (1), we have $wt(v) \leq 2p$.

(ii) Let $v \in V_i$ is nice vector of length l , then by Definition 4.4.3,

$$v = \sum_{j=k}^{k+l} \alpha_j\left(e_{i+p^r+jp^{r-1}} + e_{i+p^r+(j-1)p^{r-1}} - e_{i+jp^{r-1}} - e_{i+(j-1)p^{r-1}}\right),$$

where $0 \neq \alpha_j \in F_q, k \geq 1, l \geq 0, k+l \leq p-1$.

Then,

$$v = \alpha_k\left(e_{i+p^r+(k-1)p^{r-1}} - e_{i+(k-1)p^{r-1}}\right) + \alpha_{k+l}\left(e_{i+p^r+(k+l)p^{r-1}} - e_{i+(k+l)p^{r-1}}\right)$$

$$+\sum_{j=k}^{k+l-1}(\alpha_j + \alpha_{j+1})\left(e_{i+p^r+jp^{r-1}} - e_{i+jp^{r-1}}\right). \quad (2)$$

Since v is nice vector, so $\alpha_j \neq 0$ and the sum in (2) has l terms, therefore $4 \leq wt(v) \leq 2l + 4$.

Lemma 4.4.8. If l, k, λ are integers satisfying $0 \leq l \leq p-1$, $1 \leq k \leq p-l-1$ and $2 \leq \lambda \leq l+2$, then the number of nice vectors in V_i is $\binom{l}{\lambda-2}(q-1)(q-2)^{\lambda-2}$.

Proof. For any nice vector $v \in V_i$ such that length of v is l and weight 2λ , then by equation (2),

$$v = \alpha_k\left(e_{i+p^r+(k-1)p^{r-1}} - e_{i+(k-1)p^{r-1}}\right)$$
$$+ \alpha_{k+l}\left(e_{i+p^r+(k+l)p^{r-1}} - e_{i+(k+l)p^{r-1}}\right)$$
$$+ \sum_{j=k}^{k+l-1}\left(\alpha_j + \alpha_{j+1}\right)\left(e_{i+p^r+jp^{r-1}} - e_{i+jp^{r-1}}\right),$$

where $\alpha_j \in F_q$ are non zero for $k \le j \le k + l$.

Now we observe that the weight of v is 2λ if and only if out of a total of l sums $(\alpha_j + \alpha_{j+1}), 0 \le j \le k + l - 1$, exactly $\lambda - 2$ are non zero. That is possible if and only if there exists $i_1, i_2, \dots, i_{\lambda-2} \le k + l - 1$ such that $(\alpha_{i_1} + \alpha_{i_2}) \ne 0, (\alpha_{i_2} + \alpha_{i_3}) \ne 0, \dots, (\alpha_{i_{\lambda-2}} + \alpha_{i_{k+l}}) \ne 0$ and $\alpha_j + \alpha_{j+1} = 0$, otherwise. We observe that the total number of choices of such a nice element v is $\binom{l}{\lambda - 2}(q-1)(q-2)^{\lambda-2}$.

Observation 4.4.9. In the above lemma the number of nice vectors is independent of the choice of the initial point.

Definition 4.4.10. For any integer $\lambda \ge 0$, define

$$N(\lambda) = \begin{cases} 1, & \text{if } \lambda = 0, \\ 0, & \text{if } 1 \le \lambda \le 3 \text{ or } \lambda \ge 2p + 1, \\ \sum_{t\ge1} \sum_{(\lambda_1,\lambda_2,\dots,\lambda_t)\in B_t(\lambda)} \sum_{(l_1,l_2,\dots,l_t)\in C_t(\lambda_1,\lambda_2,\dots,\lambda_t)} \eta, & \text{otherwise.} \end{cases}$$

Lemma 4.4.11. Let λ be an integer such that $2 \le \lambda \le p$. Then, for each $i, 1 \le i \le p^{r-1}$, the number of vectors in V_i having weight 2λ are exactly $N(\lambda)$.

Proof. Let $A_i(2\lambda)$ be the set of all codewords in V_i having weight 2λ. Let $W_i(\lambda_1, \lambda_2, \dots, \lambda_t; l_1, l_2, \dots, l_t)$ be the set of all $v \in V_i$ such that $v = \sum_{j=1}^{t} v_j$, $v_1, v_2, \dots, v_t$ is a chain in V_i and $wt(v_j) = 2\lambda_j, l(v_j) = l_j$ for $1 \le j \le t$. Then,

$$wt(v) = wt\left(\sum_{j=1}^{t} v_j\right) = \sum_{j=1}^{t} wt(v_j) = \sum_{j=1}^{t} 2\lambda_j = 2\lambda.$$

We claim that $A_i(2\lambda)$ is the disjoint union of $W_i(\lambda_1, \lambda_2, \dots, \lambda_t; l_1, l_2, \dots, l_t)$.

i.e.

$$A_i(2\lambda) = \bigcup_{t\geq 1} \bigcup_{(\lambda_1,\lambda_2,\ldots,\lambda_t)\in B_t(\lambda)} \bigcup_{(l_1,l_2,\ldots,l_t)\in C_t(\lambda_1,\lambda_2,\ldots,\lambda_t)} W_i(\lambda_1,\lambda_2,\ldots,\lambda_t; l_1,l_2,\ldots,l_t). \quad (3)$$

Let v be an arbitrary vector of W_i. Then by the above discussion $wt(v) = 2\lambda$. Consequently, $v \in A_i(2\lambda)$. Thus the union on right hand side is the sub set of $A_i(2\lambda)$.

Now, let v be an arbitrary element of $A_i(2\lambda)$, then $wt(v) = 2\lambda$.

By using Remark 4.4.5, we get $v = \sum_{j=1}^{t} v_j$, $v_1, v_2, \ldots, v_t$ is a chain in V_i and $wt(v_j) = 2\lambda_j, l(v_j) = l_j$, for $1 \leq j \leq t$. Then by Lemma 4.4.7, $4 \leq \lambda_j \leq 2p, l_j \geq \lambda_j - 2$ for all j. Also

$$\sum_{j=1}^{t} l_j = \sum_{j=1}^{t} \big(E(v_j)\big) - I(v_j))$$

$$= \sum_{j=2}^{t} \big(E(v_{j-1})\big) - I(v_j)) + E(v_t) - I(v_1).$$

As, $E(v_t) \leq p-1, I(v_1) \geq 1$, i.e.$-I(v_1) \leq -1$ and $\big(I(v_j) - E(v_{j-1})\big) \geq 2$,

i.e. $\big(E(v_{j-1}) - I(v_j)\big) \leq -2.$

Therefore,

$$\sum_{j=1}^{t} l_j \leq \sum_{j=2}^{t} -2 + p - 1 - 1 = p - 2t.$$

This implies that $(l_1, l_2, , \ldots, l_t) \in C_t(\lambda_1, \lambda_2, \ldots, \lambda_t)$ and $\in W_i(\lambda_1, \lambda_2, \ldots, \lambda_t; l_1, l_2, \ldots, l_t)$. It is clear that the union of right hand side of (3) is disjoint.

Now to evaluate $|W_i(\lambda_1, \lambda_2, \ldots, \lambda_t; l_1, l_2, \ldots, l_t)|$ we find out the number of chains $v_1, v_2, \ldots, v_t$ in V_i such that $wt(v_j) = 2\lambda_j, l(v_j) = l_j$ for all j. As $k_j = I(v_j)$. Then $k_1 \geq 1, k_t + l_t \leq p - 1$ and $k_{j-1} + l_{j-1} + 2 \leq k_j$ for $2 \leq j \leq t$.

For $j = 2,$ $\quad k_1 + l_1 + 2 \leq k_2,$ (4)

For $j = 3,$ $\quad k_2 + l_2 + 2 \leq k_3,$

implies $k_2 \leq k_3 - l_2 - 2.$

Using k_2 in (4), we get

$$k_1 + l_1 + 2 \leq k_3 - l_2 - 2,$$

implies $k_1 \leq k_3 - (l_1 + l_2) - 2.2$. (5)

For $j = 4$, $k_3 + l_3 + 2 \leq k_4$,

implies $k_3 \leq k_4 - l_3 - 2.$

Using k_3 in (5), we get

$$k_1 \leq k_4 - (l_1 + l_2 + l_3) - 2.3 \ . \quad (6)$$

Continuing in this way for $j = t$, we get

$$k_1 \leq k_t - (l_1 + l_2 + l_3 + \cdots + l_{t-1}) - 2(t-1). \quad (7)$$

But $k_t \leq p - 1 - l_t$ and $k_1 \geq 1$. Using (4) to (7) inequalities, we get

$$k_1 \leq p - 1 - (l_1 + l_2 + l_3 + \cdots + l_{t-1} + l_t) - 2(t-1).$$

implies

$$1 \leq k_1 \leq p - (l_1 + l_2 + l_3 + \cdots + l_{t-1} + l_t) - 2t + 1.$$

By the above discussion the number of choices for k_1 is

$$\sum_{k_1=1}^{p-\sum_{j=1}^{t} l_j - 2t+1} 1.$$

Similarly, the number of choices for initial point k_2 of v_2 is

$$\sum_{k_2=k_1+l_1+2}^{p-\sum_{j=2}^{t} l_j - 2(t-1)+1} 1.$$

Therefore, total number of choices for initial points of $v_1, v_2, \dots, v_t$ is

$$= \sum_{k_1=1}^{p-\sum_{i=1}^{t} l_i - 2t+1} \quad \sum_{k_2=k_1+l_1+2}^{p-\sum_{i=2}^{t} l_i - 2(t-1)+1} \cdots \sum_{k_{t-1}=k_{t-2}+l_{t-2}+2}^{p-\sum_{i=t-1}^{t} l_i - 3} \quad \sum_{k_t=k_{t-1}+l_{t-1}+2}^{p-l_t-1} 1\, .$$

By using Lemma 4.4.8, the number of nice vectors

v_j of length l_j weight λ_j and having a fixed initial point k_j is given by

$$\binom{l_j}{\lambda_j - 2}(q-1)(q-2)^{\lambda_j - 2} \text{ for each } j, 1 \le j \le t.$$

By using Notation 4.4.6, we get

$$|W_i(\lambda_1,\ \lambda_2, \dots, \lambda_t; l_1,\ l_2, \dots, l_t)| = \eta. \tag{8}$$

Using (3) and (8),

$$|A_i(2\lambda)| = N(2\lambda) \text{ for } 2 \le \lambda \le p.$$

Theorem 4.4.12. Let F_q be the finite field with q elements; p, q be two odd primes with gcd(p,q)= 1 and $m \ge 1$ be an integer. If the multiplicative order of q modulo $2p^m$, then the weight distribution $A_{2w}^{(2p^r)}$, $w \ge 0$, of the minimal cyclic code $\mathbb{M}_1^{(2p^r)}$is given by

$$A_{2w}^{(2p^r)} = \sum_{(w_1, w_2, \dots, w_{p^{r-1}})} \prod_{i=1}^{p^{r-1}} N(w_i), \text{ where } \sum_{i=1}^{p^{r-1}} w_i = w.$$

Proof. Let $A(2w)$ be the set of codewords in $\mathbb{M}_1^{(2p^r)}$ of weight $2w, w \ge 0$. By Remark 4.4.2, $\mathbb{M}_1^{(2p^r)} \cong V_1 \oplus V_2 \oplus \dots \oplus V_{p^{r-1}}$, where V_i is the vector subspace of $F_q^{2p^r}$ spanned by $e_{i+p^r+jp^{r-1}} + e_{i+p^r+(j-1)p^{r-1}} - e_{i+jp^{r-1}} - e_{i+(j-1)p^{r-1}}$ for $1 \le i \le p^{r-1}$ and $1 \le j \le p-1$.

Let x be any element of $\mathbb{M}_1^{(2p^r)}$ of weight $2w$. Then, by the above discussion x corresponds $v \in V_1 \oplus V_2 \oplus \dots \oplus V_{p^{r-1}}$ such that $v = \sum_{i=1}^{p^{r-1}} v_i$ and $wt(v_i) = 2w_i$, satisfying $w = \sum_{i=1}^{p^{r-1}} w_i$. To determine the number of elements x in $\mathbb{M}_1^{(2p^r)}$having weight $2w$, we have to determine the number of v_i in V_i such that $wt(v_i) = 2w_i$. By Lemma 4.4.11, the number of v_i having weight $2w_i$ in V_i is $N(2w_i)$ for each $i, 1 \le i \le p^{r-1}$.

If we fix w_i, satisfying $w = \sum_{i=1}^{p^{r-1}} w_i$ for each $i, 1 \le i \le p^{r-1}$. Then by the above discussion, the number of codewords of weight $2w_i$ is $\prod_{i=1}^{p^{r-1}} N(2w_i)$. Consequently, $A_{2w}^{(2p^r)} = \sum_{(w_1, w_2, \dots, w_{p^{r-1}})} \prod_{i=1}^{p^{r-1}} N(2w_i), w = \sum_{i=1}^{p^{r-1}} w_i$.

This completes the proof of the Theorem.

Case (ii) The multiplicative order of q modulo $2p^m$ is $p^d(0 \leq d \leq m-1)$.

Lemma 4.4.13. Let $o(q)_{2p} = t$ and $t = \frac{\varphi(2p)}{k}$, where k is any integer. If $q \equiv 1 \pmod{2p}$ and $2p^2$such that p does not divide $q-1$, then $o(q)_{2p^m} = tp^{m-1}$ for all $m \geq 1$.

Proof. As $q \equiv 1 \pmod{2p}$, $2p^2$and p does not divide $q-1$ gives $q^t = 1 + 2p\lambda$, where $2p$ does not divide λ. Then for any integer i, $q^{tp^i} = 1 + 2p^{i+1}\lambda_i$, where $2p$ does not divide λ_i.

Let $o(q)_{2p^m} = h_m$, then for $i = m-1, q^{tp^{m-1}} = 1 + 2p^m\lambda_{m-1}$ implies $q^{tp^{m-1}} \equiv 1 (\bmod 2p^m)$. Thus h_m divides tp^{m-1}.

Also $q^{h_m} \equiv 1(\bmod 2p^m)$ yields that $q^{h_m} \equiv 1(\bmod 2p)$, which gives t divides h_m. Therefore, let $h_m = tp^d$for some d, $0 \leq d \leq m-1$.

As $o(q)_{2p^m} = h_m$ implies $q^{h_m} \equiv 1(\bmod 2p^m)$ i.e. $q^{tp^d} \equiv 1(\bmod 2p^m)$. But $q^{tp^d} \equiv 1 + 2p^{d+1}\lambda_d$ which gives that $p^{d+1} \equiv 0(\bmod p^m)$ implies $m \leq d+1$. Thus $d = m-1$. Hence, $h_m = tp^{m-1}$.

Lemma 4.4.14. Let multiplicative order of q modulo $2p^m$ is $p^d(0 \leq d \leq m-1)$. Then

$$o(q)_{2p^r} = \begin{cases} 1, & \text{if } r \leq m-d; \\ p^{r-(m-d)}, & \text{if } r > m-d. \end{cases}$$

Proof. Let $o(q)_{2p^{m-d}} = t$, then by Lemma 4.4.13, we get $o(q)_{2p^m} = tp^d$. But $o(q)_{2p^m} = p^d$. This gives, $t = 1$.

Now if $r \leq m-d$, then $o(q)_{2p^r} = 1$.

If $r > m-d$, then by using Lemma 4.4.13, we get $o(q)_{2p^r} = p^{r-(m-d)}$.

Lemma 4.4.15. If $r > m-d$ and$o(q)_{2p^r} = p^{r-(m-d)}$, then

$$\sum_{j=0}^{2p^{(m-d)}-1} \beta^{j+1}\left(e_{i+jp^{r-(m-d)}} + e_{i+jp^{r-(m-d)}+p^r}\right), 1 \leq i \leq p^{r-(m-d)},$$

constitute a basis of $\mathbb{M}_1^{(2p^r)}$ over F_q, where β is primitive $2p^{r-(m-d)}$th root of unity in F_q.

Proof. By Lemma 4.4.14, $o(q)_{2p^r} = p^{r-(m-d)}$ if $r > m-d$. Then $q-$ cyclotomic coset modulo $2p^r$ containing 1 is $C_1 = \{1,\ q,\ q^2,\ \dots\ ,q^{p^{r-(m-d)-1}}\}$.

By Definition 4.3.1, $M_1^{2p^r}(x) = \prod_{j\in C_1}(x-\alpha^j)$. We observe that $\alpha, \alpha^q, \alpha^{q^2}$, $\dots\ ,\alpha^{q^{p^{r-(m-d)}}}$ are all the roots of the $x^{p^{r-(m-d)}} - \alpha^{p^{r-(m-d)}}$. We claim that $\alpha^{p^{r-(m-d)}} \in F_q$. Let $\alpha^{-p^{r-(m-d)}} = \beta$, then β is primitive $2p^{m-d}th$ root of unity which gives $\beta^{2p^{m-d}} = 1$ and $\beta^s \neq 1$ for $s < 2p^{m-d}$. But by Lemma 4.4.14, multiplicative order of q modulo $2p^{m-d}$is 1. So, $q \equiv 1 \pmod{2p^{m-d}}$.Thus $\beta^q = \beta \in F_q$.Therefore, minimal polynomial of α is $x^{p^{r-(m-d)}} - \alpha^{p^{r-(m-d)}}$ over F_q. By Definition 4.3.2, generating polynomial of $\mathbb{M}_1^{(2p^r)}$ is

$$g(x) = \frac{x^{2p^r}-1}{x^{p^{r-(m-d)}}-\alpha^{p^{r-(m-d)}}} = \Big(\beta + \beta^2 x^{p^{r-(m-d)}} + \beta^3 x^{2p^{r-(m-d)}} + \cdots + x^{(p^{(m-d)}-1)p^{r-(m-d)}}\Big)(x^{p^r}+1).$$

By MacWilliams and Sloane [79], $\mathbb{M}_1^{(2p^r)}$ is a vector subspace of R_{2p^r} and spanned by $g(x), xg(x), \dots, x^{p^{r-(m-d)}}g(x)$. Since $R_{2p^r} \cong F_q^{2p^r}$ under the standard isomorphism, then $x^{i-1}g(x)$ corresponds to $\sum_{j=0}^{2p^{(m-d)}-1}\beta^{j+1}\Big(e_{i+jp^{r-(m-d)}} + e_{i+jp^{r-(m-d)}+p^r}\Big)$, $1 \le i \le p^{r-(m-d)}$, which completes the proof.

Theorem 4.4.16. Let multiplicative order of q modulo $2p^m$ is $p^d (0 \le d \le m-1)$.

(i) If $r \le m-d$, then weight of each non-zero codeword in $\mathbb{M}_1^{(2p^r)}$ is $2p^r$.

(ii) $r > m-d$, the weight distribution $A_w^{(2p^r)}, w \ge 0$ of $\mathbb{M}_1^{(2p^r)}$is given by

$$A_w^{(2p^r)} = \begin{cases} 0, & \text{if } 2p^{(m-d)} \text{does not divide } w; \\ \binom{p^{r-(m-d)}}{w'}(q-1)^{w'}, & \text{if } w = 2p^{(m-d)}w', 0 \le w' \le p^{r-(m-d)}. \end{cases}$$

Proof. (i) If $r \le m-d$, by Lemma 4.4.14, $o(q)_{2p^r} = 1$. Let α be a primitive root of unity in some extension of F_q .Then $\alpha^{2p^r} = 1$ and $q \equiv 1 (\text{mod}\ 2p^r)$. This implies

$q-1=\lambda 2p^r$, where λ be any scalar. Therefore, $1=\alpha^{2p^r\lambda}=\alpha^{q-1}$ i.e. $\alpha\in F_q$. The minimal polynomial of α over F_q is $x-\alpha$. Thus generating polynomial of $\mathbb{M}_1^{(2p^r)}$ is $\frac{x^{2p^r}-1}{x-\alpha}=\alpha^{2p^r-1}+\alpha^{2p^r-2}x+\alpha^{2p^r-3}x^2+\cdots+\alpha x^{2p^r-2}+x^{2p^r-1}$. Consequently, every non-zero codeword of $\mathbb{M}_1^{(2p^r)}$ is a scalar multiple of $\alpha^{2p^r-1}+\alpha^{2p^r-2}x+\alpha^{2p^r-3}x^2+\cdots+\alpha x^{2p^r-2}+x^{2p^r-1}$. This gives that the only possible nonzero weight in $\mathbb{M}_1^{(2p^r)}$ is $2p^r$.

(iii) If $r>m-d$, by Lemma 4.4.15, then basis of $\mathbb{M}_1^{(2p^r)}$ is $\sum_{j=0}^{2p^{(m-d)}-1}\beta^{j+1}\left(e_{i+jp^{r-(m-d)}}+e_{i+jp^{r-(m-d)}+p^r}\right), 1\le i\le p^{r-(m-d)}$.

Let $c\in\mathbb{M}_1^{(2p^r)}$,

then $c=\sum_{i=1}^{p^{r-(m-d)}}\sum_{j=0}^{2p^{(m-d)}-1}\alpha_i\beta^{j+1}\left(e_{i+jp^{r-(m-d)}}+e_{i+jp^{r-(m-d)}+p^r}\right)$, $\alpha_i\in F_q$. It is easy to see that $wt(c)=2p^{m-d}w'$, where $w'$is the number of nonzero α_i's. But total number of α_i's is $p^{r-(m-d)}$, so we can choose w' non-zero α_i's in $\binom{p^{r-(m-d)}}{w'}$ ways.

Consequently,

$$A_w^{(2p^r)}=\begin{cases}0, & \text{if } 2p^{(m-d)} \text{ does not divide } w;\\ \binom{p^{r-(m-d)}}{w'}(q-1)^{w'}, & \text{if } w=2p^{(m-d)}w', 0\le w'\le p^{r-(m-d)}.\end{cases}$$

4.5. Some examples.

Example 4.5.1. The weight distribution of length 50.

Let $q=13, p=5$ and $m=2$, then $o(13)_{50}=20=\varphi(50)$.

Using Notations 4.4.6, for $t=1$, we get

$$a_{(l_1)}=\sum_{k_1=1}^{5-l_1-1}1=(4-l_1).$$

For $t=2$,

$$a_{(l_1,l_2)}=\sum_{k_1=1}^{5-(l_1+l_2)-3}\sum_{k_2=k_1+l_1+2}^{5-l_2-1}1=\frac{(3-l_1-l_2)(2-l_1-l_2)}{2}.$$

By Definition 4.4.10, we have

$$N(2) = \sum_{l_1=0}^{3} a_{(l_1)} \binom{l_1}{0} (13-1)$$

$$= 12 \sum_{l_1=0}^{3} (4 - l_1)$$

$$= 120,$$

$$N(3) = \sum_{l_1=1}^{3} a_{(l_1)} \binom{l_1}{1} (13-1)\,(13-2)$$

$$= 132 \sum_{l_1=1}^{3} (4 - l_1) l_1$$

$$= 1320,$$

$$N(4) = \sum_{l_1=2}^{3} a_{(l_1)} \binom{l_1}{2} (13-1)\,(13-2)^2$$

$$+ \sum_{\substack{l_1 \geq 0 \\ l_2 \geq 0}}^{l_1+l_2 \leq 1} a_{(l_1,l_2)} \binom{l_1}{0} \binom{l_2}{0} (13-1)^2$$

$$= 1452 \sum_{l_1=2}^{3} (4 - l_1) \binom{l_1}{2} + 144 \sum_{l_2=0}^{1} \frac{(3 - l_1 - l_2)(2 - l_1 - l_2)}{2}$$

$$+144(1)$$

$$= 7980,$$

$$N(5) = \sum_{l_1=3}^{3} a_{(l_1)} \binom{l_1}{3} (13-1)(13-2)^3$$

$$+ \sum_{\substack{l_1 \geq 1 \\ l_2 \geq 0}}^{l_1+l_2 \leq 1} a_{(l_1,l_2)} \binom{l_1}{1} \binom{l_2}{0} (13-1)^2 (13-2)$$

$$+\sum_{\substack{l_1\geq 0\\ l_2\geq 1}}^{l_1+l_2\leq 1} a_{(l_1,l_2)}\binom{l_1}{0}\binom{l_2}{1}(13-1)^2(13-2)$$

$$= 19140.$$

Now the weight distribution of minimal cyclic codes $\mathbb{M}_1^{(50)}$ are given below:

$A_0^{(50)} = N(0) = 1,$

$A_2^{(50)} = 0 = A_k^{(50)}$, k is odd,

$A_4^{(50)} = \frac{5!}{4!}N(2) = 600,$

$A_6^{(50)} = \frac{5!}{4!}N(3) = 6600,$

$A_8^{(50)} = \frac{5!}{4!}N(4) + \frac{5!}{3!\,2!}N(2)^2 = 183900,$

$A_{10}^{(50)} = \frac{5!}{4!}N(5) + \frac{5!}{3!}N(2)N(3) = 3263700,$

$A_{12}^{(50)} = \frac{5!}{3!\,2!}N(3)^2 + \frac{5!}{3!}N(2)N(4) + \frac{5!}{3!\,2!}N(2)^3 = 53856000,$

$A_{14}^{(50)} = \frac{5!}{3!}N(2)N(5) + \frac{5!}{3!}N(3)N(4) + \frac{5!}{2!\,2!}N(2)^2N(3) = 826848000,$

$$A_{16}^{(50)} = \frac{5!}{3!\,2!}N(4)^2 + \frac{5!}{3!}N(5)N(3) + \frac{5!}{2!\,2!}N(2)N(3)^2 + \frac{5!}{2!\,2!}N(2)^2N(4) + \frac{5!}{4!}N(2)^4 = 11898900000,$$

$$A_{18}^{(50)} = \frac{5!}{3!}N(4)N(5) + \frac{5!}{2!\,2!}N(2)^2N(5) + \frac{5!}{2!}N(2)N(3)N(4) + \frac{5!}{2!\,3!}N(3)^3 + \frac{5!}{3!}N(2)^3N(3) = 155784024000,$$

$$A_{20}^{(50)} = \frac{5!}{2!\,3!}N(5)^2 + \frac{5!}{2!\,2!}N(2)N(4)^2 + \frac{5!}{2!}N(2)N(3)N(5) + \frac{5!}{2!\,2!}N(3)^2N(4) + \frac{5!}{2!\,2!}N(2)^2N(3)^2 + \frac{5!}{5!}N(2)^5 + \frac{5!}{3!}N(2)^3N(4) = 1885442841000,$$

$$A_{22}^{(50)} = \frac{5!}{2!\,2!}N(3)^2N(5) + \frac{5!}{3!}N(2)^3N(5) + \frac{5!}{2!}N(2)N(4)N(5) + \frac{5!}{2!\,2!}N(3)N(4)^2 + \frac{5!}{2!}N(2)^2N(3)N(4) + \frac{5!}{3!}N(2)N(3)^3 + \frac{5!}{4!}N(2)^4N(3)$$

$$= 21272945760000,$$

$$A_{24}^{(50)} = \frac{5!}{2!\,2!}N(2)N(5)^2 + \frac{5!}{2!}N(2)^2N(3)N(5) + \frac{5!}{2!}N(3)N(4)N(5) + \frac{5!}{2!\,3!}N(4)^3 + \frac{5!}{2!\,2!}N(2)^2N(4)^2 + \frac{5!}{2!}N(2)N(3)^2N(4) + \frac{5!}{4!}N(2)^4N(4) + \frac{5!}{4!}N(3)^4 + \frac{5!}{2!\,3!}N(2)^3N(3)^2 = 221509483900000,$$

$$A_{26}^{(50)} = \frac{5!}{2!\,2!}N(3)N(5)^2 + \frac{5!}{2!\,2!}N(4)^2N(5) + \frac{5!}{2!}N(2)^2N(4)N(5) + \frac{5!}{2!}N(2)N(3)^2N(5) + \frac{5!}{4!}N(2)^4N(5) + \frac{5!}{2!}N(2)N(3)N(4)^2 + \frac{5!}{3!}N(3)^3N(4) + \frac{5!}{3!}N(2)^3N(3)N(4) + \frac{5!}{2!\,3!}N(2)^2N(3)^3$$

$$= 2110528308000000,$$

$$A_{28}^{(50)} = \frac{5!}{2!\,2!}N(4)N(5)^2 + \frac{5!}{2!\,2!}N(2)^2N(5)^2 + 5!\,N(2)N(3)N(4)N(5) + \frac{5!}{3!}N(3)^3N(5) + \frac{5!}{3!}N(2)^3N(3)N(5) + \frac{5!}{3!}N(2)N(4)^3 + \frac{5!}{2!\,2!}N(3)^2N(4)^2 + \frac{5!}{2!\,3!}N(2)^3N(4)^2 + \frac{5!}{2!\,2!}N(2)^2N(3)^2N(4) + \frac{5!}{4!}N(2)N(3)^4 = 18379729270000000,$$

$$A_{30}^{(50)} = \frac{5!}{2!\,3!}N(5)^3 + \frac{5!}{2!}N(2)N(3)N(5)^2 + \frac{5!}{2!}N(2)N(4)^2N(5) + \frac{5!}{2!}N(3)^2N(4)N(5) + \frac{5!}{3!}N(2)^3N(4)N(5) + \frac{5!}{2!\,2!}N(2)^2N(3)^2N(5) + \frac{5!}{3!}N(3)N(4)^3 + \frac{5!}{2!\,2!}N(2)^3N(3)N(4)^2 + \frac{5!}{3!}N(2)N(3)^3N(4)$$

$$+N(3)^5 = 4467026317000000000,$$

$$A_{32}^{(50)} = \frac{5!}{2!}N(2)N(4)N(5)^2 + \frac{5!}{2!\,2!}N(3)^2N(5)^2 + \frac{5!}{3!\,2!}N(2)^3N(5)^2$$
$$+\frac{5!}{2!}N(3)N(4)^2N(5) + \frac{5!}{2!}N(2)^2N(3)N(4)N(5) + \frac{5!}{3!}N(2)N(3)^3N(5)$$
$$+\frac{5!}{4!}N(4)^4 + \frac{5!}{2!\,3!}N(2)^2N(4)^3 + \frac{5!}{2!\,2!}N(2)N(3)^2N(4)^2$$
$$+\frac{5!}{4!}N(3)^4N(4) = 950057674100000000,$$

$$A_{34}^{(50)} = \frac{5!}{3!}N(2)N(5)^3 + \frac{5!}{2!}N(3)N(4)N(5)^2 + \frac{5!}{2!\,2!}N(2)^2N(3)N(5)^2$$
$$+\frac{5!}{3!}N(4)^3N(5) + \frac{5!}{2!\,2!}N(2)^2N(4)^2N(5) + \frac{5!}{2!}N(2)N(3)^2N(4)N(5)$$
$$+\frac{5!}{4!}N(3)^4N(5) + \frac{5!}{3!}N(4)^3N(2)N(3) + \frac{5!}{2!\,3!}N(3)^3N(4)^2$$
$$= 6459511779000000000,$$

$$A_{36}^{(50)} = \frac{5!}{3!}N(3)N(5)^3 + \frac{5!}{2!\,2!}N(4)^2N(5)^2 + \frac{5!}{2!\,2!}N(2)^2N(4)N(5)^2$$
$$+\frac{5!}{2!\,2!}N(2)N(3)^2N(5)^2 + \frac{5!}{2!}N(2)N(3)N(4)^2N(5) + \frac{5!}{3!}N(3)^3N(4)N(5)$$
$$+\frac{5!}{4!}N(2)N(4)^4 + \frac{5!}{2!\,3!}N(3)^2N(4)^3$$
$$= 34342949340000000000,$$

$$A_{38}^{(50)} = \frac{5!}{3!}N(3)N(5)^3 + \frac{5!}{2!\,2!}N(4)^2N(5)^2 + \frac{5!}{2!\,2!}N(2)^2N(4)N(5)^2$$
$$+\frac{5!}{2!\,2!}N(2)N(3)^2N(5)^2 + \frac{5!}{2!}N(2)N(3)N(4)^2N(5)$$
$$+\frac{5!}{3!}N(3)^3N(4)N(5) + \frac{5!}{4!}N(2)N(4)^4 + \frac{5!}{2!\,3!}N(3)^2N(4)^3$$
$$= 34342949340000000000,$$

$$A_{40}^{(50)} = \frac{5!}{4!}N(5)^4 + \frac{5!}{3!}N(2)N(3)N(5)^3 + \frac{5!}{2!\,2!}N(2)N(4)^2N(5)^2$$

$$+\frac{5!}{2!\,2!}N(3)^2N(4)N(5)^2 + \frac{5!}{3!}N(3)N(4)^3N(5) + \frac{5!}{5!}N(4)^5$$

$$= 548815311300000000000,$$

$$A_{42}^{(50)} = \frac{5!}{3!}N(2)N(4)N(5)^3 + \frac{5!}{2!\,3!}N(3)^2N(5)^3 + \frac{5!}{2!\,2!}N(3)N(4)^2N(5)^2$$

$$+\frac{5!}{4!}N(4)^4N(5) = 1568358033000000000000,$$

$$A_{44}^{(50)} = \frac{5!}{4!}N(2)N(5)^4 + \frac{5!}{3!}N(3)N(4)N(5)^3 + \frac{5!}{2!\,3!}N(4)^3N(5)^2$$

$$= 3419326550000000000000,$$

$$A_{46}^{(50)} = \frac{5!}{4!}N(3)N(5)^4 + \frac{5!}{2!\,3!}N(4)^2N(5)^3 = 5350855080000000000000,$$

$$A_{48}^{(50)} = \frac{5!}{4!}N(4)N(5)^4 = 5354767631000000000000,$$

$$A_{50}^{(50)} = N(5)^5 = 2568678006000000000000.$$

Example 4.5.2. Let $p = 5$, $q = 11$ and $o(11)_{2.5^m} = 5^d$, for some integer d.

As, $o(11)_{2.5^m} = 5^{m-1}$.

Therefore, $d = m - 1$.

Let r be a positive integer.

By Theorem 4.4.16, if $r = 1$, then the weight of each non-zero codeword in $\mathbb{M}_1^{(10)}$ is 10.

If $r = 2$, the weight distribution of $\mathbb{M}_1^{(50)}$ is given below :

$A_1^{(50)} = A_2^{(50)} = A_3^{(50)} = A_4^{(50)} = A_5^{(50)} = A_6^{(50)} = A_7^{(50)} = A_8^{(50)} = A_9^{(50)} = A_{11}^{(50)} =$
$A_{12}^{(50)} = A_{13}^{(50)} = A_{14}^{(50)} = A_{15}^{(50)} = A_{16}^{(50)} = A_{17}^{(50)} = A_{18}^{(50)} = A_{19}^{(50)} =$

$A_{21}^{(50)} = A_{22}^{(50)} = A_{23}^{(50)} = A_{24}^{(50)} = A_{25}^{(50)} = A_{26}^{(50)} = A_{27}^{(50)} = A_{28}^{(50)} = A_{29}^{(50)} =$

$A_{31}^{(50)} = A_{32}^{(50)} = A_{33}^{(50)} = A_{34}^{(50)} = A_{35}^{(50)} = A_{36}^{(50)} = A_{37}^{(50)} = A_{38}^{(50)} = A_{39}^{(50)} =$

$A_{41}^{(50)} = A_{42}^{(50)} = A_{43}^{(50)} = A_{44}^{(50)} = A_{45}^{(50)} = A_{46}^{(50)} = A_{47}^{(50)} = A_{48}^{(50)} = A_{49}^{(50)} = 0,$

$A_{0}^{(50)} = 0,$

$A_{10}^{(50)} = \binom{5}{1}(11-1) = 50,$

$A_{20}^{(50)} = \binom{5}{2}(11-1)^2 = 1000,$

$A_{30}^{(50)} = \binom{5}{3}(11-1)^3 = 10000,$

$A_{40}^{(50)} = \binom{5}{4}(11-1)^4 = 50000,$

$A_{50}^{(50)} = \binom{5}{5}(11-1)^5 = 100000.$