CHAPTER 2
\((\sigma, \delta)\)-RINGS

In this chapter we study some known definitions and results that lead to further discussion. It is divided into two sections. The first section (2.1) deals with relevant definitions and includes a relation involving endomorphism \(\sigma\) of an integral domain \(R\) and \(\delta\) a \(\sigma\)-derivation of \(R\) (Proposition (2.1.19)). In the second section (2.2), \((\sigma, \delta)\)-rings have been introduced. The main purpose of this section is to study the properties of \((\sigma, \delta)\)-rings (Theorem (2.2.3), (2.2.6), (2.2.8)).

2.1 Automorphism and Derivations

We begin this section with the definition of prime ideal:

**Definition 2.1.1:** Let \(R\) be a ring.

1. A prime ideal of \(R\) is any ideal \(P\) of \(R\) \((P \neq R)\) such that, whenever \(A\) and \(B\) are ideals of \(R\) with \(AB \subseteq P\), either \(A \subseteq P\) or \(B \subseteq P\).

2. A semiprime ideal of \(R\) is any ideal \(I\) of \(R\) \((I \neq R)\) such that, \(I\) is the intersection of prime ideals of \(R\).

**Examples 2.1.2:**

1. Let \(R = \left( \begin{array}{cc} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{array} \right) = M_2(\mathbb{Z})\).

   If \(p\) is a prime number, then the ideal \(P = M_2(p\mathbb{Z})\) is a prime ideal of \(R\). (Example (1.1) of [7]).

2. Let \(R = \mathbb{Z}_8\). Its ideals are \(\{0\}\), \(\{0, 4\}\), \(\{0, 2, 4, 6\}\) and \(\mathbb{Z}_8\). Only \(\{0, 2, 4, 6\}\) is the prime ideal and \(\{0\}\), \(\{0, 4\}\) are not the prime ideals. (Example (12.3.3) of [52]).
Definition 2.1.3:

(1) The ring $R$ is called a prime ring if zero ideal of $R$ is a prime ideal i.e., if $I, J$ are ideals of $R$ such that $IJ = 0$, then $I = 0$ or $J = 0$.

(2) The ring $R$ is called a semiprime ring if zero ideal of $R$ is a semiprime ideal.

Examples 2.1.4:

(1) Any domain is a prime ring.

(2) Any reduced ring is a semiprime ring.

Definition 2.1.5: An ideal $I$ of a ring $R$ is said to be completely semiprime if $a^2 \in I$ implies that $a \in I$, for $a \in R$.

Example 2.1.6:

(1) Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$ be a ring. Then

$$P_1 = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{pmatrix}, P_2 = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}, P_3 = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$$

are prime ideals of $R$. It can be easily seen that $P_1, P_2, P_3$ are completely semiprime ideals.

(2) Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$ be a ring. If $p$ is a prime number, then $P = M_2(p\mathbb{Z})$ is a prime ideal of $R$. But it is not a completely semiprime ideal as

$$\begin{pmatrix} \sqrt{3} & 0 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \in P.$$

But $\begin{pmatrix} \sqrt{3} & 0 \\ 0 & 0 \end{pmatrix} \notin P$. 
Definition 2.1.7: The nil radical of a ring $R$ is the set of all nilpotent elements of $R$. It is denoted by $N(R)$.

Example 2.1.8: In $R = M_2(\mathbb{R})$, $A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$, $a \in \mathbb{R}$ is nilpotent.

Hence $N(R) = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, a \in \mathbb{R} \right\}$.

Definition 2.1.9: The prime radical of a ring $R$ is the intersection of all the prime ideals of $R$. It is denoted by $P(R)$.

Example 2.1.10:

(1) Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where $F$ is a field.

Then $P(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$.

(2) Let $\mathbb{Z}_2$ be the ring of integers modulo 2 and $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Then the prime radical of $R$ is $P(R) = \{(0,0)\}$.

Note that a ring is semiprime if and only if its prime radical is zero. If $R$ is the zero ring, it has no prime ideals, and the prime radical equals $R$. If $R$ is non-zero, it has at least one maximal ideal, which is prime. So, the prime radical of a non-zero ring is a proper ideal.

Definition 2.1.11: A ring $R$ is 2-primal if and only if the set of nilpotent elements and the prime radical of $R$ are the same if and only if the prime radical is a completely semiprime ideal.

Example 2.1.12:

(1) Let $R = F[x]$ be the polynomial ring over the field $F$. Then $R$ is 2-primal.
Let \( R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \right\} \mid a, b \in \mathbb{Z}_4 \}. \) \( R \) is not a reduced ring as \( \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \) is a non-zero nilpotent element and hence \( R \) is not 2-primal.

2-primal rings originally and independently came from the context of left near rings by Birkenmeier et al. [9]. Some of the earlier results known to us about 2-primal rings (although not so called at time) and prime ideals are due to Shin [50]. Hirano [25] used the term \( N \)-ring for what we call 2-primal ring. Also a reduced ring is 2-primal and so is a commutative Noetherian ring. Part of the attraction of 2-primal rings in addition to their being a common generalization of commutative rings and rings without nilpotent elements lies in the structure of their prime ideals.

We now recall the following definitions:

**Definition 2.1.13:** A homomorphism \( \sigma : R \to R' \) of rings \( R \) and \( R' \) is called an endomorphism if \( R = R' \).

**Example 2.1.14:**

1. Let \( R = \mathbb{Z}[\sqrt{2}] \). Then \( \sigma : R \to R \) defined as
   \[
   \sigma(a + b\sqrt{2}) = a - b\sqrt{2} \text{ for } a + b\sqrt{2} \in R
   \]
   is an endomorphism of \( R \).

2. Let \( \sigma : \mathbb{Z} \to \mathbb{Z} \) be defined by \( \sigma(n) = 2n \). Then \( \sigma \) is not a ring endomorphism.

**Definition 2.1.15:** Let \( R \) be a ring, a map \( \delta : R \to R \) is called a \( \delta \)-derivation if for every \( a, b \in R \)

1. \( \delta(a + b) = \delta(a) + \delta(b) \).

2. \( \delta(ab) = \delta(a)\cdot b + a\cdot\delta(b) \).

**Example 2.1.16:** Let \( R = F[x] \), where \( F \) is a field. Define \( \delta : R \to R \) by
$$\delta(f(x)) = \frac{d}{dx}(f(x)).$$

Now

$$\delta(f(x) + g(x)) = \frac{d}{dx}(f(x) + g(x))$$
$$= \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$$
$$= \delta(f(x)) + \delta(g(x)).$$

Similarly,

$$\delta(f(x).g(x)) = \frac{d}{dx}(f(x).g(x))$$
$$= \frac{d}{dx}(f(x)).g(x) + f(x).\frac{d}{dx}(g(x))$$
$$= \delta(f(x)).g(x) + f(x).\delta(g(x)).$$

Hence $\delta$ is a derivative.

**Definition 2.1.17:** Let $R$ be a ring and $\sigma$ an endomorphism of $R$, a mapping $\delta : R \to R$ is called a $\sigma$-derivation if

1. $\delta(a + b) = \delta(a) + \delta(b)$.
2. $\delta(a \cdot b) = \delta(a) \cdot \sigma(b) + a \cdot \delta(b)$.

**Example 2.1.18:** Let $R$ be a ring and $\delta : R \to R$ any map. Let $\phi : R \to R$ be a map defined by

$$\phi(r) = \begin{pmatrix} \sigma(r) & 0 \\ \delta(r) & r \end{pmatrix}.$$ 

Then $\delta$ is a $\sigma$-derivation of $R$ if and only if $\phi$ is a homomorphism.

For any $a, b \in R$,

$$\phi(a + b) = \phi(a) + \phi(b)$$
implies
\[
\begin{pmatrix}
\sigma(a + b) & 0 \\
\delta(a + b) & a + b
\end{pmatrix} = \begin{pmatrix}
\sigma(a) & 0 \\
\delta(a) & a
\end{pmatrix} + \begin{pmatrix}
\sigma(b) & 0 \\
\delta(b) & b
\end{pmatrix}
\]
\[
= \begin{pmatrix}
\sigma(a) + \sigma(b) & 0 \\
\delta(a) + \delta(b) & a + b
\end{pmatrix}
\]
\[
= \begin{pmatrix}
\sigma(a + b) & 0 \\
\delta(a) + \delta(b) & a + b
\end{pmatrix},
\]
as \sigma is an endomorphism. Comparing both sides, we get
\[
\delta(a + b) = \delta(a) + \delta(b).
\]
Also,
\[
\phi(a,b) = \phi(a)\cdot \phi(b)
\]
implies
\[
\begin{pmatrix}
\sigma(ab) & 0 \\
\delta(ab) & ab
\end{pmatrix} = \begin{pmatrix}
\sigma(a) & 0 \\
\delta(a) & a
\end{pmatrix} \begin{pmatrix}
\sigma(b) & 0 \\
\delta(b) & b
\end{pmatrix}
\]
\[
= \begin{pmatrix}
\sigma(a)\sigma(b) + 0\delta(b) & \sigma(a)0 + 0b \\
\delta(a)\sigma(b) + a\delta(b) & ab
\end{pmatrix}
\]
\[
= \begin{pmatrix}
\sigma(a)\sigma(b) & 0 \\
\delta(a)\sigma(b) + a\delta(b) & ab
\end{pmatrix}
\]
\[
= \begin{pmatrix}
\sigma(ab) & 0 \\
\delta(a)\sigma(b) + a\delta(b) & ab
\end{pmatrix}.
\]
Equating corresponding terms on both sides, we have
\[
\delta(ab) = \delta(a)\sigma(b) + a\delta(b).
\]
Thus \(\delta\) is a \(\sigma\)-derivation.

We now state and prove a relation involving endomorphism \(\sigma\) of an integral domain \(R\) and \(\delta\) a \(\sigma\)-derivation of \(R\).
Proposition 2.1.19: Let $R$ be an integral domain, $\sigma$ an endomorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$. Then for $0 \neq u \in R$, $\sigma(u) + \delta(u) \neq 0$.

Proof. Let $0 \neq u \in R$, we show that $\sigma(u) + \delta(u) \neq 0$. Let for $0 \neq u$,
\[ \sigma(u) + \delta(u) = 0 \]
which implies that
\[ \delta(u) = -\sigma(u). \]

We know that for $a, b \in R$,
\[ \delta(ab) = \delta(a)\sigma(b) + a\delta(b) = -\sigma(a)\sigma(b) + a(-\sigma(b)) \]
which gives
\[ -\sigma(ab) = -[a + \sigma(a)]\sigma(b). \]
Since $\sigma$ is an endomorphism of $R$, we have
\[ -\sigma(a)\sigma(b) = -[a + \sigma(a)]\sigma(b). \]
As $R$ is an integral domain, this gives
\[ \sigma(a) = a + \sigma(a). \]
Therefore, $a = 0$, which is not possible. Hence the result. $\square$

We now give an example to show that the above Proposition does not hold if $R$ is not an integral domain.

Example 2.1.20: Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Define $\sigma : R \to R$ by
\[ \sigma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix} \text{ for } a, b, c \in F. \]
Then $\sigma$ is an automorphism of $R$. Choose $C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in R$ and define $\delta : R \to R$ by
\[ \delta(A) = AC - C\sigma(A) \text{ for each } A \in R. \]
Then $\delta$ is a $\sigma$-derivation of $R$. Now choose $0 \neq U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R$. Clearly, $\sigma(U) + \delta(U) = 0$.

### 2.2 $(\sigma, \delta)$-Rings and Completely Semiprime Ideals

In this section, we introduce $(\sigma, \delta)$-ring and study its properties in the context of completely semiprime ideals and 2-primal rings [Theorem (2.2.3), (2.2.6) and (2.2.8)]. Here a ring $R$ always means an associative ring with identity $1 \neq 0$, unless otherwise stated. The ring of integers is denoted by $\mathbb{Z}$, the field of rational numbers is denoted by $\mathbb{Q}$ and $P(R)$ denotes the prime radical of $R$.

We begin with the following definition:

**Definition 2.2.1:** Let $R$ be a ring. Let $\sigma$ be an endomorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$. Then $R$ is said to be a $(\sigma, \delta)$-ring if $a(\sigma(a) + \delta(a)) \in P(R)$ implies that $a \in P(R)$ for $a \in R$.

**Example 2.2.2:**

1. Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$. Then $P(R) = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$.

Let $\sigma : R \rightarrow R$ be defined by

$$\sigma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix},$$

for all $a, b, c \in \mathbb{Z}$.

Then it can be seen that $\sigma$ is an endomorphism of $R$.

Define $\delta : R \rightarrow R$ by

$$\delta(a) = a - \sigma(a),$$

for all $a \in R$.

Clearly, $\delta$ is a $\sigma$-derivation of $R$. 

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Now let \( A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \).

Also \( A[\sigma(A) + \delta(A)] \in P(R) \)

implies that

\[
\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \left\{ \sigma\left( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) + \sigma\left( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) - \sigma\left( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) \right\} \in P(R)
\]

or

\[
\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \left\{ \left( \begin{array}{c} a \\ -b \end{array} \right) + \left( \begin{array}{c} a \\ b \end{array} \right) - \sigma\left( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) \right\} \in P(R)
\]

which gives on simplification

\[
\begin{pmatrix} a^2 & ab + bc \\ 0 & c^2 \end{pmatrix} \in P(R) = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{pmatrix}.
\]

Hence \( a^2 = 0, c^2 = 0 \) i.e. \( a = 0, c = 0 \).

Therefore, \( A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in P(R) \).

Thus \( R \) is a \((\sigma, \delta)\)-ring.

(2) Let \( R = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \). Then \( R \) is a commutative reduced ring. Define an automorphism \( \sigma : R \to R \) by

\[
\sigma((a, b)) = (b, a) \text{ for } a, b \in \mathbb{Z}_2.
\]

Also \( \delta : R \to R \) defined by

\[
\delta((a, b)) = (a - b, 0) \text{ for } a, b \in \mathbb{Z}_2
\]

is a \( \sigma \)-derivation of \( R \). Here \( P(R) = \{0\} \). But \( R \) is not a \((\sigma, \delta)\)-ring, for take \((a, b) = (0, 1)\).

We prove a relation between \((\sigma, \delta)\)-ring and 2-primal rings in the form of following Theorem:
Theorem 2.2.3: Let $R$ be a Noetherian integral domain which is also an algebra over $\mathbb{Q}$. Let $\sigma$ be an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$. If $R$ is a ($\sigma, \delta$)-ring, then $R$ is 2-primal.

Proof. Let $R$ be a ($\sigma, \delta$)-ring. We know that a reduced ring is 2-primal. We use the principle of Mathematical induction to prove that $R$ is a reduced ring. Let for $x \in R$, $x^n = 0$. We use induction on $n$ and show that $x = 0$. Result is trivially true for $n = 1$, as

$$x^n = x^1 = a(\sigma(a) + \delta(a)) = 0.$$ 

Now Proposition (2.1.19), implies that $a = 0$. Hence $x = 0$. Therefore, the result is true for $n = 1$. Let us assume that the result is true for $n = k$, i.e. $x^k = 0$ which implies that $x = 0$. Let $n = k + 1$. Then $x^{k+1} = 0$ which implies that

$$a^{k+1}(\sigma(a) + \delta(a))^{k+1} = 0.$$ 

Again by Proposition (2.1.19), we get $a = 0$. Hence $x = 0$. Therefore, the result is true for $n = k + 1$ also. Thus the result is true for all $n$. □

However, the converse of the above Theorem is not true as seen in the following example.

Example 2.2.4: Let $R = F(x)$, the field of rational polynomials in one variable, $x$. Then $R$ is 2-primal with $P(R) = \{0\}$. Let $\sigma : R \to R$ be an endomorphism defined by

$$\sigma(f(x)) = f(0).$$

For $r \in R$, $\delta_r : R \to R$ is a $\sigma$-derivation defined as

$$\delta_r(a) = ar - r\sigma(a) \text{ for } a \in R.$$ 

Then $R$ is not a ($\sigma, \delta$)-ring. For take $f(x) = xa + b, r = \frac{-b}{xa}$.

Towards the proof of the next Theorem, we require the following:

J. Krempa (1996, [33]) has investigated the relation between minimal prime ideals and completely prime ideals of a ring $R$. With this he proved the following:
Theorem 2.2.5: For a ring $R$ the following conditions are equivalent:

1. $R$ is reduced.
2. $R$ is semiprime and all minimal prime ideals of $R$ are completely prime.
3. $R$ is a sub-direct product of domains.

With this we prove that:

Theorem 2.2.6: Let $R$ be a Noetherian integral domain which is also an algebra over $\mathbb{Q}$. Let $\sigma$ be an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$. If $R$ is a $(\sigma, \delta)$-ring, then $P(R)$ is completely semiprime.

Proof. As proved in Theorem (2.2.3), $R$ is a reduced ring and by using Theorem (2.2.5), the result follows. \qed

However, the converse of the above Theorem is not true as shown in Example (2.2.7).

Example 2.2.7: Let $F$ be a field, $R = F \times F$. Let $\sigma : R \to R$ be an automorphism defined as

$$
\sigma((a, b)) = (b, a), \text{ for all } a, b \in F.
$$

Here $P(R)$ is a completely semiprime ring, as $R$ is a reduced ring. For $r \in F$, define $\delta_r : R \to R$ by

$$
\delta_r((a, b)) = (a, b)r - r\sigma((a, b)) \text{ for } a, b \in F.
$$

Then $\delta_r$ is a $\sigma$-derivation of $R$. Also $R$ is not a $(\sigma, \delta)$-ring. For take $A = (1, -1), r = \frac{1}{2}$. 