CHAPTER 1
INTRODUCTION

1.1 Background and Motivation

The concept of Noetherian rings was first introduced by Amalie Emmy Noether [1] (23 March, 1882 – 14 April, 1935), a German mathematician known for her ground breaking contribution to Abstract Algebra and Theoretical Physics. The development of Abstract Algebra, which is one of the most distinctive innovations of twentieth century Mathematics, is largely due to her - in published papers, in lectures and in personal influence on her contemporaries. Two of the most basic objects in Abstract Algebra are Groups and Rings.

In this epoch, Noether became famous for her deft use of ascending or descending chain conditions. Ascending and descending chain conditions can be applied to many types of mathematical objects although on the surface they might not seem to be very powerful. Noether showed how to exploit them to maximum advantage. For example, how to show that every set of sub-objects has a maximal/minimal element or that a complex object can be generated by a smaller number of elements, which are often crucial steps in a proof.

Many types of objects in Abstract Algebra can satisfy chain conditions and usually if they satisfy an ascending chain condition, they are called Noetherian in her honour. Recall that a ring $R$ is said to be a Noetherian ring if it satisfies the ascending chain condition on ideals. There are other equivalent formulations of the definition of a Noetherian ring. The notion of a Noetherian ring is of fundamental importance in both commutative and non-commutative ring theory, due to the role it plays in simplifying the ideal structure of a ring. For instance, the ring of integers and the polynomial ring over a field are both Noetherian rings, and consequently, such theorems as the Lasker-Noether theorem, the Krull Intersection theorem, and the Hilbert’s
Basis theorem hold for them.

For the non-commutative rings, it is necessary to distinguish between three very similar concepts viz; a ring $R$ is left-Noetherian if it satisfies the ascending chain condition on left ideals, a ring $R$ is right-Noetherian if it satisfies the ascending chain condition on right ideals and it is Noetherian if it is both left- and right-Noetherian. For commutative rings, all three concepts coincide, but in general they are different.

In Mathematics, especially in the field of Abstract Algebra, polynomial ring is a ring formed from the set of polynomials in one or more indeterminate or variables with coefficients in another ring, often a field. There is considerable interest in studying if and how certain properties of rings are preserved under polynomial extensions. If a ring $R$ has some property one would like to know whether the ring $R[x]$ also enjoys that property. For some properties, such as commutativity or being a domain or being a division ring, the issue is trivial. However in many cases, very interesting research has been generated. In 1972, for example, R. C. Shock proved that the uniform dimension of the regular module is preserved by the polynomial ring, a result that has garnered the appellation ‘Shock’s Theorem’. Polynomial rings have influenced much of the Mathematics, from the Hilbert’s Basis theorem, to the construction of the splitting fields, and to the understanding of a linear operator. Many important conjectures involving polynomial rings, such as Serre’s problem, have influenced the study of other rings, and have influenced even the definition of other rings, such as group rings and rings of formal power series. A closely related notion is that of the ring of polynomial functions on a vector space.

For studying the non-commutative aspects of the theory, the assumption that the indeterminate $x$ commutes with the base ring is relaxed. This can be done by two well known means. The first leads to Ore extensions while the latter leads to differential polynomial rings. We refer reader to Goodearl and Warfield (2004, [23]) for more details. This definition of non-commutative
polynomial rings was first introduced by Ore (1933, [47]), who combined earlier ideas of Hilbert (in case $\delta = 0$) and Schlessinger (in the case $\sigma = I$). Little work appeared on Ore extensions for some time after that.

Ever since the appearance of Ore’s fundamental paper (1933, [47]), Ore extensions have played an important role in the non-commutative ring theory and many non-commutative ring theorists have investigated Ore extensions from different points of view such as: Ideal theory, Order theory, Galois theory, Homological algebras and so on.

In commutative ring theory, three basic classes of rings are reduced rings, integral domains and fields. The defining condition for these are conditions on the elements of a ring and not commutativity. When we move from commutative rings to non-commutative rings an alternate way of generalizing an “element-wise” condition should be to replace the role of elements by that of ideals. By making these changes judiciously in the basic definitions, we are led to the notions of semiprime rings, prime rings and (left and right) primitive rings.

Also in commutative ring theory, the notion of prime ideals play a very important role and are closely tied to multiplicative closed sets. The compliment of a prime ideal is closed with respect to multiplication and given a (non-empty) multiplicative closed set $S$, an ideal disjoint from $S$ and maximal w.r.t this property is always a prime ideal.

Prime ideals are useful in the study of non-commutative rings. They are not defined, as one might suspect from experience in commutative Algebra, as simply those ideals that leave a domain. A more relaxed definition was first introduced by W. Krull (1929, [34]). The set of ideals $I$ such that $R/I$ is a domain turns out to be too restrictive for use in non-commutative rings $R$. Such ideals are known as completely prime ideals or, alternatively, strongly prime ideals. Many rings posses no completely prime ideals; for instance the ring of $n \times n$ matrices over a field, unless $n = 1$. 
For the commutative rings, however, one can check easily that the prime ideals and the completely prime ideals coincide. The set of prime ideals is denoted by $Spec(R)$.

In [29, 30] primes of Ore extensions over commutative Noetherian rings were considered. In [12, 17], prime ideals disjoint from the coefficient rings of Ore extensions of derivation type were described. The case of Ore extensions of automorphism type has been dealt recently in [3, 15]. Leroy and Matczuk (1991, [39]) studied prime ideals of Ore extensions, which have zero intersection with coefficient ring. The methods they used are based on [40].

The classification of prime ideals is an inodorous and difficult task for many rings. Ring theorists, therefore most often restrict their attention to certain types of prime ideals or to certain contexts in which the prime ideals are especially beneficial or easy to study. The various adjectives that describe prime ideals testify to their ubiquitous nature across many branches of Mathematics, including Number theory, Geometry and Topology.

The aim of the work in this thesis is to find the nature of prime ideals (completely prime ideals, completely semiprime ideals, minimal prime ideals, etc.) of certain Ore extensions and Noetherian rings.

1.2 Literature Survey

The study of prime ideals gives a good information of the ring structure as a whole. For example, one can measure the ‘size’ of commutative ring by forming chains of prime ideals in the ring and determining the longest such chain which is called the Krull dimension of the ring. Prime ideals are also closely tied in with other major topics such as the theory of unique factorization, the technique of localization and algebraic geometry. These topics, in turn, lead to a host of additional applications and topics in Algebra and beyond and for research.
Now let $R$ be a Noetherian ring, $\sigma$ an endomorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$, unless otherwise stated. Some of the results relevant to the topic are:

Gabriel proved in (1971, [19]) that if $R$ is a right Noetherian ring which is also an algebra over $\mathbb{Q}$ and $P$ is a prime ideal of $R[x;\delta]$, then $P \cap R$ is a prime ideal of $R$.

Shin showed in (1973, Proposition (1.11) of [50]) that a ring $R$ is 2-primal if and only if every minimal prime ideal $P \subset R$ is completely prime (i.e. $R/P$ is a domain). Irving (1979, [29]) has shown that the prime ideals of $A = R[x;\sigma]$ can be completely described in terms of certain ideals of $R$.

A lot of work on minimal prime ideals has been done by Goodearl and Warfield (2004, [23]). Goodearl and Letzter (1994, [22]) have proved that if $R$ is a Noetherian ring, then for each prime ideal $P$ of $O(R)$, the prime ideals of $R$ minimal over $P \cap R$ are contained within a single $\sigma$-orbit of $\text{Spec}(R)$.

Kwak in (2003, [37]) establishes a relation between a 2-primal ring and a $\sigma(\ast)$-ring. The property is also extended to the Ore extension $R[x;\sigma]$.

Bhat (2010, [7]) has found a relation between Ore extension of a completely prime ideal and the Ore extension of a ring.

Marubayashi et al. (2012, [41]) have investigated the class of minimal prime ideals in $R$ where $\delta = 0$. For the case $\delta \neq 0$ but it is inner, Goodearl (1992, [21]) showed the existence of an isomorphism between $D[x;\sigma,\delta]$ and $D[y;\sigma]$, where $D$ is a commutative Dedekind domain and $R = D[x;\sigma,\delta]$, the Ore extension over $D$.

Andrunakievic and Rjabuhin (1968, [2]) have proved that for a ring $R$, $R$ is reduced; $R$ is semiprime and all minimal prime ideals of $R$ are completely prime; and $R$ is a sub-direct product of domains are equivalent.
Ore extensions (Skew polynomial rings) have attracted the attention of many ring theorists and they have been investigated from different points of view. The majority of the previous work on Ore extensions $R[x; \sigma, \delta]$, with the notable exception of a paper of Irving (1979, [30]), has concentrated on the two unmixed cases, in which either $\sigma = I$ or $\delta = 0$. However, the surge in the quantum groups and quantized algebras (see, e.g., [16, 31, 36, 44]) has brought renewed interest in general Ore extensions, due to the fact that many of these quantized algebras and their representations can be expressed in terms of Ore extensions.

In 1970, Brewer and Heinzer [10] for the commutative case proved that for each prime $P \subseteq R$, the ideal $P[x] \subseteq R[x]$ is prime. Prime ideals in $R[x; \sigma, \delta]$ for $R$ non-commutative Noetherian ring are investigated in [22]. More recent proofs by Faith 2000 also offer techniques which could be used for non-commutative rings.

The above discussion leads to the question, how prime ideals behave in various types of ring extensions; in particular, the Ore extensions and this thesis concerns the same question.

### 1.3 Structure of Work

The main purpose of the present thesis is to carry out the study of prime ideals of certain Ore extensions over Noetherian rings.

The first section of chapter 1 is devoted to background and motivation with a brief development of Noetherian rings, Ore extensions, reduced rings and prime ideals. The second section is the literature survey of the relevant topics with emphasis on how prime ideals behave on certain ring structures.

In chapter 2, composed of two sections, we recall some notions like semiprime ideals, completely semiprime ideals, derivations, prime radical, nil radical and 2-primal rings. We state and prove a relation involving endomorphism $\sigma$ of an integral domain $R$ and $\delta$ a $\sigma$-derivation of $R$. We also define $(\sigma, \delta)$-rings
(Definition (2.2.1)) and find its prime ideals.

In chapter 3, \((\sigma, \delta)\)-rigid rings (Definition (3.1.1)) and weak \((\sigma, \delta)\)-rigid rings (Definition (3.1.2)) are introduced. In this chapter the properties of these rings are investigated in the context of prime ideals. We also find the relationship between \((\sigma, \delta)\)-rings, matrix rings and rings introduced in this chapter.

Chapter 4 gives a structure of minimal prime ideals and completely prime ideals of a ring \(R\), where \(R\) is a \((\sigma, \delta)\)-ring (Section (4.1)) and of weak \((\sigma, \delta)\)-rigid ring (Section (4.2)).

In chapter 5, Ore extensions, \((\sigma, \delta)\)-rings and weak \((\sigma, \delta)\)-rigid rings are discussed. We give a necessary and sufficient condition for a commutative Noetherian integral domain which is a \((\sigma, \delta)\)-ring to be an Ore extension (Section (5.1)). We also prove a similar condition for a weak \((\sigma, \delta)\)-rigid ring (Section (5.2)).

Finally, in chapter 6, we give the conclusion of our work and some areas related to this research for future work.