CHAPTER 6
CONCLUSION AND FUTURE WORK

Ore extensions have both a “ring theoretical aspect”: characterization of simplicity, description of prime ideals, passage of properties from the base ring to the Ore extensions (often with the aim of giving examples of a non left/right symmetric behavior) and a more arithmetical aspect mainly related to the factorization of polynomials, computation of the roots and also in relation with special matrices, in particular Vandermonde and Wronskian matrices. The link between these two aspects is given, in particular, by special types of polynomial (invariant, semi-invariant, irreducible, Wedderburn, fully reducible, etc.). Another important feature of the Ore extensions is their relation with Differential equations and Operator theory. This was the origin of their study even before their formal definition given by Ore.

In this thesis we have considered these non-commutative polynomials, Ore extensions. We have used these polynomials to find out the prime ideals of Noetherian rings under various conditions. We have defined \( (\sigma, \delta) \)-rings, \( (\sigma, \delta) \)-rigid rings and weak \( (\sigma, \delta) \)-rigid rings, studied their prime ideals and found necessary and sufficient conditions for them to be an Ore extension. The main results of this thesis are listed as:

In chapter 2, we give a relation between an endomorphism and its derivation for an integral domain (Theorem (2.1.19)). Also \( (\sigma, \delta) \)-ring is discussed and the nature of its completely semiprime ideal is elaborated.

In chapter 3, we prove that a weak \( (\sigma, \delta) \)-rigid ring \( R \) and the upper triangular matrix ring \( T_n(R) \) over \( R \) are equivalent, for any positive integer \( n \). Also their prime ideals are investigated.

In chapter 4, we discuss minimal prime ideals and completely prime ideals of rings under consideration. In fact we find necessary and sufficient condition
for a minimal prime ideal to be a completely prime ideal for a Noetherian integral domain which is a \((\sigma, \delta)\)-ring (Theorem (4.2.1)). Also we find necessary and sufficient condition for a minimal prime ideal to be a completely prime ideal for a Noetherian integral domain which is a 2-primal weak \((\sigma, \delta)\)-rigid ring (Theorem (4.3.1)). Again, let \(R\) be a Noetherian, integral domain which is also an algebra over \(\mathbb{Q}\). Let \(\sigma\) be an automorphism of \(R\) and \(\delta\) a \(\sigma\)-derivation of \(R\). Then \(R\) a weak \((\sigma, \delta)\)-rigid ring implies that \(N(R)\) is completely semiprime (Theorem (4.3.2)).

Theorem (5.2.5) of chapter 5, gives a necessary and sufficient condition for an Ore extension to be 2-primal. Also if \(R\) is a commutative Noetherian integral domain which is also an algebra over \(\mathbb{Q}\), \(\sigma\) an automorphism of \(R\) and \(\delta\) a \(\sigma\)-derivation of \(R\) such that \(R\) is a \((\sigma, \delta)\)-ring, then \(O(N(R)) = N(O(R))\) (Theorem (5.3.1)). In this chapter, we have also shown that a commutative Noetherian 2-primal integral domain is a weak \((\sigma, \delta)\)-rigid ring if and only if so is \(O(R) = R[x; \sigma, \delta]\) (Theorem (5.3.2)).

There are many interesting questions left for future work.

(1) Does \(R\) a commutative Noetherian \((\sigma, \delta)\)-ring imply that \(P(R)\) is completely semiprime?

(2) Is \(P(O(R)) = O(P(R))\), for a Noetherian ring \(R\) which is also a \((\sigma, \delta)\)-ring?

(3) Is a commutative Noetherian ring a \((\sigma, \delta)\)-ring if and only if so is \(O(R) = R[x; \sigma, \delta]\)?

(4) Characterization of Minimal prime ideals and completely prime ideals of matrix rings over \((\sigma, \delta)\)-rings.