CHAPTER 5
$(\sigma, \delta)$-RINGS AND THEIR EXTENSIONS

One of the most active and important research areas in non-commutative Algebra is the investigation of Ore extensions (also called skew polynomial rings). Ore extensions have a number of important structural properties which make them mathematically rich, correspond to real applications and also allow for effective and efficient algorithms. Skew polynomial rings in several variables with coefficients in a field $K$ were first introduced by Noether and Schmeidler (1920, [46]) and they were later systematically studied by Ore [47] in the 1930’s both in the context of differential equations and as operators on finite fields. Since then, these rings have been extensively studied, for example, for characterizing various kinds of radicals (Jacobson and Baer) and global and Krüll dimensions of such rings, for constructing finite-dimensional Algebras, classifying all valuations of these Algebras, etc. They have been successfully applied in many areas including solving Ordinary Differential Equations (Bronstein and Prtkovsek, [11]; Chyzak and B.Salvy, [14]; Van Hoeij, [27] and Singer, [51]; etc.), Control theory (Chyzak-Quadrat-Robertz, [13]; Fliess-Mounier, [18]; Gluesing-Luerssen, [20]; etc.), and Coding theory (McEliece, [43]; Piret, [49]; etc.).

5.1 Preliminaries

Let $R$ be a ring, $\sigma$ an automorphism of $R$, $\delta$ a $\sigma$-derivation of $R$ and $x$ an indeterminate. The Ore polynomial ring $S = R[x; \sigma, \delta]$ is built as follows:

Elements of $S$ are written as polynomials (right) of the form $a_0 + xa_1 + \ldots + x^n a_n$, where $n$ is a non-negative integer and $a_i \in R$. It is often convenient to write such an expression as a sum $\Sigma_i x^i a_i$, leaving it understood that the summation runs over a finite sequence of non-negative integers $i$, or by thinking of it as an infinite sum in which almost all of the coefficients $a_i$ are zero, as followed in McConnell and Robson (2001, [42]). Define addition in $S$ in
the usual way:

\[(\sum_i x^i a_i) + (\sum_i x^i b_i) = \sum x^i(a_i + b_i).\]

As for multiplication, the coefficients multiply together as they do in \(R\), and the powers of \(x\) multiply following the usual rules for exponents. The product of a power \(x^i\) with an element \(a \in R\) (in that order) is the single-term \(x^i a\). It is in a product of the form \(ax^i\) that the twist enters, which are multiplied using the distributive law and the Ore commutation rule

\[ax = x\sigma(a) + \delta(a). \tag{5.1.1}\]

This can be used to write all elements of \(S\) as right polynomials.

One can check that the Ore commutation rule (5.1.1) leads to a valid multiplication rule making \(S\) into a ring. This verification is somewhat tedious, especially to check the associative law. In checking that \((ab)x = a(bx)\) holds for all \(a, b \in R\), we first rewrite each expression as a right polynomial using (5.1.1) and then equate coefficients. The \(\sigma\)-derivation law viz

\[\delta(ab) = \delta(a)\sigma(b) + a\delta(b), \tag{5.1.2}\]

is the result of equating coefficients of \(x\) on each side. Thus the law (5.1.2) is a necessary constraint in order to make \(S\) into a ring. Also note that by equating the constant terms on each side of \((ab)x = a(bx)\), it follows that \(\sigma \in \text{End}(R)\). Together with the additivity of \(\sigma\) and \(\delta\), we obtain necessary and sufficient conditions for a valid ring structure on \(S\). We refer reader to Goodearl and Warfield (2004, [23]) for more details.

**Definition 5.1.1:** Let \(R\) be a ring, \(\sigma\) an endomorphism of \(R\) and \(\delta\) a \(\sigma\)-derivation of \(R\). The Ore extension \(R[x; \sigma, \delta]\) is the set of polynomials over \(R\), written as \(S = R[x; \sigma, \delta] = \{f = \sum_{i=0}^{n} x^i a_i : a_i \in R, n \in N\}\) provided

(a) \(S\) is a ring, containing \(R\) as a sub-ring.
(b) \(x\) is an element of \(S\).
(c) \(S\) is a free left \(R\)-module with basis \(\{1, x, x^2, \ldots\}\).
(d) \(ax = x\sigma(a) + \delta(a)\) for every \(a \in R\).
Such a ring is called an Ore extension of $R$ (honoring O. Ore, who first systematically studied the general case). We denote $R[x; \sigma, \delta]$ by $O(R)$.

Some mathematicians prefer Ore extensions to have left-hand coefficients. To achieve this, one starts with a ring $R$, an endomorphism $\sigma$ of $R$, and a left $\sigma$-derivation $\delta$ of $R$. The corresponding Ore extension is a free left $R$-module with a basis $\{1, x, x^2, \ldots\}$, where $xa = \sigma(a)x + \delta(a)$, for all $a \in R$. We follow the notation of an Ore extension as in McConnell and Robson [42].

The two archetypical cases of Ore extension are:

1. If $\sigma = I$, the identity map on $R$, then $R[x; \sigma, \delta] = R[x; \delta]$ is called differential operator ring or Ore extension of derivation type. It is denoted by $D(R)$. Here multiplication is defined subject to the relation
   \[ ax = xa + \delta(a), \text{ for all } a \in R. \]

2. If $\delta = 0$, then $R[x; \sigma] = R[x; \sigma]$ is called Ore extension of endomorphism type. It is also denoted by $S(R)$. Here multiplication is defined subject to the relation
   \[ ax = x\sigma(a), \text{ for all } a \in R. \]

The following example (5.1.2) shows that Ore extension is not a commutative ring.

**Example 5.1.2:** Let $f = xa + b$ and $g = xc + d \in R[x; \sigma, \delta]$. Then $f.g \neq g.f$, for all $f, g \in R[x; \sigma, \delta]$. Thus, it is not a commutative ring.

### 5.2 Ore Extensions over $(\sigma, \delta)$-Rings

We now investigate the conditions under which the Ore extension of a minimal prime ideal of a $(\sigma, \delta)$-ring $R$ is a completely prime ideal of Ore extension of ring $R$. For this we need the following result of Bhat, 2010, [7].

**Proposition 5.2.1:** Let $R$ be a ring. Let $\sigma$ be an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$. Then:
(1) For any completely prime ideal $P$ of $R$ with $\sigma(P) = P$ and $\delta(P) \subseteq P$, $O(P)$ is a completely prime ideal of $O(R)$.

(2) For any completely prime ideal $U$ of $O(R)$, $U \cap R$ is a completely prime ideal of $R$.

Proof. See Theorem (2.4) of [7].

**Theorem 5.2.2:** Let $R$ be a Noetherian, integral domain which is also an algebra over $\mathbb{Q}$. Let $\sigma$ be an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$ such that $R$ is a $(\sigma, \delta)$-ring and $\delta(P(R)) \subseteq P(R)$. Let $P \in \text{Min.Spec}(R)$ be such that $\sigma(P) = P$, then $O(P)$ is a completely prime ideal of $O(R)$.

Proof. $R$ is 2-primal by Theorem (2.2.8) and so by Proposition (4.2.3), $\delta(P) \subseteq P$ and as in proof of Proposition (4.2.3) above, $P$ is a completely prime ideal of $R$. Now use Proposition (5.2.1) and the proof is complete.

**Remark 5.2.3:** Let $\sigma$ be an endomorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$ such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$. Then $\sigma$ can be extended to an endomorphism say $\overline{\sigma}$ of $R[x; \sigma, \delta]$ by

$$\overline{\sigma}(\sum_{i=0}^{m} x^i a_i) = \sum_{i=0}^{m} x^i \sigma(a_i).$$

Also $\delta$ can be extended to a $\overline{\sigma}$-derivation say $\overline{\delta}$ of $R[x; \sigma, \delta]$ by

$$\overline{\delta}(\sum_{i=0}^{m} x^i a_i) = \sum_{i=0}^{m} x^i \delta(a_i).$$

We note that if $\sigma(\delta(a)) \neq \delta(\sigma(a))$, for all $a \in R$ then above does not hold.

For example, take $f(x) = xa + b$, $g(x) = xc + d$ for $a, b, c, d \in R$. Then

\[
f(x).g(x) = (xa + b)(xc + d) = xa.xc + xa.d + b.xc + bd = x(ax)c + xa.d + (bx)c + bd = x[x\sigma(a) + \delta(a)]c + xa.d + [x\sigma(b) + \delta(b)]c + bd = x^2\sigma(a)c + x\delta(a)c + xad + x\sigma(b)c + \delta(b)c + bd.\]

Now
\[\overline{\delta}(f(x)g(x))\]
\[= \overline{\delta}(x^2\sigma(a)c + x\delta(a)c + x\delta(b)c + \delta(b)c + bd)\]
\[= x^2\delta(\sigma(a)c) + x\delta(\delta(a)c) + x\delta(ad) + x\delta(\sigma(b)c) + \delta(\delta(b)c) + \delta(bd)\]
\[= x^2(\delta(\sigma(a))\sigma(c) + \sigma(a)\delta(c)) + x(\delta(\delta(a))\sigma(c) + \delta(a)\delta(c)) + x(\delta(\delta(b))\sigma(d) + a\delta(d))\]
\[x(\delta(\sigma(b))\sigma(c) + \sigma(b)\delta(c)) + \delta(\delta(b))\sigma(c) + \delta(b)\delta(c) + \delta(b)\delta(d) + b\delta(d).\]

(5.2.1)

And

\[\overline{\delta}(f(x)\sigma(g(x)) + f(x)\overline{\delta}(g(x)))\]
\[= (x\delta(a) + \delta(b))(x\sigma(c) + \sigma(d)) + (xa + b)(x\delta(c) + \delta(d))\]
\[= x\delta(a)x\sigma(c) + x\delta(a)\sigma(d) + \delta(b)x\sigma(c) + \delta(b)\sigma(d) + xa\delta(c) + x\delta(d)\]
\[+ bx\delta(c) + b\delta(d)\]
\[= x(x\sigma(\delta(a)) + \delta(\delta(a)))\sigma(c) + x\delta(a)\sigma(d) + (x\sigma(\delta(b)))\sigma(c)\]
\[+ \delta(b)\sigma(d) + x(x\sigma(a) + \delta(a))\delta(c) + x\sigma(d) + (x\sigma(b) + \delta(b))\sigma(c) + b\delta(d)\]
\[= x^2\sigma(\delta(a))\sigma(c) + x\delta(\delta(a))\sigma(c) + x\delta(a)\sigma(d) + x\sigma(\delta(b))\sigma(c) + \delta(\delta(b))\sigma(c)\]
\[+ \delta(b)\sigma(d) + x^2\sigma(a)\delta(c) + x\delta(a)\delta(c) + xa\delta(d) + x\sigma(b)\delta(c) + \delta(b)\delta(c) + b\delta(d).\]

(5.2.2)

From (5.2.1) and (5.2.2), it follows that \(\overline{\delta}\) is a \(\sigma\)-derivation if

\[\sigma(\delta(a)) = \delta(\sigma(a)), \text{ for all } a \in R.\]

**Theorem 5.2.4:** Let \(R\) be a Noetherian integral domain which is also an algebra over \(\mathbb{Q}\). Let \(\sigma\) be an automorphism of \(R\) and \(\delta\) a \(\sigma\)-derivation of \(R\). Assume that \(\sigma(\delta(a)) = \delta(\sigma(a)), \text{ for all } a \in R\). Let \(R\) be a \((\sigma, \delta)\)-ring. Then \(U \in \text{Min.Spec}(R)\) such that \(\sigma(U) = U\) and \(\delta(U) \subseteq U\) implies that \(UO(R) = U[x; \sigma, \delta]\) is a completely prime ideal of \(O(R) = R[x; \sigma, \delta]\).

**Proof.** Since \(R\) is a \((\sigma, \delta)\)-ring, therefore by Theorem (2.2.6), \(P(R)\) is a completely semiprime ideal of \(R\). Let \(U \in \text{Min.Spec}(R)\). Then by Theorem (4.2.1), \(\sigma(U) + \delta(U) = U\) and \(U\) is completely prime. Now we note that \(\sigma\) can be extended to an automorphism say \(\overline{\sigma}\) of \(R/U\) and \(\delta\) to a \(\overline{\sigma}\)-derivation say \(\overline{\delta}\) of \(R/U\) (on the same lines as in Remark (5.2.3)). Then by Exercise (2ZA) of Goodearl and Warfield [23], it is known that

\[O(R)/UO(R) \simeq (R/U)[x; \overline{\sigma}, \overline{\delta}]\]

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and hence $UO(R)$ is a completely prime ideal of $O(R)$.

To prove that a $(\sigma, \delta)$-integral domain is an Ore extension and vice versa, we begin in this section with the following theorem:

**Theorem 5.2.5:** Let $R$ be a Noetherian integral domain which is also an algebra over $\mathbb{Q}$. Let $\sigma$ be an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$ such that $R$ is a $(\sigma, \delta)$-ring, $\delta(P(R)) \subseteq P(R)$ and $\sigma(P) = P$, for all $P \in \text{Min.Spec}(R)$. Then $O(R)$ is 2-primal if and only if $O(P(R)) = P(O(R))$.

**Proof.** By Theorem (5.2.2), it follows that $P(O(R)) \subseteq O(P(R))$. Therefore, it only remains to show that $O(P(R)) \subseteq P(O(R))$. Let $$f(x) = \sum_{i=0}^{n} x^i a_i \in O(P(R))$$ which implies that $a_i \in P(R)$. $R$ is 2-primal, therefore, $a_i$ is nilpotent and thus $a_i \in N(O(R)) = P(O(R))$ for each $0 \leq i \leq n$ and hence $f(x) \in P(O(R))$. Thus $O(P(R)) = P(O(R))$.

Conversely, we have to show that $O(R)$ is 2-primal. Let $$f(x) = \sum_{i=0}^{n} x^i a_i \in O(R), a_i \neq 0$$ be such that $$(f(x))^2 \in P(O(R)) = O(P(R)).$$ We will show that $f(x) \in P(O(R))$. Now leading coefficient of $(f(x))^2$, $$\sigma^{2n-1}(a_n) \in P(R) \subseteq P,$$ where $P$ is the minimal prime ideal of $R$. Using Theorem (4.2.1), it follows that $P$ is a completely prime ideal of $R$. Hence $a_n \in P$ which implies that $a_n \in P(R)$. But $P(R)$ is $\delta$-invariant and $P$ is $\sigma$-stable. Therefore, for all $P \in \text{Min.Spec}(R)$, $$(\sum_{i=1}^{n} x^i a_i)^2 \in P(O(R)) = O(P(R))$$ and as shown above $a_{n-1} \in P(R)$. Repeating the same process $n$-times we have, $$a_i \in P(R), \text{ for all } i, 0 \leq i \leq n.$$
Therefore, \( f(x) \in O(P(R)) = P(O(R)) \). Hence \( P(O(R)) \) is a completely semiprime ideal of \( O(R) \). Thus \( O(R) \) is 2-primal. \( \square \)

**Theorem 5.2.6:** Let \( R \) be a Noetherian integral domain which is also an algebra over \( \mathbb{Q} \). Let \( \sigma \) be an automorphism of \( R \) and \( \delta \) a \( \sigma \)-derivation of \( R \) such that \( R \) is a \((\sigma,\delta)\)-ring, \( \delta(P(R)) \subseteq P(R) \) and \( \sigma(P) = P \), for all \( P \in \text{Min.Spec}(R) \). Then \( O(R) \) is 2-primal.

**Proof.** Let \( P_1 \in \text{Min.Spec}(R) \). Then by hypothesis \( \sigma(P_1) = P_1 \) and hence by Theorem (3) of [6], \( O(P_1) \in \text{Min.Spec}(O(R)) \). Also \( \sigma(P \cap R) = P \cap R \) implies that \( P \cap R \in \text{Min.Spec}(R) \) (by Theorem (3) of [6]). Thus \( O(P(R)) = P(O(R)) \) and the result follows by Theorem (5.2.5). \( \square \)

**Corollary 5.2.7:** Let \( R \) be a Noetherian integral domain which is also an algebra over \( \mathbb{Q} \). Let \( \sigma \) an automorphism of \( R \) and \( \delta \) a \( \sigma \)-derivation of \( R \) such that \( R \) is a \((\sigma,\delta)\)-ring. Also \( \sigma(\delta(a)) = \delta(\sigma(a)) \), for all \( a \in R \) and \( \sigma(P) = P \), for all \( P \in \text{Min.Spec}(R) \). Then \( O(R) \) is 2-primal.

**Proof.** Since a \((\sigma,\delta)\)-ring is 2-primal. Therefore, by Proposition (3) of [6], \( \delta(P_1) \subseteq P_1 \) where \( P_1 \) is a minimal prime ideal of \( R \) such that \( \sigma(P_1) = P_1 \).
Therefore, \( \delta(P(R)) \subseteq P(R) \) and the result follows by Theorem (5.2.6). \( \square \)

**Theorem 5.2.8:** Let \( R \) be a Noetherian integral domain which is also an algebra over \( \mathbb{Q} \). Let \( \sigma \) an automorphism of \( R \) and \( \delta \) a \( \sigma \)-derivation of \( R \) such that \( R \) is a \((\sigma,\delta)\)-ring. Assume also that \( \sigma(\delta(a)) = \delta(\sigma(a)) \), for all \( a \in R \). Then \( R[x;\sigma,\delta] \) is 2-primal Noetherian.

**Proof.** The result follows by Corollary (5.2.7), if we show that every minimal prime ideal of \( R \) is \( \sigma \)-stable. Let \( P = P_1 \) be a minimal prime ideal of \( R \) such that \( \sigma(P) \neq P \). Let \( P_2, P_3, \ldots, P_n \) be other minimal prime ideals of \( R \) and on renumbering let \( \sigma(P) = P_n \) (because \( \sigma(P) \) is also a minimal prime ideal). Let \( a \in \cap_{i=1}^{n-1} P_i \), then \( \sigma(a) \in P_n \). Also \( \delta(a) \in P_n \) (by Theorem (4.2.4)). So

\[
a(\sigma(a) + \delta(a)) \in \cap_{i=1}^{n-1} P_i = P(R)
\]

which implies that \( a \in P(R) \). Thus \( \cap_{i=1}^{n-1} P_i \subseteq P_n \) implies that \( P_i \subseteq P_n \) for some \( i \neq n \), which is not possible by definition of minimal prime ideals. Hence our supposition is wrong and the result follows. \( \square \)
**Theorem 5.2.9:** Let $R$ be a commutative Noetherian 2-primal integral domain which is also an algebra over $\mathbb{Q}$. Let $\sigma$ an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$. Assume also that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$. Then $R$ is a $(\sigma, \delta)$-ring if and only if $O(R) = R[x; \sigma, \delta]$ is a $(\sigma, \delta)$-ring.

**Proof.** Let $R$ be a $(\sigma, \delta)$-ring. Also by Theorem (5.2.5), $O(P(R)) = P(O(R))$. We show that $R[x; \sigma, \delta]$ is a $(\sigma, \delta)$-ring. Let $f \in O(R)$ say $f = \sum_{i=0}^{m} x^i a_i$ be such that $f[\sigma(f) + \delta(f)] \in P(O(R))$. We use induction on $m$ to prove the result. For $m = 1$, $f = xa_1 + a_0$. Now

$$f[\sigma(f) + \delta(f)] \in P(O(R))$$

implies that

$$(xa_1 + a_0)[\sigma(xa_1 + a_0) + \delta(xa_1 + a_0)] \in P(O(R)) = O(P(R))$$

i.e.,

$$x^2 \sigma^2(a_1) + x \sigma(\delta(a_1)) + x \sigma(a_0) \sigma(a_1) + \delta(\sigma(a_0)) + x a_1 \sigma(a_0) + a_0 \sigma(a_0) +$$

$$+ x a_1 x \delta(a_1) + xa_1 \delta(a_0) + a_0 x \delta(a_1) + a_0 \delta(a_0) \in O(P(R))$$

or

$$x^2 \sigma^2(a_1) + x \sigma(\delta(a_1)) + x \sigma(a_0) \sigma(a_1) + \delta(\sigma(a_0)) + x a_1 \sigma(a_0) + a_0 \sigma(a_0) +$$

$$+ x \sigma(a_1) \delta(a_1) + xa_1 \delta(a_0) + (x \sigma(a_0) + \delta(a_0)) \delta(a_1) + a_0 \delta(a_0) \in O(P(R)).$$

Now coefficient of leading term $\sigma(a_1)(\sigma(a_1) + \delta(a_1)) \in P(R)$, which implies $a_1 \in P(R)$, as in proof of Proposition (3) of [35] (using the fact that $R$ is 2-primal). Also coefficient of constant term

$$a_0(\sigma(a_0) + \delta(a_0)) + \delta(a_0) (\sigma(a_1) + \delta(a_1)) \in P(R)$$

(5.2.3)

Now $a_1 \in P(R)$, $\sigma(P(R)) = P(R)$. Therefore, $\sigma(a_1) \in P(R)$. Also $R$ is 2-primal, using Proposition (1.1) of [5], $\delta(a_1) \in P(R)$. Since $P(R)$ is an ideal, so $\delta(a_0)(\sigma(a_1) + \delta(a_1)) \in P(R)$. Therefore, (5.2.3) implies that $a_0(\sigma(a_0) + \delta(a_0)) \in P(R)$ which gives $a_0 \in P(R)$, as $R$ is a $(\sigma, \delta)$-ring. Hence $f \in O(P(R)) = P(O(R))$. 

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Suppose the result is true for \( m = k \). We prove for \( m = k + 1 \). Now

\[
f[\sigma(f) + \delta(f)] \in P(O(R))
\]

implies that

\[
(x^{k+1}a_{k+1} + ... + a_0)[x^{k+1}\sigma(a_{k+1}) + ... + \sigma(a_0) + x^{k+1}\delta(a_{k+1}) + ... + \delta(a_0)]
\]

\[
\in P(O(R)) = O(P(R))
\]
or

\[
x^{2k+2}\sigma^{k+2}(a_{k+1}) + x^{2k+1}\sigma^k(a_{k+1})\sigma(a_k) + x^{2k+1}\sigma^{k+1}(a_k)\sigma(a_{k+1}) + ... \\
+ x^{2k+2}\sigma^{k+1}(a_{k+1})\delta(a_{k+1}) + ... \in O(P(R))
\]

which gives on rearranging

\[
x^{2k+2}(\sigma^{k+2}(a_{k+1}) + \sigma^{k+1}(a_{k+1})\delta(a_{k+1}) + \text{term of } x^{2k+1} + g[\sigma(g) + \delta(g)] \\
\in O(P(R))
\]

where \( g = \sum_{i=0}^{k} x^ia_i \). Hence \( \sigma^{k+2}(a_{k+1}) + \sigma^{k+1}(a_{k+1})\delta(a_{k+1}) \in P(R) \) which implies that \( a_{k+1} \in P(R) \). Also coefficient of \( x^{2k+1} \in P(R) \) and so

\[
g[\sigma(g) + \delta(g)] \in P(R).
\]

But degree of \( g \) is \( k \), therefore, by induction hypothesis, the result is true for all \( m \).

Conversely, let \( O(R) \) be a \((\sigma, \delta)\)-ring. Then by definition so is \( R \). \( \square \)

### 5.3 Ore Extensions over weak \((\sigma, \delta)\)-rigid rings

Let \( R \) be a ring, \( \sigma \) an endomorphism of \( R \) and \( \delta \) a \( \sigma \)-derivation of \( R \). Recall that \( R \) is said to be a \((\sigma, \delta)\)-ring if \( a(\sigma(a) + \delta(a)) \in P(R) \) implies that \( a \in P(R) \) for \( a \in R \). We also recall that \( R \) is said to be a weak \((\sigma, \delta)\)-rigid ring if \( a(\sigma(a) + \delta(a)) \in N(R) \) implies and is implied by \( a \in N(R) \) for \( a \in R \), where \( N(R) \) is the set of nilpotent elements of \( R \).
In this section a relationship between the set of nilpotent elements of a com-
mutative Noetherian integral domain \( R \) and that of its Ore extension is dis-
cussed. Also we give a necessary and sufficient condition for a commutative
Noetherian integral domain and its Ore extension to be a weak \((\sigma, \delta)\)-rigid
ring.

**Theorem 5.3.1:** Let \( R \) be a commutative Noetherian integral domain which
is also an algebra over \( \mathbb{Q} \). Let \( \sigma \) be an automorphism of \( R \) and \( \delta \) a \( \sigma \)-derivation
of \( R \) such that \( R \) is a \((\sigma, \delta)\)-ring. Then \( O(N(R)) = N(O(R)) \).

**Proof.** By Theorem (2.2.3), we know that \( R \) a \((\sigma, \delta)\)-ring is 2-primal. Also
\( O(N(R)) \subseteq N(O(R)) \). We will show that \( N(O(R)) \subseteq O(N(R)) \). Let
\( f = \sum_{i=0}^{m} x^i a_i \in N(O(R)) \). Then \( (f)(O(R)) \subseteq N(O(R)) \), and \( (f)(R) \subseteq N(O(R)) \). Let \( (f)(R))^k = 0, k > 0 \). Then equating leading term to zero,
we get

\[
(x^m a_m R)^k = 0.
\]

After simplification equating leading term to zero, we get

\[
x^{km} \sigma^{(k-1)m}(a_m R) \cdot \sigma^{(k-2)m}(a_m R) \cdot \sigma^{(k-3)m}(a_m R) \cdots (a_m R) = 0.
\]

Therefore,

\[
\sigma^{(k-1)m}(a_m R) \cdot \sigma^{(k-2)m}(a_m R) \cdot \sigma^{(k-3)m}(a_m R) \cdots (a_m R) = 0 \subseteq P,
\]

for all \( P \in \text{Min. Spec}(R) \). This implies that

\[
\sigma^{(k-j)m}(a_m R) \subseteq P, \text{ for some } j, 1 \leq j \leq k.
\]

Therefore, by using Theorem (4.2.1)

\[
a_m R \subseteq \sigma^{-(k-j)m}(P) \subseteq \sigma(P) + \delta(P) = P.
\]

So we have \( a_m R \subseteq P \), for all \( P \in \text{Min. Spec}(R) \). Therefore, \( a_m \in P(R) \) and \( R 
\)

being 2-primal implies that \( a_m \in N(R) \). Now \( x^m a_m \in O(N(R)) \subseteq N(O(R)) \)

implies that \( \sum_{i=0}^{m-1} x^i a_i \in N(O(R)) \) and with the same process in a finite
number of steps, it can be seen that

\[
a_i \in P(R) = N(R), 0 \leq i \leq m - 1.
\]

Therefore, \( f \in O(N(R)) \). Hence \( N(O(R)) \subseteq O(N(R)) \) and result follows.

\( \square \)
Recall [48] that a ring $R$ with an endomorphism $\sigma$ is weak $\sigma$-rigid if $a\sigma(a) \in N(R)$ implies and is implied by $a \in N(R)$ for $a \in R$. We now state and prove Propositions which are needed to prove Theorem (5.3.4). We include the proof of Proposition (3) of [35] here for ready reference.

**Proposition 5.3.2:** Let $R$ be a Noetherian ring. Let $\sigma$ be an automorphism of $R$. Then $R$ is a weak $\sigma$-rigid ring implies that $N(R)$ is completely semiprime.

**Proof.** First of all we show that $\sigma(N(R)) = N(R)$. We have $\sigma(N(R)) \subseteq N(R)$ as $\sigma(N(R))$ is a nilpotent ideal of $R$. Now for any $n \in N(R)$, there exists $a \in R$ such that $n = \sigma(a)$. So

$$I = \sigma^{-1}(N(R)) = \{a \in R \text{ such that } \sigma(a) = n \in N(R)\}$$

is an ideal of $R$. Now $I$ is nilpotent, therefore $I \subseteq N(R)$, which implies that $N(R) \subseteq \sigma(N(R))$. Hence $\sigma(N(R)) = N(R)$.

Now let $R$ be a weak $\sigma$-rigid ring. We will show that $N(R)$ is completely semiprime. Let $a \in R$ be such that $a^2 \in N(R)$. Then

$$a\sigma(a)\sigma(a) = a\sigma(a)\sigma(a)\sigma^2(a) \in \sigma(N(R)) = N(R).$$

Therefore $a\sigma(a) \in N(R)$ and hence $a \in N(R)$. So $N(R)$ is completely semiprime.

Also we have the following:

**Proposition 5.3.3:** Let $R$ be a Noetherian $\mathbb{Q}$-algebra. Let $\delta$ be a derivation of $R$. Then $\delta(P(R)) \subseteq P(R)$.

**Proof.** See Proposition (1.1) of [5].

Before we state and prove the next Theorem of this section, we have the following as in Remark (5.2.3).

Let $\sigma$ be an endomorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$ such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$. Then $\sigma$ can be extended to an endomorphism say $\sigma$. 60
of $R[x;\sigma,\delta]$ by

$$\sigma(\sum_{i=0}^{m} x^i a_i) = \sum_{i=0}^{m} x^i \sigma(a_i).$$

Also $\delta$ can be extended to a $\sigma$-derivation say $\overline{\delta}$ of $R[x;\sigma,\delta]$ by

$$\overline{\delta}(\sum_{i=0}^{m} x^i a_i) = \sum_{i=0}^{m} x^i \delta(a_i).$$

We note that if $\sigma(\delta(a)) \neq \delta(\sigma(a))$, for all $a \in R$ then above does not hold. For example take $f(x) = xa + b$, $g(x) = xc + d$ for $a, b, c, d \in R$.

The proof of the next Theorem is on the same lines as in Theorem (5.2.9).

**Theorem 5.3.4:** Let $R$ be a commutative Noetherian 2-primal integral domain which is also an algebra over $\mathbb{Q}$. Let $\sigma$ an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$. Assume that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$. Then $R$ is a weak $(\sigma,\delta)$-rigid ring if and only if $O(R) = R[x;\sigma,\delta]$ is a weak $(\sigma,\overline{\delta})$-rigid ring.

**Proof.** Let $R$ be a weak $(\sigma,\delta)$-rigid ring. Then by Theorem (3.2.8), $R$ is a $(\sigma,\delta)$-ring. Also by Theorem (5.3.1), $O(N(R)) = N(O(R))$. We show that $R[x;\sigma,\delta]$ is a weak $(\sigma,\overline{\delta})$-rigid ring. Let $f \in O(R)$ say $f = \sum_{i=0}^{m} x^i a_i$ be such that $f[\sigma(f) + \overline{\delta}(f)] \in N(O(R))$. We use induction on $m$ to prove the result. For $m = 1$, $f = xa_1 + a_0$. Now

$$f[\sigma(f) + \overline{\delta}(f)] \in N(O(R))$$

implies that

$$(xa_1 + a_0)[\sigma(xa_1 + a_0) + \overline{\delta}(xa_1 + a_0)] \in N(O(R)) = O(N(R))$$

i.e.,

$$x^2\sigma^2(a_1) + x\delta(a_1)\sigma(a_1) + x\sigma(a_0)\sigma(a_1) + \delta(a_0)\sigma(a_1) + xa_1\sigma(a_0) + a_0\sigma(a_0) + xa_1x\delta(a_1) + xa_1\delta(a_0) + a_0x\delta(a_1) + a_0\delta(a_0) \in O(N(R))$$

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or

\[x^2\sigma^2(a_1) + x\delta(a_1)\sigma(a_1) + x\sigma(a_0)\sigma(a_1) + \delta(a_0)\sigma(a_1) + xa_1\sigma(a_0) + a_0\sigma(a_0) + x(x\sigma(a_1) + \delta(a_1))\delta(a_1) + xa_1\delta(a_0) + (x\sigma(a_0) + \delta(a_0))\delta(a_1) + a_0\delta(a_0) \in O(N(R)).\]

Now coefficient of leading term \(\sigma(a_1)(\sigma(a_1) + \delta(a_1))\) \(\in N(R)\), which implies \(a_1 \in N(R)\), as in proof of Proposition (5.3.2). Also coefficient of constant term

\[a_0(\sigma(a_0) + \delta(a_0)) + \delta(a_0)(\sigma(a_1) + \delta(a_1)) \in N(R) \tag{5.3.1}\]

Now \(a_1 \in N(R), \sigma(N(R)) = N(R)\), by Proposition (5.3.2). Therefore, \(\sigma(a_1) \in N(R)\). Also \(R\) is 2-primal, using Proposition (5.3.3), \(\delta(a_1) \in N(R)\). Since \(N(R)\) is an ideal, so \(\delta(a_0)(\sigma(a_1) + \delta(a_1)) \in N(R)\). Therefore, (5.3.1) implies that \(a_0(\sigma(a_0) + \delta(a_0)) \in N(R)\) which gives \(a_0 \in N(R)\), as \(R\) is a weak \((\sigma, \delta)\)-rigid ring. Hence \(f \in O(N(R)) = N(O(R))\).

Suppose the result is true for \(m = k\). We prove for \(m = k + 1\). Now \(f[\sigma(f) + \delta(f)] \in N(O(R))\)

implies that

\[(x^{k+1}a_{k+1} + ... + a_0)[x^{k+1}\sigma(a_{k+1}) + ... + \sigma(a_0) + x^{k+1}\delta(a_{k+1}) + ... + \delta(a_0)] \in N(O(R)) = O(N(R))\]

or

\[x^{2k+2}\sigma^{k+2}(a_{k+1}) + x^{2k+1}\sigma^k(a_{k+1})\sigma(a_k) + x^{2k+1}\sigma^{k+1}(a_k)\sigma(a_{k+1}) + ... + x^{2k+2}\sigma^{k+1}(a_{k+1})\delta(a_{k+1}) + ... \in O(N(R))\]

which gives on rearranging

\[x^{2k+2}(\sigma^{k+2}(a_{k+1}) + \sigma^{k+1}(a_{k+1})\delta(a_{k+1})) + \text{term of } x^{2k+1} + g[\sigma(g) + \delta(g)] \in O(N(R))\]
where $g = \sum_{i=0}^{k} x^i a_i$. Hence $\sigma^{k+2}(a_{k+1}) + \sigma^{k+1}(a_{k+1}) \delta(a_{k+1}) \in N(R)$ which implies that $a_{k+1} \in N(R)$. Also coefficient of $x^{2k+1} \in N(R)$ and so $g[\sigma(g) + \delta(g)] \in N(R)$. But degree of $g$ is $k$, therefore, by induction hypothesis, the result is true for all $m$.

Conversely, let $O(R)$ be a weak $(\sigma, \delta)$-rigid ring. Clearly, then so is $R$. □